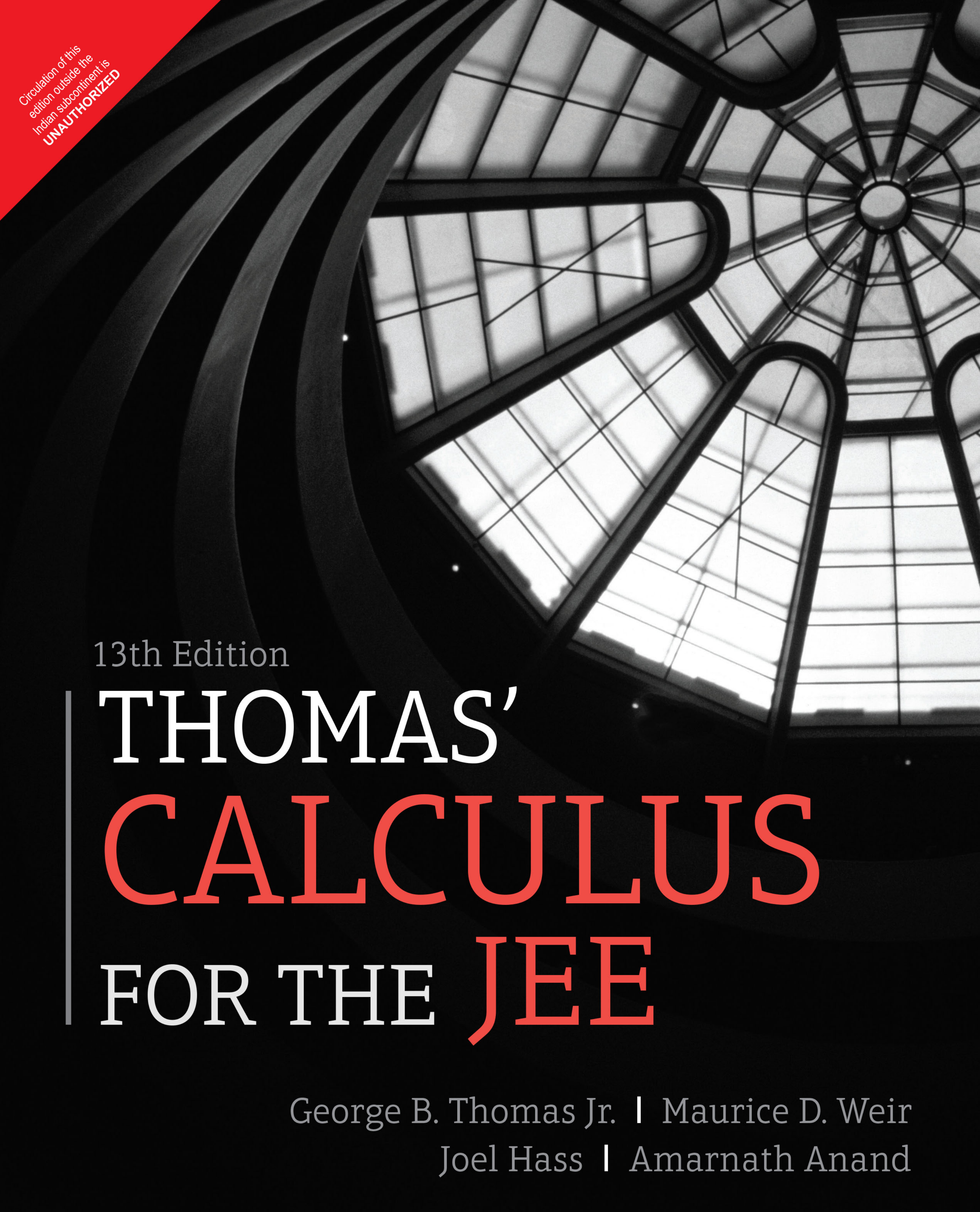

13th Edition

THOMAS' CALCULUS FOR THE JEE

George B. Thomas Jr. | Maurice D. Weir
Joel Hass | Amarnath Anand

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THOMAS' CALCULUS for the JEE

Thirteenth Edition

Based on the original work by

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as revised by

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PEARSON

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Preface to the Adaptation

Calculus is a branch of mathematics in which we study how things change and the rate at which things change. Calculus, as we know today, was developed in the later half of the 17th century by two great mathematicians, Gottfried Leibniz and Isaac Newton. It provides a framework for modeling a system in which there are variations, and it also provides different tools to predict such models. Calculus introduces concepts and tools to describe and analyse different functions. There are two main branches of calculus: differential calculus and integral calculus. This book presents a complete study of calculus to the readers. This book is meant for theory as well as practice and clarifies difficult concepts for the readers.

Sometimes, students only memorise how to solve problems without knowing its basics. This book helps students to gain knowledge and understand the concepts with clarity; thus, this book is mainly targeted for students who are preparing for IIT-JEE. For cracking such a prestigious exam, we need a very good and sound knowledge of calculus. The content of this book is as per the IIT-JEE syllabus. After every section, a rich collection of questions based on the concepts are provided. A student must attempt all the questions to master the concepts. At the end of every chapter, exercises have been added that are in accordance with the latest pattern of IIT-JEE examinations. These include single choice questions, multiple choice questions, passage type questions, matrix match type questions and integer type questions.

The quality of questions in these exercises is planned keeping in mind the level and requirement of IIT-JEE exam. This book also contains numerous solved examples which will help students in applying the concepts learned. The content of the book is well organized and user friendly. All suggestions for improvement are welcome. All the best to students for their bright future.

Amarnath Anand

Acknowledgments

I am thankful to Pearson Education for the efforts put in publishing this book on Calculus. My mother, Smt. Bhawani Anand, has always been a source of inspiration for me and a blessing in my life. I am indebted to my family members for their love and support.

I would like to thank all my colleagues for their advice and my students for their feedback and faith in me. I am also grateful and fortunate for being part of a highly prestigious institution for JEE preparation in India, wherein I have had an opportunity to interact with some of the finest minds in the industry.

Finally, I would like to thank my wife. This work would not have been possible without her support and sacrifice.

Amarnath Anand

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About the Adaptor

Amarnath Anand, a B.Tech., from IIT Delhi, has extensive experience in teaching mathematics at Apex JEE training institutions at Kota. With more than 11 years of experience in teaching mathematics, he has helped and guided a large pool of students to succeed in various premier engineering entrance examinations like IIT-JEE, AIEEE's and BITSAT. His unique method of teaching and intuitive solutions to even the most complex problems has established him as a popular faculty among his students.

1

Functions

OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are pictured as graphs, how they are combined and transformed, and ways they can be classified. We review the trigonometric functions.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these function ideas.

Functions: Domain, Codomain and Range

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels at constant speed along a straight-line path depends on the elapsed time.

In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we might call x. We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

In this notation, the symbol f represents the function, the letter x is the **independent variable** representing the input value of f, and y is the **dependent variable** or output value of f at x.

> **DEFINITION** A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line.

The set Y is called as codomain and the function is generally denoted as f: $X \to Y$ or $X \xrightarrow{f} Y$, which is read as f is a function from set X to set Y. Here, the definition emphasizes on two things:

1. No element in set X is left in the process.
2. Every element in set X is assigned a single element in set Y.

Image and Pre-image

If an element $x \in X$ is associated with an element $y \in Y$ under the rule f, then y is called as image or functional image of x and x is called as pre-image of y under the rule f. We can symbolically write it as $y = f(x)$.

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r (so r, interpreted as a length, can only be positive in this formula). When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, which is called the **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

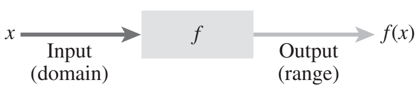


FIGURE 1.1 A diagram showing a function as a kind of machine.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions of a real variable we consider are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key.

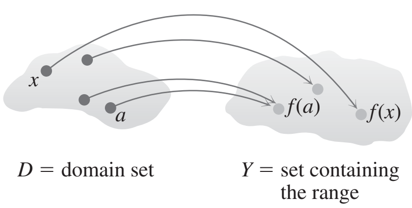


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates an element of the domain D with a unique or single element in the set Y. In Figure 1.2, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on. Notice that a function can have the same *value* at two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

EXAMPLE 1 Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

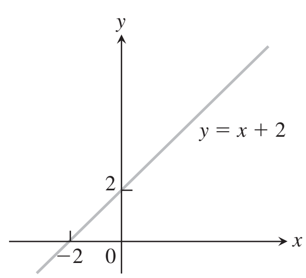


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

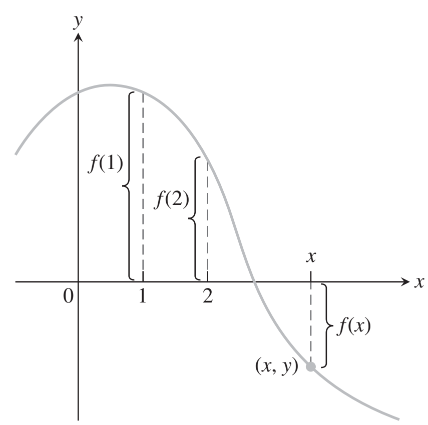


FIGURE 1.4 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

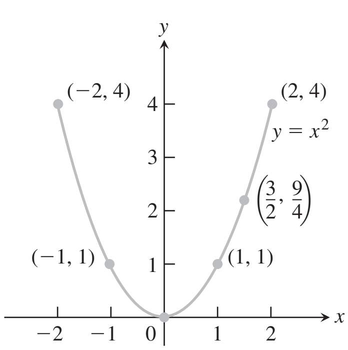
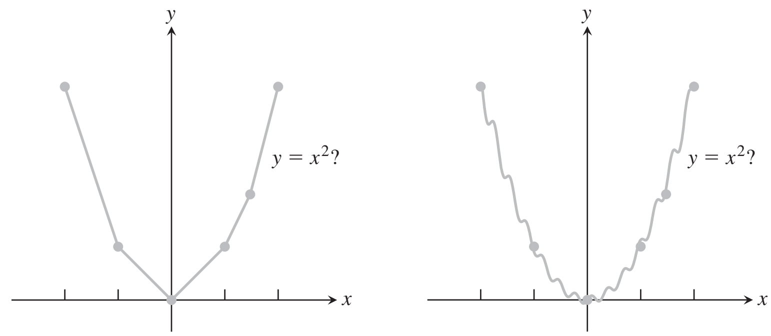


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

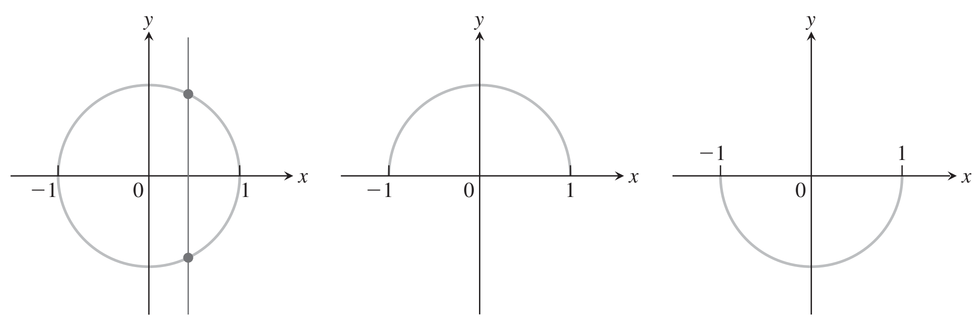


To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.6a, however, does contain the graphs of functions of x, such as the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.6b and 1.6c).



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

FIGURE 1.6 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

Increasing and Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

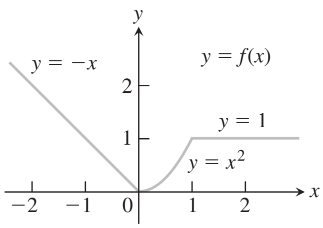


FIGURE 1.7 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 3).

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded) and by definition never consists of a single point (Appendix 1).

EXAMPLE 3 The function graphed in Figure 1.7 is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions. ■

Important Functions

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**. Figure 1.8a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope $m = 0$ (Figure 1.8b). A linear function with positive slope whose graph passes through the origin is called a *proportionality* relationship.

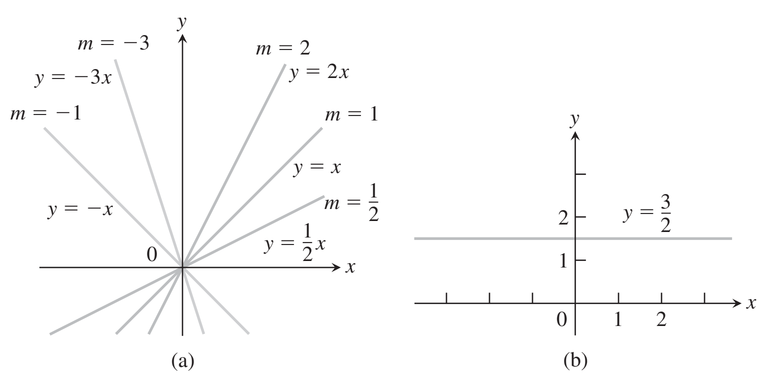


FIGURE 1.8 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

> **DEFINITION** Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.9. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$, and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y-axis; those with odd powers are symmetric

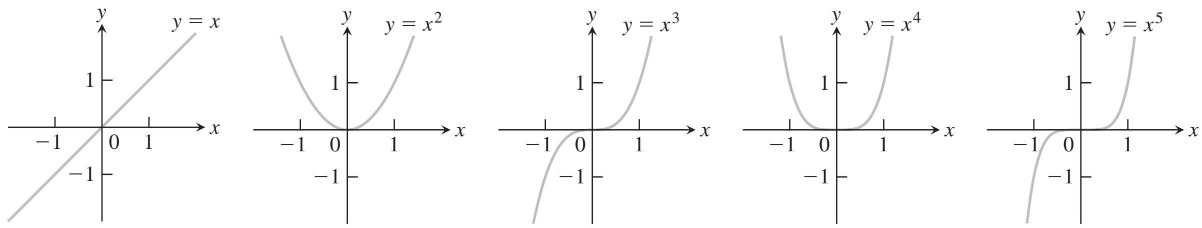


FIGURE 1.9 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

(b) $a = -1 \quad \text{or} \quad a = -2.$

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.10. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y-axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

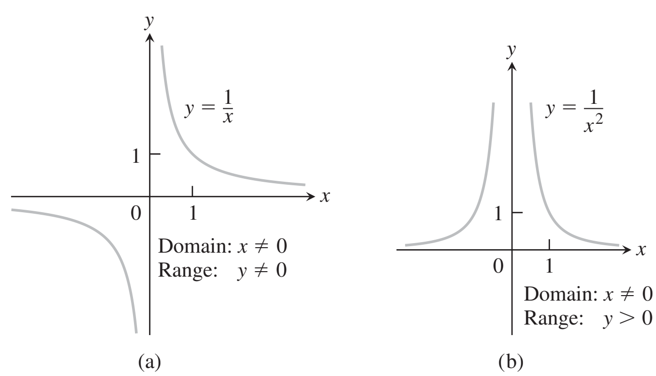


FIGURE 1.10 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \text{ and } \frac{2}{3}.$

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.11, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2,

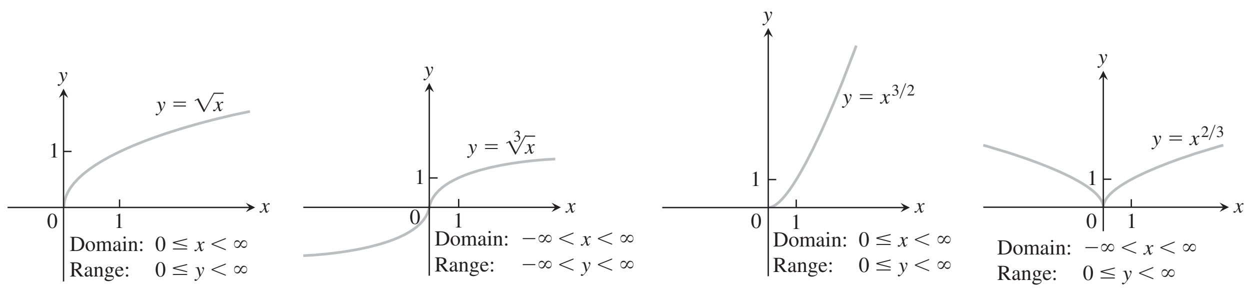


FIGURE 1.11 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.12 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

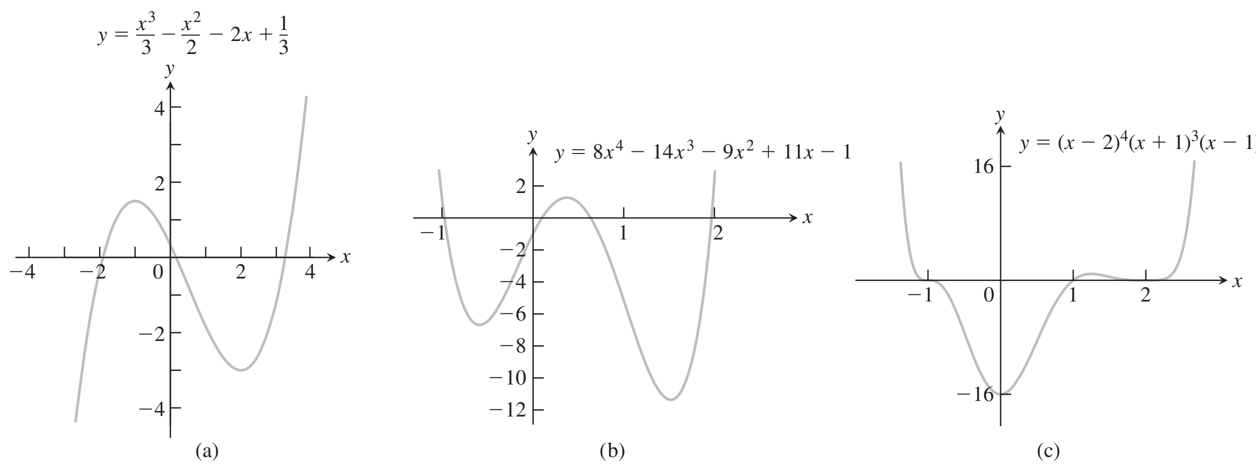


FIGURE 1.12 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.13.

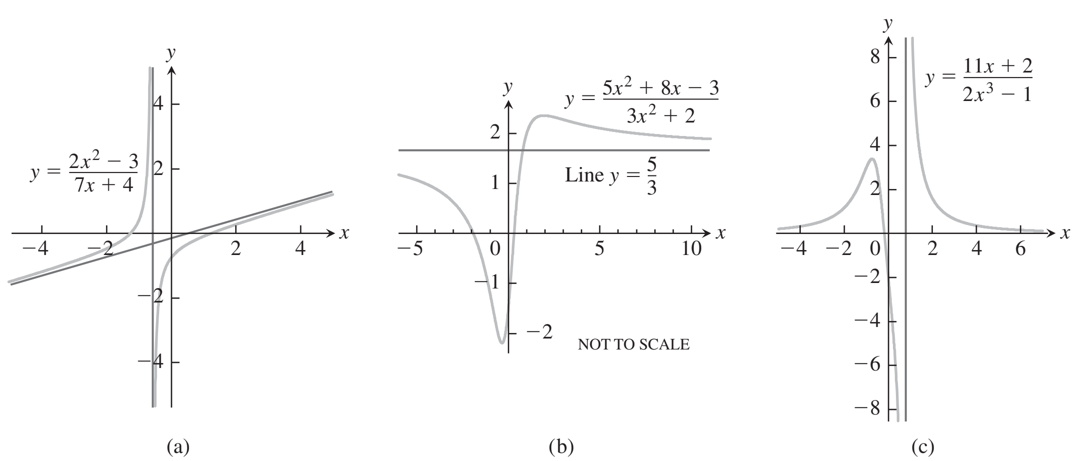


FIGURE 1.13 Graphs of three rational functions. The straight red lines approached by the graphs are called *asymptotes* and are not part of the graphs. We discuss asymptotes in Section 2.6.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.14 displays the graphs of three algebraic functions.

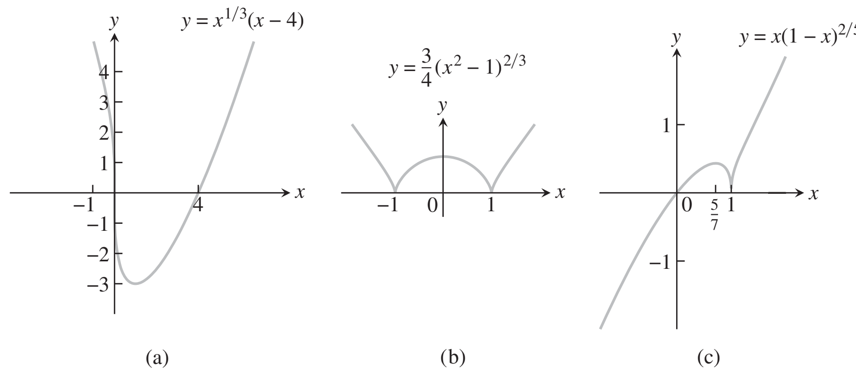


FIGURE 1.14 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.15.

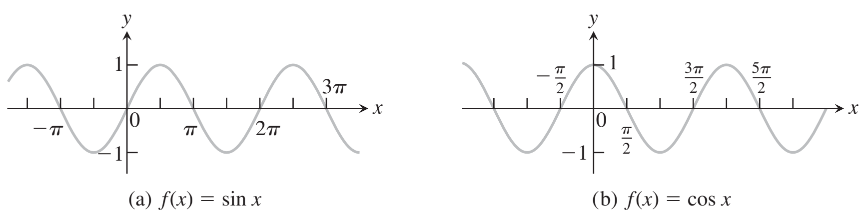


FIGURE 1.15 Graphs of the sine and cosine functions.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1.16.

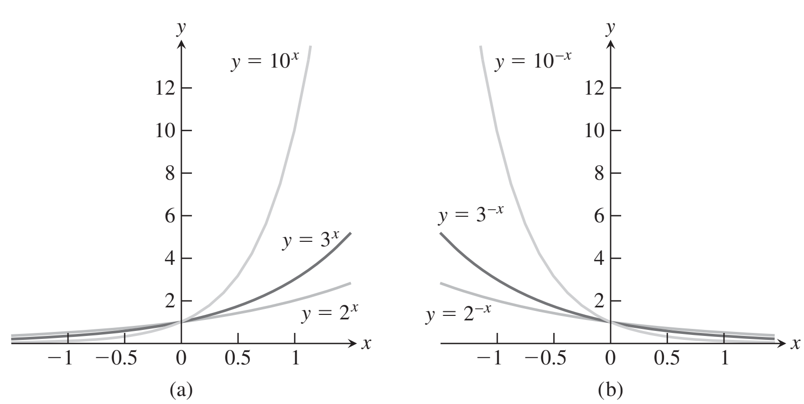


FIGURE 1.16 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions. Figure 1.17 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

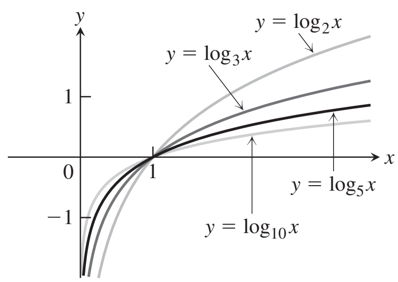


FIGURE 1.17 Graphs of four logarithmic functions.

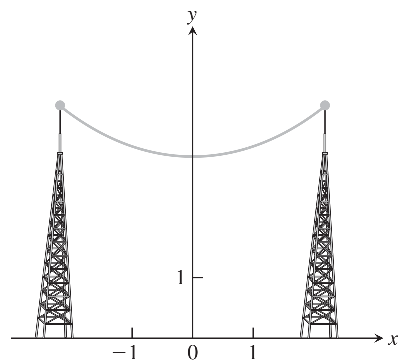


FIGURE 1.18 Graph of a catenary or hanging cable. (The Latin word *catena* means "chain.")

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a **catenary**. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.18).

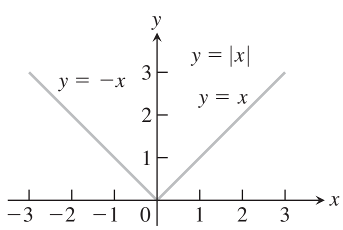


FIGURE 1.19 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \qquad \text{First formula} \\ -x, & x < 0, \qquad \text{Second formula} \end{cases}$$

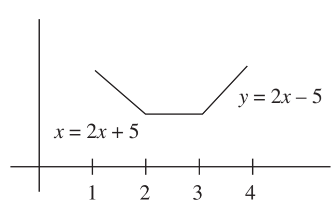
whose graph is given in Figure 1.19. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled.

Clearly from Figure 1.19, domain of $|x|$ is R and range is $[0, \infty)$

EXAMPLE 4 Sketch the graph of $f(x) = |x - 2| + |x - 3|$ when we are given more than one modulus as in this problem. First we can frame cases such that we can easily open modulus clearly we can see $|x - 2|$ changes its definition about $x = 2$ and $|x - 3|$ changes its definition about $x = 3$. Hence, the complete function $|x - 2| + |x - 3|$ can be opened as

$$|x-2| + |x-3| = \begin{cases} (x+2)+(x-3) = 2x-5 & x \geq 3 \\ (x-2)+(3-x) = 1 & 2 \leq x < 3 \\ (2-x)+(3-x) = 5-2x & x < 2 \end{cases}$$

Hence is can be seen that this piecewise-defined function has three different definitions for different values of x.



■

Signum function

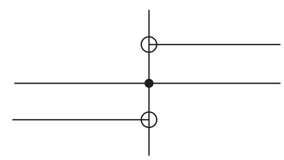
A function $y = f(x) = \text{sgn}(x)$ is defined as follows:

$$\text{Sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

It can be also written as

$$\text{Sgn}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Graph of the function is shown in the following figure.



Hence, the domain of $f(x)$ is $x \in R$ and range includes three values –1, 0, and 1.

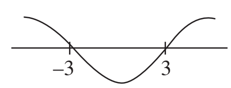
EXAMPLE 5 Sketch the graph of $y = \text{Sgn}(x^2 - 9)$

Solution Since sgn (x) depends upon the sign of x

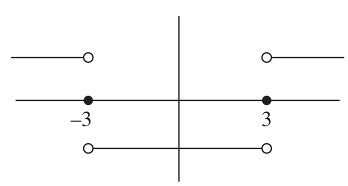
$$\text{Hence, Sgn}(x^2 - 9) \begin{cases} 1 & x^2 - 9 > 0 \\ 0 & x^2 - 9 = 0 \\ -1 & x^2 - 9 < 0 \end{cases}$$

We need to check the sign of $x^2 - 9$ which is shown below.

$$(x^2 - 9) = (x - 3)(x + 3)$$



$$\therefore \quad \text{Sgn}(x^2 - 9) \begin{cases} 1 & x \in (-\infty, -3) \cup (3, \infty) \\ 0 & x = \pm 3 \\ -1 & x \in (-3, 3) \end{cases}$$



■

Greatest Integer Function

The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.20 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$

Hence, domain of $[x]$ is $x \in R$ and range is a set of integers.

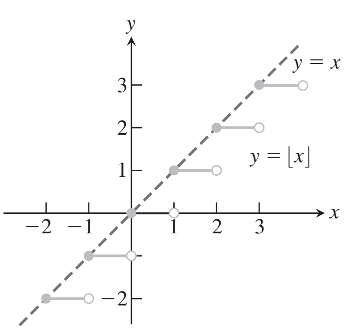


FIGURE 1.20 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

Properties of Greatest Integer Function

1. $[[x]] = [x]$

Proof: It is clear $[x]$ is some integer say "I" so [I] = I

2. $[x + m] = [x] + m$ (where "m" is an integer)

Proof: Let $x = I + f$ (where I is an integer and $f \in [0, 1)$)

Now $[x + m] = [I + m + f] = I + m = [x] + m$

3. $[x] + [-x] = \begin{cases} -1 & x \notin I \\ 0 & x \in I \end{cases}$

Proof: Let $x \notin I$

$\therefore$ $x = I + f$ (where I is an integer and $f \in [0, 1)$)

$\therefore$ $[x] = I$

Now $-x = -I - f$

$\Rightarrow$ $-x = (-I - 1) + (1 - f)$

Now $1 - f \in (0, 1)$

$\therefore$ $[-x] = -I - 1$

Hence $[x] + [-x] = I + (-I - 1) = -1$

Now, if $x = I \Rightarrow [x] = I$

and $-x = -I \Rightarrow [-x] = -I$

$\therefore$ $[x] + [-x] = 0$

4. $x - 1 < [x] \le x$

Proof: $x = I + f$ (where I is an integer and $f \in [0, 1)$)

$\therefore$ $[x] = I$

$\therefore$ $I \le I + f \Rightarrow$ $[x] \le x$ (equality holds if $f = 0$)

Now, $I + f - 1 > I$ (as $f - 1$ is a negative quantity)

$\therefore$ $x - 1 < [x]$

5. $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] \forall\, x \in R$

Proof:

Case I: Let $x = 2I + f$ (where I is an integer and $f \in [0, 1)$)

$\therefore$ L.H.S $= 2I$

R.H.S. $= \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] = \left[I + \frac{f}{2}\right] + \left[I + \frac{f+1}{2}\right]$

$= I + I \left(\because \frac{f+1}{2} \text{ and } \left[\frac{1}{2}, 1\right)\right)$

$= 2I$

Case II: Let $x = (2I + 1) + f$ (where I is an integer and $f \in [0, 1)$)

L.H.S. $= 2I + 1$

R.H.S. $= \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] = \left[I + \frac{f+1}{2}\right] + \left[I + 1 + \frac{f}{2}\right]$

$I + I + 1 = 2I + 1$

which completes the proof.

Fractional Part Function

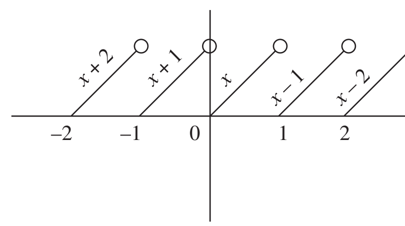
Every real number x can be expressed as sum of its integral part denoted by $[x]$ and its fractional part denoted by $\{x\}$.

$$\therefore \quad x = [x] + \{x\} \Rightarrow \{x\} = x - [x]$$

Now,

$$y = \{x\} = x - [x] = \begin{cases} x+2 & x \in [-2,-1) \\ x+1 & x \in [-1,0) \\ x & x \in [0,1) \\ x-1 & x \in [1,2) \\ x-2 & x \in [2,3) \end{cases}$$

Hence, its graph is



From the above graph it is clear that function repeats itself after a length of 1. Hence, it is a periodic function with period 1. Also, domain of function is real number set and range is [0, 1). Properties of fractional part function:

1. $\{x + n\} = \{x\}$ (where $n \in I$)

Proof: $x = I + f$ (where I is an integer and $f \in [0, 1)$)

$\therefore \quad x + n = I + n + f \Rightarrow \{x + n\} = \{I + n + f\} = \{f\} = \{x\}$

2. $[\{x\}] = 0$ (where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ denotes fractional part function.)

Proof: Let $x = I + f$ (where I is an integer and $f \in [0, 1)$)

Now $\quad \{x\} = \{I + f\} = f$

$\therefore \quad [\{x\}] = [f] = 0$ (As $f \in [0, 1)$)

3. $\{[x]\} = 0$

Proof: Clearly fractional part of an integer is 0.

4. $\{x\} + \{-x\} = \begin{cases} 1 & x \notin I \\ 0 & x \in I \end{cases}$

Proof: Since $[x] + [-x] = -1 \;\forall\, x \notin I \Rightarrow x - \{x\} - x - \{-x\} = -1 \Rightarrow \{x\} + \{-x\} = 1$

Least Integer Function

The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.21 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot that charges \$1 for each hour or part of an hour.

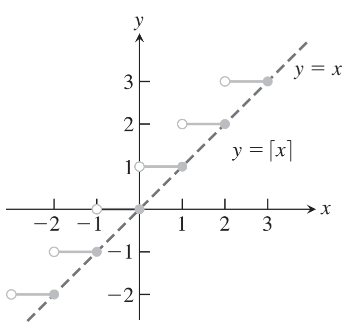


FIGURE 1.21 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 5).

Example of Domain and Range

EXAMPLE 6 The domain of definition of $f(x) = \dfrac{\log_2(x+3)}{x^2+3x+2}$ is

(a) $R - \{-1, -2\}$ (b) $(-2, -\infty)$

(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

Solution Since logarithm is involved, so $(x + 3)$ must be positive as well as denominator must be nonzero.

Hence, $x + 3 > 0 \Rightarrow x > -3$

and $x^2 + 3x + 2 \neq 0$

$\Rightarrow (x + 1)(x + 2) \neq 0$

$\Rightarrow x \neq -1$ and -2

$\therefore$ Domain of $f(x)$ is $(-3, \infty) - \{-2, -1\}$

$\therefore$ (d) is correct answer ■

EXAMPLE 7 Number of integers in the domain of function

$$f(x) = \log\left\{\log_{|\cos x|}(x^2 - 8x + 23) - \frac{3}{\log_2|\cos x|}\right\} \text{ is}$$

(a) 0 **(b)** 1 **(c)** 2 **(d)** More than two

Solution $f(x) = \log\{\log_{|\cos x|}(x^2 - 8x + 23) - 3\log_{|\cos x|}2\}$

$$= \log\left\{\log_{|\cos x|}\left(\frac{x^2 - 8x + 23}{8}\right)\right\}$$

Now, for $f(x)$ to be defined.

1. $\log_{|\cos x|}\left(\dfrac{x^2 - 8x + 23}{8}\right) > 0$

2. $\dfrac{x^2 - 8x + 23}{8} > 0$

3. $|\cos x| \neq 0$ and 1

Now, solving condition (1)

$$\frac{x^2 - 8x + 23}{8} < 1 \quad (\because |\cos x|, \text{ i.e., base is less than } 1)$$

$\Rightarrow x^2 - 8x + 23 < 8$

$\Rightarrow x^2 - 8x + 15 < 0$

$\Rightarrow (x - 3)(x - 5) < 0$

$\Rightarrow x \in (3, 5)$

Solving condition (2), we get

$$\frac{x^2 - 8x + 23}{8} > 0$$

$\Rightarrow (x - 4)^2 + 7 > 0$

$\Rightarrow x \in R$

Solving condition (3), we get

$$|\cos x| \neq 0 \text{ and } 1 \quad \Rightarrow \quad \cos x \neq 0 \text{ and } \pm 1$$

$\therefore \quad x \neq (2n + 1)\dfrac{\pi}{2}$ and $n\pi \ (n \in I)$

Taking intersection of all three conditions, we get

$$x \in (3, 5) - \left\{\pi, \frac{3\pi}{2}\right\}$$

$\therefore$ No. of integers in domain of $f(x)$ is 1.

$\therefore$ Correct answer is (b) ■

EXAMPLE 8 The domain of definition of the function $y(x)$ is given by the equation $2^x + 2^y = 2$, is

(a) $0 < x \le 1$ **(b)** $0 \le x \le 1$ **(c)** $-\infty < x \le 0$ **(d)** $-\infty < x < 1$

Solution Clearly the function is expressed implicitly

Now $\quad 2^y = 2 - 2^x$

As left-hand side is always positive. Hence, right-hand side is also positive.

$\therefore \quad 2 - 2^x > 0$

$\Rightarrow \quad 2^x < 2$

$\Rightarrow \quad x < 1$

$\therefore \quad x \in (-\infty, 1)$

$\therefore$ (d) is correct answer. ■

EXAMPLE 9 Range of the function $f(x) = \dfrac{x^2+x+2}{x^2+x+1}$; $(x \in R)$ is

(a) $(1, \infty)$ **(b)** $\left(1, \dfrac{11}{7}\right)$ **(c)** $\left(1, \dfrac{7}{3}\right]$ **(d)** $\left(1, \dfrac{7}{5}\right)$

Solution $f(x) = \dfrac{x^2+x+2}{x^2+x+1}$

Method 1:

We can first sketch the graph

$\because$ Domain of $f(x)$ is R. To check the monotonicity

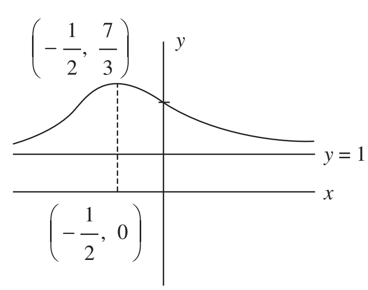
$$f'(x) = \frac{(x^2+x+1)(2x+1) - (x^2+x+2)(2x+1)}{(x^2+x+1)^2}$$

$$\Rightarrow \quad f'(x) = \frac{-(2x+1)}{(x^2+x+1)^2}$$

$\therefore$ for $x > -\dfrac{1}{2} \Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing

and for $x < -\dfrac{1}{2} \Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing

Clearly as $x \to \pm\infty \Rightarrow f(x) \to 1$



$\therefore$ Range of $f(x)$ is $\left(1, \dfrac{7}{3}\right]$

Method 2:

$$y = \frac{x^2 + x + 2}{x^2 + x + 1}$$

$\Rightarrow \quad x^2y + xy + y = x^2 + x + 2$ (after cross multiplication)

$$\Rightarrow \quad (y - 1)\,x^2 + (y - 1)x + (y - 2) = 0$$

Clearly $x \in R$ and let $y - 1 \neq 0 \Rightarrow y \neq 1$; we can see a quadratic with real roots.

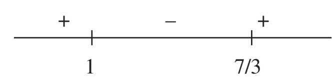
$$\therefore \quad D \geq 0$$

$$\Rightarrow \quad (y - 1)^2 - 4(y - 1)\,(y - 2) \geq 0$$

$$(y - 1)\,(y - 1 - 4y + 8) \geq 0$$

$$\therefore \quad (y - 1)\,(-3y + 7) \geq 0$$

$$\Rightarrow \quad (y - 1)\,(3y - 7) \leq 0$$



$$\therefore \quad y \in \left[1, \frac{7}{3}\right]$$

$$\because \quad y \neq 1$$

$$\therefore \quad y \in \left(1, \frac{7}{3}\right]$$

Now, we have assumed $y \neq 1$ for our convenience so that theory of equation can be applied. So, we must check corresponding to $y = 1$, there exists any x or not

$$\therefore \quad 1 = \frac{x^2 + x + 2}{x^2 + x + 1}$$

$\Rightarrow \quad 2 = 1$ (which is absurd)

So, there is no x for which $y = 1$.

$\therefore$ Range of $f(x)$ is $\left(1, \frac{7}{3}\right]$

Method 3:

$f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$ which can be simplified as $f(x) = 1 + \dfrac{1}{x^2 + x + 1}$

Now, we need to calculate the range of $\dfrac{1}{x^2 + x + 1}$

Now, $\quad x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$

$\therefore \quad x^2 + x + 1 \in \left[\frac{3}{4}, \infty\right) \qquad \frac{1}{x^2 + x + 1} \in \left(0, \frac{4}{3}\right]$

$\therefore$ Range of $f(x)$ is $\left(1, \frac{7}{3}\right]$

$\therefore$ Correct answer is (c) ■

EXAMPLE 10 The number of integers not in domain of $f(x) = \dfrac{1}{[|x-1|]+[|7-x|]-6}$ (where $[\cdot]$ denotes greatest integer function)

(a) 3 **(b)** 5 **(c)** 7 **(d)** 9

Solution For $f(x)$ to be defined, only denominator should be nonzero, i.e., $[|x-1|] + [|7-x|] - 6 \neq 0$. So we can first find out for what values of x the above expression becomes zero and can exclude it from real number set. So, let us solve the following equation.

$$[|x-1|] + [|x-7|] = 6 \text{ (As } |x-7| = |7-x| \ \forall\, x \in R)$$

Since the equation contains modulus, we can frame "3" cases.

Case I: $x \geq 7$

$$[x-1] + [x-7] = 6$$

$\Rightarrow$ $[x] - 1 + [x] - 7 = 6$ $\quad (\because\ [x+n] = [x] + n \ \forall\, n \in I)$

$\Rightarrow$ $2[x] = 14$

$\Rightarrow$ $[x] = 7$

$\Rightarrow$ $x \in [7, 8)$ which satisfies the above condition.

Case II: $x \in (1, 7)$

$\therefore$ $[x-1] + [7-x] = 6$

$\Rightarrow$ $[x] - 1 + 7 + [-x] = 6$

$\Rightarrow$ $[x] + [-x] = 0$

$\Rightarrow$ $x \in I$

Since $x \in (1, 7)$. Hence possible values of x are 2, 3, 4, 5, and 6.

Case III: $x \leq 1$

$\therefore$ $[1-x] + [7-x] = 6$

$\Rightarrow$ $8 + 2[-x] = 6$

$\Rightarrow$ $[-x] = -1$

$\Rightarrow$ $-x \in [-1, 0)$

$\Rightarrow$ $x \in (0, 1]$

which satisfies the above condition.

$\therefore$ Domain of (x) is $x \in R - (0, 1] \cup \{2, 3, 4, 5, 6\} \cup [7, 8)$

Hence, no. of integers not in domain of (x) is 7.

$\therefore$ (C) is correct answer. ■

EXAMPLE 11 In which of the following cases there does not exist any function:

(a) $f^2(x^2) + f(2^x) + 1 = 0 \ \forall\, x \in R$

(b) $f^2(x^2 - x) - 4f(2x-2) + x^2 + x = 0 \ \forall\, x \in R$

(c) $f(x) + f(1/x) = 2x \ \forall\, x \in R - \{0\}$

(d) $f(\sin x) + f(\cos x) = \forall\, x \in R$

Solution Checking option wise

(a) $f^2(x)^2 + f(2^x) + 1 = 0$

Put $x = 2 \Rightarrow f^2(4) + f(4) + 1 = 0$, so there exist no value of $f(4)$ which will satisfy the equation. Hence not valid for $x \in R$

(b) $f^2(x^2 - x) - 4f(2x - 2) + x^2 + x = 0$

Put $x = 1$

$$f^2(0) - 4f(0) + 2 = 0$$

$\Rightarrow$ $f(0)$ has two values. Hence cannot be a function.

(c) $f(x) + f(1/x) = 2x$

Put $x \to 1/x$

$\therefore$ $$f(1/x) + f(x) = 2/x.$$

$\therefore$ $$2x = 2/x$$

$\Rightarrow$ $$x = \pm 1$$

i.e., function is not defined for $x \in R - \{0\}$

(d) $f(\sin x) + f(\cos x) = x$

Put x as $\dfrac{\pi}{2} - x$

$\therefore$ $$f(\cos x) + f(\sin x) = \frac{\pi}{2} - x$$

i.e., $$x = \frac{\pi}{2} - x$$

$\Rightarrow$ $$x = \frac{\pi}{4}.$$

Hence not defined for $x \in R$.

$\therefore$ Correct answers are (a), (b), (c), and (d). ■

EXAMPLE 12 Let $A = \sin^2\theta + \cos^4\theta$, then for all real values of θ

(a) $1 \le A \le 2$ **(b)** $\dfrac{3}{4} \le A \le 1$ **(c)** $\dfrac{13}{16} \le A \le 1$ **(d)** $\dfrac{3}{4} \le A \le \dfrac{13}{16}$

Solution $A = 1 - \cos^2\theta + \cos^4\theta$

$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2$$

A_{max} can be achieved when $\left(\cos^2\theta - \dfrac{1}{2}\right)^2$ is minimum, i.e., $\cos^2\theta = \dfrac{1}{2}$

$\therefore$ $A_{min} = \dfrac{3}{4} \Rightarrow \dfrac{3}{4} \le A \le 1$

$\therefore$ correct answer is (b) ■

EXAMPLE 13 Range of the function $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 3\ \forall\ x \in [-4, 2]$ is

(a) $[0, 363]$ **(b)** $[2, 363]$ **(c)** $[-2, 363]$ **(d)** $(-\infty, 363]$

Solution $f(x) = (x^2 + 5x + 4)(x^2 + 5x + 6) + 3$

Put $x^2 + 5x + 4 = t$

$$(x^2 + 5x + 4)(x^2 + 5x + 6) + 3 = t(t + 2) + 3 = (t + 1)^2 + 2$$

$\therefore$ Minimum value of $f(x) = 2$ when $t = -1$

i.e., $x^2 + 5x + 5 = 0$

$$x = \frac{-5 \pm \sqrt{5}}{2}$$

Clearly $f(x)$ will achieve maximum for $x = 2$

$\therefore$ $$f(x)|_{\max} = 3 \cdot 4 \cdot 5 \cdot 6 + 3 = 363$$

$\therefore$ Correct answer is (b). ■

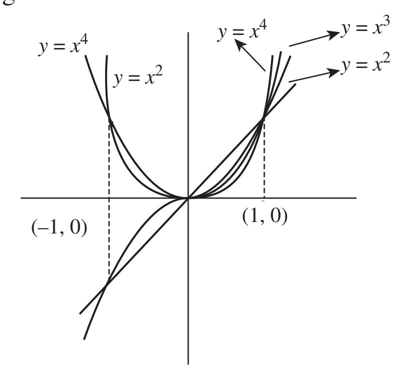
EXAMPLE 14 For each real x, let $f(x) = \max\{x, x^2, x^3, x^4\}$, then which of the following options is/are correct ?

(a) $f(x) = x^4\ \forall\ x \le -1$ **(b)** $f(x) = x^2\ \forall\ -1 < x \le 0$

(c) $f\left(\frac{1}{2}\right) = \frac{1}{2}$ **(d)** $f\left(\frac{1}{2}\right) = \frac{1}{4}$

Solution Give, $f(x) = \max\ \{x, x^2, x^3, x^4\}$

This function is defined by sketching all the functions involved in the comparison on same scale and then selecting the maximum.



From the graph, it is clear that

$$f(x) = \begin{cases} x^4 & x \le -1 \\ x^2 & - < x \le 0 \\ x & 0 \le x \le 1 \\ x^4 & 1 \le x \end{cases}$$

$\therefore$ Correct answers are (a), (b) and (c). ■

EXAMPLE 15 Let $f(x) = |x - p| + |x - 15| + |x - p - 15|$, where $p \in (0,15)$, then the minimum value of $f(x)\ \forall\ x \in [p, 15]$ is

(a) 0 **(b)** 15 **(c)** 30 **(d)** does not exist

Solution Clearly this is related with defining $f(x)$ under the given circumstances:

$$|x - p| = x - p \text{ (as } x \ge p)$$

$$|x - 15| = 15 - x \text{ (as } x \le 15|$$

$|x - p - 15| = 15 + p - x$ (as p is positive and $x - 15 \le 0$

$\therefore$ $f(x) = 30 - x$

$\therefore$ for $f(x)|_{\min} = 15$ when $x = 15$

$\therefore$ Correct answer is (b). ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

> **DEFINITIONS** A function $y = f(x)$ defined in a symmetrical interval about origin is an
>
> **even function of *x*** if $f(-x) = f(x)$,
> **odd function of *x*** if $f(-x) = -f(x)$,
>
> for every x in the function's domain.

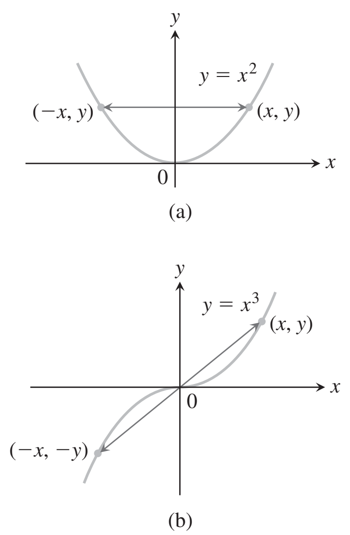


FIGURE 1.22 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

The names *even* and *odd* come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the *y*-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.22a). A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.22b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.

EXAMPLE 16 Prove that $f(x) = \dfrac{(1+2^x)^2}{2^x}$ is an even function.

Solution $$f(-x) = \frac{(1+2^{-x})^2}{2^{-x}} = \frac{\left(1+\dfrac{1}{2^x}\right)^2}{2^{-x}} = \frac{(2^x+1)^2}{2^{2x}\cdot 2^{-x}} = \frac{(2^x+1)^2}{2^x} = f(x)$$

$\therefore$ $f(x)$ is even as it is also defined in symmetrical interval.

EXAMPLE 17 Prove that $f(x) = \ln(x + \sqrt{1+x^2})$ is an odd function.

Solution $$f(-x) = \ln(-x + \sqrt{1+x^2}) = \ln\left(\frac{\left(-x+\sqrt{1+x^2}\right)\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}}\right) = \ln\left(\frac{1}{x+\sqrt{1+x^2}}\right)$$

$$= -\ln(x + \sqrt{1+x^2}) = -f(x)$$

Aliter:

Also, $$f(-x) + f(x) = \ln(-x + \sqrt{1+x^2}) + \ln(x + \sqrt{1+x^2})$$

$$= \ln(1 - x^2 - x^2) = 0$$

$\Rightarrow$ $$f(-x) = -f(x)$$

$\therefore$ $f(x)$ is an odd function ■

Properties of Odd and Even Function

1. A function may be neither odd nor even, for example, $f(x) = \sin x + \cos x$. $f(-x) = -\sin x + \cos x$. Hence it is neither even nor odd.
2. Inverse of an even function is not defined and an even function cannot be monotonic.
3. Every function defined on a symmetric interval can be uniquely expressed as sum of odd and even function.

For example,

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd}}$$

Suppose $$h(x) = \frac{f(x)+f(-x)}{2}$$

$\Rightarrow$ $$h(-x) = \frac{f(-x)+f(x)}{2}$$

$\therefore$ $h(-x) = h(x)$; Hence it is an even part.

Now, let $$g(x) = \frac{f(x)-f(-x)}{2}$$

$\Rightarrow$ $$g(-x) = \frac{f(-x)-f(x)}{2} = -g(x)$$

$\therefore$ $g(x)$ is an odd part.

4. A function defined for $x \in R$ and which is odd as well as even at the same time is $f(x) = 0$. As it is satisfying $f(-x) = -f(x)\ \forall\ x \in R$ as well as $f(-x) = f(x)\ \forall\ x \in R$.

5. Any nonzero constant function is even. For example, $f(x) = 2\ \forall\ x \in R$ is even as $f(-x) = 2$. Hence $f(-x) = f(x)\ \forall\ x \in R$.

6. If $f(x)$ and $g(x)$ are functions defined on symmetric interval, then we have

$f(x)$	$g(x)$	$f(x)+g(x)$	$f(x)-g(x)$	$f(x)\times g(x)$	$f(x)/g(x)$	$f(g(x))$	$g(f(x))$
Odd	Odd	Odd	Odd	Even	Even	Odd	Odd
Even	Even	Even	Even	Even	Even	Even	Even
Odd	Even	N.O.N.E	N.O.N.E	Odd	Odd	Even	Even
Even	Odd	N.O.N.E	N.O.N.E	Odd	Odd	Even	Even

where N.O.N.E. denotes neither odd nor even.

Few of the proofs are done here and remaining can be worked out in the same manner.

If $f(x)$ is odd and $g(x)$ is odd (where $f(x)$ and $g(x)$ are defined in symmetric interval), then.

1. $h(x) = f(x) + g(x)$

$\Rightarrow$ $h(-x) = f(-x) + g(-x) = -(f(x) + g(x)) = -h(x)$

Hence $f(x) + g(x)$ is odd.

2. $h(x) = f(x) - g(x)$

$\Rightarrow$ $h(-x) = f(-x) - g(-x) = -f(x) + g(x) = -(f(x) - g(x)) = -h(x)$

Hence $f(x) - g(x)$ is an odd function.

3. $h(x) = f(x) \times g(x)$

$\Rightarrow$ $h(-x) = f(-x) \times g(-x) = (-f(x)) \times (-g(x)) = f(x) \times g(x) = h(x)$

Hence, $f(x) \times g(x)$ is an even function.

4. $h(x) = \dfrac{f(x)}{g(x)}$

$\Rightarrow$ $$h(-x) = \frac{f(-x)}{g(-x)} = \frac{(-f(-x))}{(-g(-x))} = \frac{f(x)}{g(x)}$$

Hence, $\dfrac{f(x)}{g(x)}$ is an even function.

5. $h(x) = f(g(x))$

$\Rightarrow \quad h(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -h(x).$

Hence, $f(g(x))$ is an odd function

Similarly other proofs can be worked out.

EXAMPLE 18 Which of the following function(s) is/are odd function(s)?

(a) $f(x) = \dfrac{x}{2^x - 1} + \dfrac{x}{2}$

(b) $f(x) = \cos^2 x + \cos^2\left(\dfrac{\pi}{3} + x\right) - \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

(c) $f(x) = \dfrac{\sin^2 x}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}}$ (where $[\cdot]$ denotes greatest integer function)

(d) $f(x) \begin{cases} x\,|x|, & x \le -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x\,|x| & x \ge 1 \end{cases}$ (where $[\cdot]$ denotes greatest integer function.)

Solution

(a) $f(-x) = \dfrac{-x}{2^{-x} - 1} - \dfrac{x}{2}$

$$= \frac{x \cdot 2^x}{2^x - 1} - \frac{x}{2}$$

Now, in cases when we cannot directly conclude whether $f(-x) = -f(x)$ or $f(-x) = f(x)$ then it is better clearly,

$$f(-x) + f(x) = \frac{x}{2^x - 1} + \frac{x}{2} + \frac{x \cdot 2^x}{2^x - 1} - \frac{x}{2} = \frac{x \cdot (1+2)^x}{2^x - 1} \neq 0$$

Also, $$f(-x) - f(x) = \frac{x \cdot 2^x}{2^x - 1} - \frac{x}{2} - \frac{x}{2^x - 1} - \frac{x}{2} - \frac{x(2^x - 1)}{2^x - 1} - x = x - x = 0$$

$\Rightarrow \quad f(-x) = f(x)$

$\Rightarrow \quad f(x)$ is even function.

(b) $f(x) = \cos^2 x + \cos^2\left(\dfrac{\pi}{3} + x\right) - \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

$$= 1 - \sin^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \frac{2}{2}\cos x \cos\left(x + \frac{\pi}{3}\right)$$

$$= 1 + \left(\cos^2\left(x + \frac{\pi}{3}\right) - \sin^2 x\right) - \frac{1}{2}\left(\cos\left(2x + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right)$$

$$= 1 + \cos\left(2x + \frac{\pi}{3}\right) \cdot \cos\frac{\pi}{3} - \frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{4}$$

(as $\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$

$$= \frac{3}{4} + \frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right) = \frac{3}{4}$$

Hence $f(x)$ is an even function.

(c) $f(x) = \dfrac{\sin^2 x}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}} = \dfrac{\sin^2 x}{\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}} \left[\text{As } \left[\dfrac{x}{\pi} + 1\right] = \left[\dfrac{x}{\pi}\right] + 1\right]$

Now, $f(-x) = \dfrac{\sin^2 x}{\left[-\dfrac{x}{\pi}\right] + \dfrac{1}{2}}$

Case I: If $\left[-\dfrac{x}{\pi}\right] = -\left[\dfrac{x}{\pi}\right]$ and $x = \pi.I$

$\therefore \quad f(-x) = \dfrac{\sin^2 x}{-\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}}$

$\because \quad x = \pi.I$

$\therefore \quad f(-x) = 0$ and $f(x) = 0$

$\forall \dfrac{x}{\pi} = I$

Case II: If $\dfrac{x}{\pi} \neq \text{I}$ (where I denotes integer)

$\left[-\dfrac{x}{\pi}\right] + \left[\dfrac{x}{\pi}\right] = -1$ (as $\dfrac{x}{\pi}$ is not integer)

$\therefore \quad \Rightarrow \left[-\dfrac{x}{\pi}\right] = -1 - \left[\dfrac{x}{\pi}\right]$

Now, $f(-x) = \dfrac{\sin^2 x}{\left[-\dfrac{x}{\pi}\right] + \dfrac{1}{2}} = \dfrac{\sin^2 x}{-1 - \left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}} = -\left(\dfrac{\sin^2 x}{\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}}\right) = -f(x)$

$\therefore \quad f(-x) = -f(x) \ \forall \ x \notin \pi \cdot I$

$= -f(x) = 0 \ \forall \ x \in \pi \cdot I$

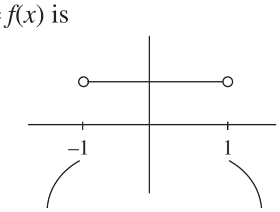
$\therefore \quad f(-x) = -f(x) \ \forall \ x \in R$

$\therefore$ $f(x)$ is an odd function.

(d) $f(x) \begin{cases} x\,|\,x\,| & x \le -1 \\ [1+x]+[1-x] & -1 < x < 1 \\ -x\,|\,x\,| & x \ge 1 \end{cases} \Rightarrow f(x) \begin{cases} x^2 & x \le -1 \\ 2+[x]+[-x] & -1 < x < 1 \\ -x^2 & x \ge 1 \end{cases}$

$\therefore \quad f(x) \begin{cases} -x^2 & x \le -1 \\ 1 & -1 < x < 1 \\ -x^2 & x \ge 1 \end{cases}$

$\therefore$ Graph of function $y = f(x)$ is



$\therefore$ Graph is symmetric about y-axis which suggests $f(-x) = f(x)$. Hence function is even.

$\therefore$ Correct answer is (c). ■

EXAMPLE 19 The absolute value of sum of all possible values of α for which the equation $|2x + 2|x| + 2\alpha| + |2x - 2|x| - 2\alpha| = 2016$ has exactly three solutions is

(a) –504 **(b)** 0 **(c)** 504 **(d)** 1008

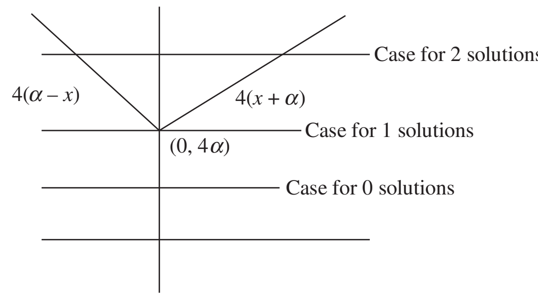
Solution $f(x) = |2x + 2|x| + 2\alpha| + |2x - 2|x| - 2\alpha|$

We can first sketch the graph of function $f(x)$

Case I: Suppose $\alpha \geq 0$

Now, $f(x) = \begin{cases} |4x+2\alpha|+|2\alpha| & x \geq 0 \\ |2\alpha|+|4x-2\alpha| & x < 0 \end{cases} \Rightarrow f(x) = \begin{cases} 4x+4\alpha & x \geq 0 \\ 2\alpha-4x+2\alpha & x < 0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} 4(x+\alpha) & x \geq 0 \\ 4(\alpha-x) & x < 0 \end{cases}$



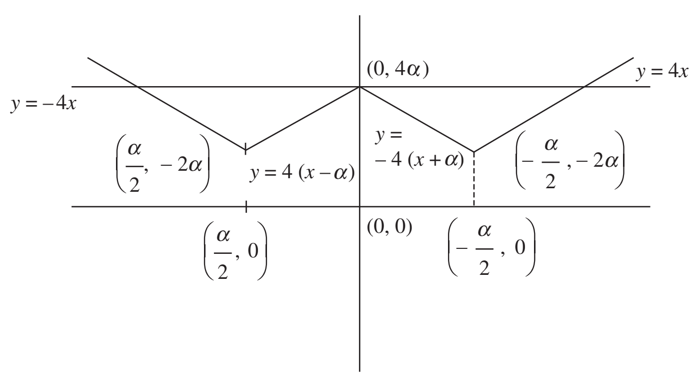
Clearly the number of real solutions of $f(x) = g(x)$ is the point of intersection $y = f(x)$ and $y = g(x)$. Now, the solution of $|2x + |2x| + 2\alpha| + |2x - 2|x| - 2\alpha| = 2016$ is 2 or 1 or 0 for $\alpha \geq 0$. As, if we draw $g(x) = 2016$. This will be a straight line parallel to x-axis and hence it will intersect the graph at atmost two points. Therefore, three solutions are not possible.

Case II: If $\alpha < 0$

$$f(x)\begin{cases} |4x+2\alpha|+|2\alpha| & x \geq 0 \\ |2\alpha|+|4x-2\alpha| & x < 0 \end{cases}$$

$$\Rightarrow \quad f(x)\begin{cases} 4x+2\alpha-2\alpha & x \geq -\dfrac{\alpha}{2} \\ -4x-2\alpha-2\alpha & 0 \leq x < -\dfrac{\alpha}{2} \\ -2\alpha+4x-2\alpha & \dfrac{\alpha}{2} \leq x < 0 \\ -2\alpha-4x+2\alpha & x < \dfrac{\alpha}{2} \end{cases}$$

$\therefore$ Graph of $f(x) = |2x + 2|x| + 2\alpha| + |2x - 2|x| - 2\alpha|$ is



$\therefore$ $|2x + 2\,|\,x\,| + 2\alpha\,| + |2x - 2|\,x\,| - 2\alpha| = 2016$ has three solutions if $-4\alpha = 2016$ $\Rightarrow \alpha = -504$

$\therefore$ Sum of absolute values of possible values of α is 504.

$\therefore$ Correct answer is (c).

Aliter:

Let $f(x) = |2x + 2\,|\,x\,| + 2\alpha| + |2x - 2|\,x\,| - 2\alpha\,|$

At $f(-x) = |-2x + 2|\,x\,| + 2\alpha\,| + |-2x - 2|\,x\,| - 2\alpha|$

$$= |2x - 2|\,x\,| - 2\alpha| + |2x + 2\,|\,x\,| + 2\alpha| = f(x)$$

Hence $f(x)$ is an even function. As we need three solutions for $f(x) = 2016$ of the solutions graph of $f(x)$ is symmetrical about x-axis. Also $y = 2016$ is parallel to x-axis and they must intersect at three points. Also number of point of intersection of $y = f(x)$ and $y = 2016$ on the left and right of y-axis must be same. Hence number of solutions will be three only when $x = 0$ is a solutions of equation; hence it will satisfy the equation. But $x = 0$ will not guarantee the number of solutions to be 3 as it is only the necessary condition. If $x = 0$ is a solution of the above equation, it means number of solutions of the equation can be 1, 3, 5, …

Now, putting $x = 0$, gives

$$|2\alpha| + |-2\alpha| = 2016$$

$$\Rightarrow \quad |\alpha| = 504$$

$$\Rightarrow \quad \alpha = \pm 504$$

For $\alpha = 504$, the equation becomes

$$|2x + 2|\,x\,| + 1008| + |2x - 2|\,x\,| - 1008\,| = 2016$$

$$\Rightarrow \quad |x + |\,x\,| + 504| + |x - |x| - 504| = 1008$$

For $x \geq 0$ $|2x + 504| + |504| = 1008$

$$\Rightarrow \quad 2x + 504 = 504$$

$$\Rightarrow \quad x = 0$$

Now for $x < 0$ $|504| + |2x - 504| = 1008$.

$$\Rightarrow \quad 504 - 2x + 504 = 1008$$

$\Rightarrow$ $x = 0$ (rejected as we have considered $x < 0$)

Hence the only solution of the equation is $x = 0$.

So, the equation does not possess three solutions for $\alpha = 504$.

Now, for $\alpha = -504$, the equation becomes

$$|2x + 2|\,x\,| - 1008\,| + |2x - 2|\,x\,| + 1008 = 2016$$

when $x \geq 0$

$$\Rightarrow \quad |4x - 1008| + 1008 = 2016$$

$$\Rightarrow \quad |4x - 1008| = 1008$$

$$\Rightarrow \quad 4x - 1008 = \pm(1008)$$

$$\Rightarrow \quad 4x = 2016 \quad \text{or} \quad 4x = 0$$

$$\Rightarrow \quad x = 504 \quad \text{or} \quad x = 0$$

$\therefore$ Both are acceptable as $x \geq 0$

$$|-1008| + |4x + 1008| = 2016$$

$\Rightarrow \quad 4x + 1008 = \pm(1008)$

$\Rightarrow \quad 4x = 0 \quad \text{or} \quad 4x = -2016$

$\Rightarrow \quad x = 0$ (Rejected as $x < 0$) or $x = -504$ (accepted)

Hence the solutions are $x = -504$, 0 and 504. Therefore $\alpha = -504$ is the only solution.

$\therefore$ Absolute value of sum of all possible values of α is 504. ■

EXAMPLE 20 If f: [–25, 25] $\to R$ defined by $f(x) = \left[\frac{x^2}{a}\right]\sin x + x^4$ (where [·] denotes greatest integer function) is an even function then complete set of values of a is:

(a) $(-\infty, \infty)$ **(b)** $[625, \infty)$ **(c)** $(625, \infty)$ **(d)** None of these

Solution $\because \quad f(x) = \left[\frac{x^2}{a}\right]\sin x + x^4$ is an even function

$\therefore \quad f(-x) = f(x) \ \forall\ x \in [-25, 25]$

$\Rightarrow \quad \left[\frac{x^2}{a}\right]\sin(-x) + x^4 = \left[\frac{x^2}{a}\right]\sin x + x^4$

$\Rightarrow \quad 2\left[\frac{x^2}{a}\right]\sin x = 0 \ \forall\ x \in [-25, 25]$

Now, $\sin x$ cannot be identically 0 $\forall\ x \in [-25, 25]$

Hence $\left[\frac{x^2}{a}\right] = 0 \ \forall\ x \in [-25, 25]$, which is possible if $0 \le \frac{x^2}{a} < 1 \ \forall\ x \in [-25, 25]$

$\therefore \quad a > 625$

$\Rightarrow \quad a \in (625, \infty)$

Hence, correct answer is (c). ■

EXAMPLE 21 If "f" is an even function, find the real value of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$.

Solution Clearly is function is even then $f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x)$

$$\Rightarrow \quad x = \frac{(x+1)}{(x+2)}$$

$$\text{and} \quad \frac{x+1}{x+2} = -x$$

$$\Rightarrow \quad x^2 + 2x = x + 1$$

$$\Rightarrow \quad x + 1 = -x^2 - 2x$$

$$\Rightarrow \quad x^2 + x - 1 = 0$$

$$\Rightarrow \quad x^2 + 3x + 1 = 0$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad x = \frac{-3 \pm \sqrt{5}}{2}$$

$\therefore$ Possible values of x are $\frac{-3-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{\sqrt{5}-3}{2}, \frac{\sqrt{5}-1}{2}$ ■

Exercises 1.1

Functions

In Exercises 1–4, find the domain and range of each function.

1. $f(x) = 1 - \sqrt{x}$

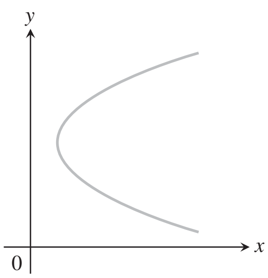
2. $g(x) = \sqrt{x^2 - 3x}$

3. $f(t) = \dfrac{4}{3 - t}$

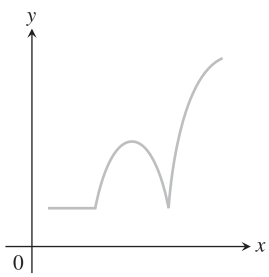
4. $G(t) = \dfrac{2}{t^2 - 16}$

In Exercises 5 and 6, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

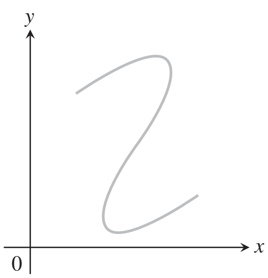
5. a.



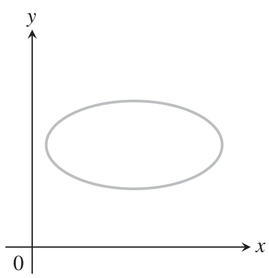
b.



6. a.



b.



Finding Formulas for Functions

7. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

8. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.

9. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

10. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

11. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x.

12. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y.

Functions and Graphs

Find the natural domain and graph the functions in Exercises 13–18.

13. $f(x) = 5 - 2x$

14. $f(x) = 1 - 2x - x^2$

15. $g(x) = \sqrt{|x|}$

16. $g(x) = \sqrt{-x}$

17. $F(t) = t/|t|$

18. $G(t) = 1/|t|$

19. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

20. Find the range of $y = 2 + \dfrac{x^2}{x^2 + 4}$.

21. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$ **b.** $y^2 = x^2$

22. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 23–26.

23. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

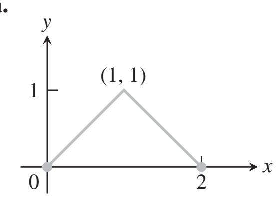
24. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

25. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

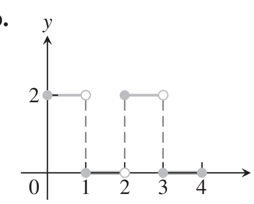
26. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

Find a formula for each function graphed in Exercises 27–30.

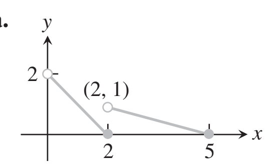
27. a.



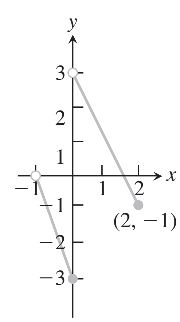
b.



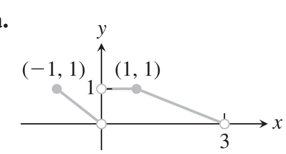
28. a.



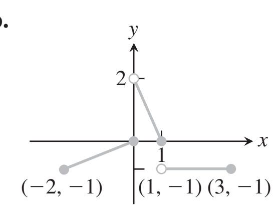
b.



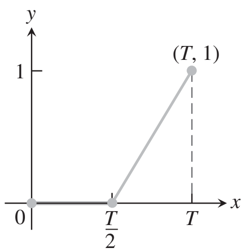
29. a.



b.



30. a.



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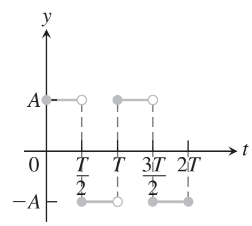


The Greatest and Least Integer Functions

31. For what values of x is

a. $\lfloor x \rfloor = 0$? **b.** $\lceil x \rceil = 0$?

32. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

33. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

34. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Increasing and Decreasing Functions

Graph the functions in Exercises 35–44. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

35. $y = -x^3$ **36.** $y = -\dfrac{1}{x^2}$

37. $y = -\dfrac{1}{x}$ **38.** $y = \dfrac{1}{|x|}$

39. $y = \sqrt{|x|}$ **40.** $y = \sqrt{-x}$

41. $y = x^3/8$ **42.** $y = -4\sqrt{x}$

43. $y = -x^{3/2}$ **44.** $y = (-x)^{2/3}$

Even and Odd Functions

In Exercises 45–56, say whether the function is even, odd, or neither. Give reasons for your answer.

45. $f(x) = 3$ **46.** $f(x) = x^{-5}$

47. $f(x) = x^2 + 1$ **48.** $f(x) = x^2 + x$

49. $g(x) = x^3 + x$ **50.** $g(x) = x^4 + 3x^2 - 1$

51. $g(x) = \dfrac{1}{x^2 - 1}$ **52.** $g(x) = \dfrac{x}{x^2 - 1}$

53. $h(t) = \dfrac{1}{t - 1}$ **54.** $h(t) = |t^3|$

55. $h(t) = 2t + 1$ **56.** $h(t) = 2|t| + 1$

Theory and Examples

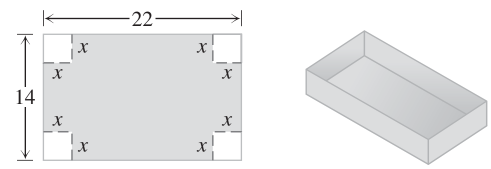
57. The variable s is proportional to t, and $s = 25$ when $t = 75$. Determine t when $s = 60$.

58. Kinetic energy The kinetic energy K of a mass is proportional to the square of its velocity v. If $K = 12{,}960$ joules when $v = 18$ m/sec, what is K when $v = 10$ m/sec?

59. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.

60. Boyle's Law Boyle's Law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7\ \text{lb/in}^2$ when $V = 1000\ \text{in}^3$, then what is V when $P = 23.4\ \text{lb/in}^2$?

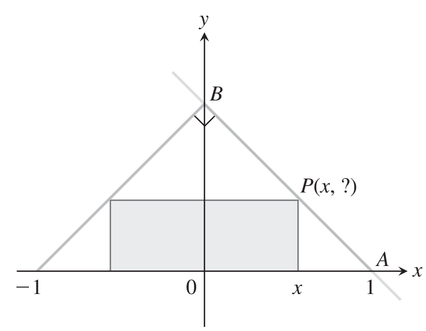
61. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 in. by 22 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



62. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

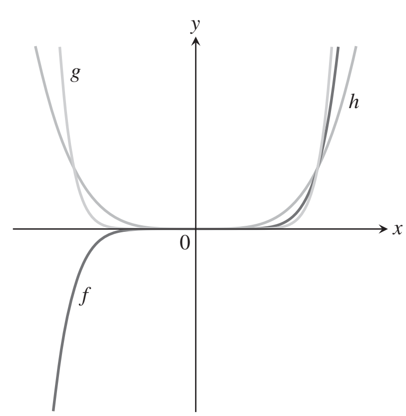
a. Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b. Express the area of the rectangle in terms of x.

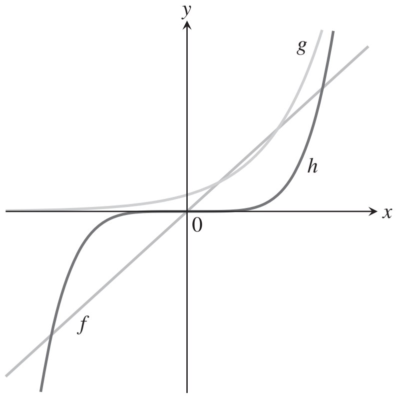


In Exercises 63 and 64, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

63. a. $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



64. **a.** $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



T **65.** **a.** Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

T **66.** **a.** Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

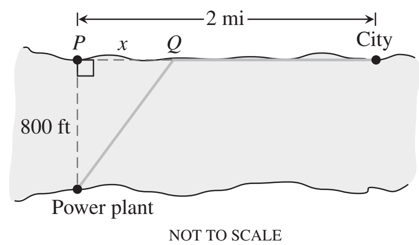
$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b. Confirm your findings in part (a) algebraically.

67. Three hundred books sell for \$40 each, resulting in a revenue of (300)(\$40) = \$12,000. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.

68. A pen in the shape of an isosceles right triangle with legs of length x ft and hypotenuse of length h ft is to be built. If fencing costs \$5/ft for the legs and \$10/ft for the hypotenuse, write the total cost C of construction as a function of h.

69. Industrial costs A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



a. Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

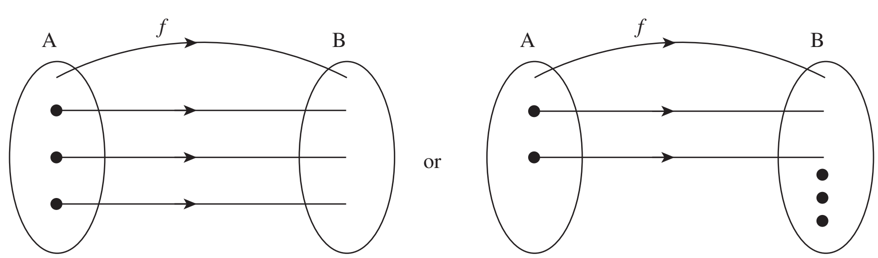
b. Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

1.2 Classification of Functions and Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are classified as well as combined or transformed to form new functions.

Classification of Function

One-one function Let f: A $\rightarrow$ B be a function such that every element in set A has distinct image in set B, then f is called as one-one function; they can be also called as injective mapping. Diagrammatically, it can be shown as:



Different ways of testing for function to be one-one function:

1. for $x_1, x_2 \in$ A and $f(x_1), f(x_2) \in$ B, symbolically we can write

$$f(x_1) = f(x_2) \quad \Leftrightarrow \quad x_1 = x_2$$

or

$$f(x_1) \neq f(x_2) \quad \Leftrightarrow \quad x_1 \neq x_2$$

EXAMPLE 1 Prove that $f(x) = x^3 + 2$ is one-one function.

Solution Let $x_1, x_2 \in$ A

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad x_1^3 + 2 = x_2^3 + 2$$

$$\Rightarrow \quad (x_1 - x_2)\,(x_1^2 + x_1x_2 + x_2^2) = 0$$

$$\Rightarrow \quad x_1 = x_2$$

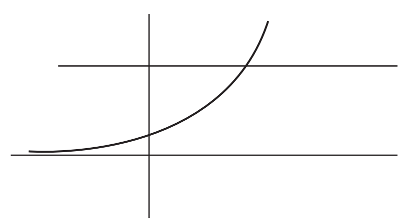
Hence a unique solution $x_1 = x_2$

$\therefore$ $f(x)$ is one-one function.

2. Graphically, we can say if any line parallel to x-axis intersects the graph of function atmost once then it is called one-one function. ■

EXAMPLE 2 Check graphically whether the function $f(x) = e^x$ is one-one or not?

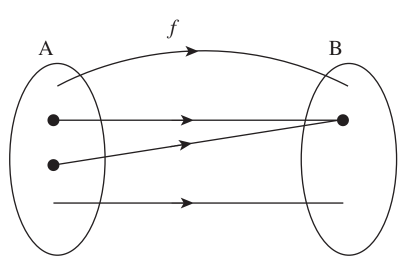
Solution It can be seen if a line $y = R$ is drawn for different values of R, the line intersects the graph at 1 point or O point. Hence one-one function

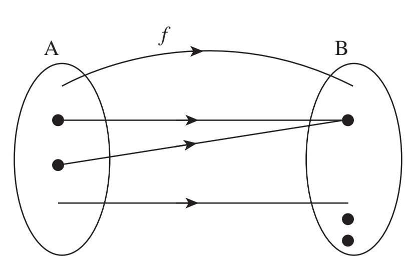


■

Method of Monotonicity If a function $f = f(x)$ is continuous and monotonic on an interval I, i.e., increasing or decreasing then $f(x)$ is one-one function monotonicity will discussed later in this book.

Many-one Function A function f: A $\to$ B is said to be many-one function if two or more element in A have same image in B. A function is said to be many-one if it is not one-one diagrammatically, a many-one function can be shown as:

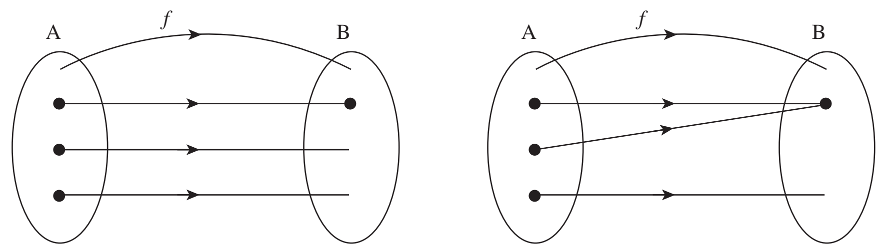




Onto Function

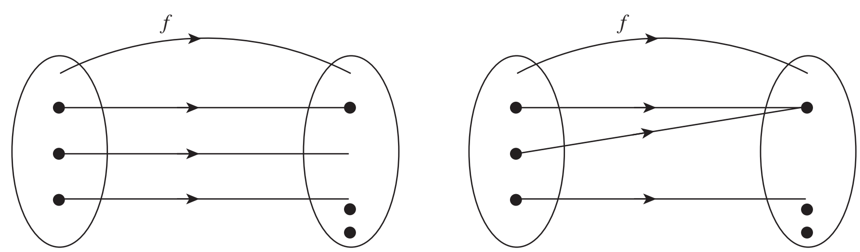
If the function f: A $\rightarrow$ B is such that each element in B (co-domain) is the image of at least one element in A, then f is said to be onto function. Onto functions are also known as surjective functions.

Diagrammatically surjective mapping can be shown as:



Into Function

If f: A $\rightarrow$ B is such that there exists at least one element in co-domain which is not the image of any element in domain, then f is said to be an into function. Diagrammatically into function can be shown as:



Working rule for a function to check onto and into function:

If range of a function = co-domain of a function then it is said to be onto function.

EXAMPLE 5 If f: $R \rightarrow R$ is defined by $f(x) = x^4 - x + 1$. Is $f(x)$ onto or into?

Solution Clearly $f(x)$ an even degree polynomial so range of the function is subset of real number set

$\therefore$ Range $\neq$ co-domain. Hence function is into.

Clearly, a function can be any of these four types.

(i) One-one onto function
(ii) One-one into function
(iii) Many-many onto function
(iii) Many-many into function

Note: A one-one onto function is also known as bijective mapping. It is also named as invertible, nonsingular, or biuniform function. ■

Example on Classification of Function

EXAMPLE 6 The function f: $[0, \infty) \to [0, \infty)$ defined by $f(x) = \dfrac{x}{1+x}$ is

(a) one-one and onto **(b)** one-one but not onto
(c) onto but not one-one **(d)** neither one-one nor onto

Solution Given $f(x) = \dfrac{x}{1+x} = \dfrac{x+1-1}{x+1} = 1 - \dfrac{1}{x+1}$

$\because$ $f'(x) = \dfrac{1}{(x+1)^2} > 0$. Hence function is increasing $\forall\, x \ge 0$

$\therefore$ Function is one-one as function is increasing on $[0, \infty)$. Hence range is

$$f(0) = 0 \text{ and } \lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1$$

$\therefore$ Range of function is $[0, 1)$ which is not co-domain. Hence function is into.

$\therefore$ Correct answer is (b). ■

EXAMPLE 7 Let the function f: $R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is

(a) one-to-one and onto **(b)** one-to-one but not onto
(c) onto but not one-one **(d)** neither one-to-one nor onto.

Solution Given $f(x) = 2x + \sin x$

$$f'(x) = 2 + \cos x > 0 \qquad (\text{As } -1 \le \cos x \le 1\ \forall\, x \in R)$$

Hence function is increasing $\forall\, x \in R$ and therefore one-one.

Now the range of the function is $(-\infty, \infty)$ as

When $x \to -\infty$, $f(x) \to -\infty$ as well as for $x \to \infty$, $f(x) \to \infty$

$\therefore$ Function is one-one as well as onto

$\therefore$ Correct answer is (a) ■

EXAMPLE 8 Consider $f: R \to R$ and $g: R \to R$ defined as $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$,

$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x, & \text{if } x \text{ is rational} \end{cases}$ then $(f(x) - g(x))$ is

(a) one-one and onto **(b)** neither one-one nor onto
(c) many one and onto **(d)** one-one and onto

Solution $f(x) - g(x) = \begin{cases} -x, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

Clearly, the function is onto as it can be seen as x varies over real number set the function $f(x) - g(x)$ takes values $-x$ for $x \in$ rational and x for $x \in$ irrational so the range is R. Similarly $\forall\, x_1, x_2 \in$ R; $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence function is one-one and onto

$\therefore$ Correct answer is (a). ■

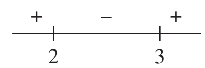
EXAMPLE 9 The function f: $[0, 3] \to [1, 29]$ defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

(a) one-one and onto **(b)** onto but not one-one

(c) one-one but not onto **(d)** neither one-one nor onto

Solution $f(x) = 2x^3 - 15x^2 + 36x + 1$

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6\,(x - 2)\,(x - 3)$$



Clearly $f(x)$ increases in $[0, 2)$ and decreases in $(2, 3]$. Hence the function is many-one Now, clear $x = 2$ is point of maximum and minimum in $[0, 3]$ will be attained at $x = 0$ or $x = 3$

$\therefore$ $f(x)|_{\max} = f(2) = 16 - 60 + 72 + 1 = 29$

Now $f(0) = 1$ and $f(3) = 54 - 135 + 108 + 1 = 28$

$\therefore$ Range of $f(x)$ is $[1, 29]$. Hence onto function.

$\therefore$ Correct answer is (b). ■

EXAMPLE 10 Which of the following function(s) would represent a bijective mapping?

(a) $f: R \to R$ $f(x) = |\,x\,|$ sgn x (where sgn(x) denotes signum function)

(b) $g: R \to R$ $g(x) = x^{3/5}$

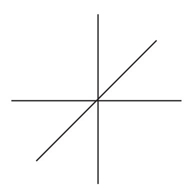
(c) $h: R \to R$ $h(x) = x^4 + 3x^2 + 1$

(d) $K: R \to R$ $K(x) = \dfrac{3x^2 - 7x + 6}{x - x^2 - 2}$

Solution

(a) $f(x) = |\,x\,|\,\text{sgn}(x) = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ x & x < 0 \end{cases}$

i.e., $f(x) = x$



Hence function is bijective.

(b) $g(x) = x^{3/5} \Rightarrow g'(x) = \dfrac{3}{5}\, x^{-2/3} > 0$. Hence increasing and one-one

Also range is $(-\infty, \infty)$. Hence bijective.

(c) $h(x) = x^4 + 3x^2 + 1$

Clearly $h(-x) = h(x)$ which suggest many-one and range of function is $[1, \infty)$. Hence function is not bijective.

(d) $K(x) = \dfrac{3x^2 + 7x + 6}{x - x^2 - 2}$

Clearly $3x^2 - 9x + 6 > 0$ (As $D = 49 - 72 = -23 < 0$) $\forall\ x \in R$

and $x - x^2 - 2 < 0\ \forall\ x \in R$ (As $D = 1 - 8 = -7 < 0$)

$\therefore$ $k(x)$ cannot take positive values. Hence the function is into as codomain is R.

Now we can differentiate the above function and can test for one-one or many-one otherwise, for objective purpose if we can show two values of x for which $K(x)$ is same then function automatically becomes many-one.

Now, for $k(x) = -1 \quad \Rightarrow \quad \dfrac{3x^2 - 7x + 6}{x - x^2 - 2} = -1$

$\Rightarrow \quad 3x^2 - 7x + 6 = -x + x^2 + 2$

$\Rightarrow \quad 2x^2 - 6x + 4 = 0$

$\quad x^2 - 3x + 2 = 0$

$\Rightarrow \quad x = 1$ and 2

$\therefore \quad K(1) = K(2) = -1.$

Hence function is many-one. Hence $K(x)$ is not bijective.

$\therefore$ Correct answers are (a) and (b). ■

EXAMPLE 11 Let $f: A \to B$ be a onto function such that $f(x) = \sqrt{(x-4) - 2\sqrt{x-5}} - \sqrt{(x-4) + 2\sqrt{x-5}}$, then set B is

(a) $[-2, 0]$ **(b)** $[0, 2]$ **(c)** $[-3, 0]$ **(d)** $[-1, 0]$

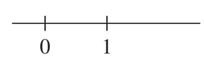
Solution Given $f(x) = \sqrt{(x-4) - 2\sqrt{x-5}} - \sqrt{(x-4) + 2\sqrt{x-5}}$

Put $\sqrt{x-5} = t$

$\therefore \quad t \in [0, \infty)$

$\Rightarrow \quad x - 4 = t^2 + 1$

$$f(t) = \sqrt{t^2 + 1 + 2t} - \sqrt{t^2 + 1 + 2t} = |t - 1| - |t + 1|$$



$\therefore \quad f(t) = (t - 1) - (t + 1) \; t \geq 1$

$\quad (1 - t) - (t + 1) \; 0 \leq t < 1$

$$\Rightarrow \quad f(t) \begin{cases} -2 & t \geq 1 \\ -2t & t \in [0, 1) \end{cases}$$

$\therefore$ Range of $f(x)$ is $[-2, 0]$

As $f(x)$ is onto $\Rightarrow$ Range = co-domain

$\therefore$ Set B is $[-2, 0]$

Hence correct answer is (a). ■

EXAMPLE 12 Let $f: R \to [0, \infty]$ defined by $f(x) = \log(\sqrt{9x^2 - 12x + \lambda} + 1)$ be a onto function, where λ is a parameter then complete set of values of λ is:

(a) $(-\infty, 4]$ **(b)** $(4, \infty]$ **(c)** $\{4\}$ **(d)** $(-\infty, \infty)$

Solution As function is onto.

Hence $0 \leq \log(\sqrt{9x^2 - 12x + \lambda} + 1) < \infty$

$\Rightarrow \quad \sqrt{9x^2 - 12x + \lambda} \geq 0$

$\therefore \quad 9x^2 - 12x + \lambda \geq 0$

Now, graph of $9x^2 - 12x + \lambda$ can be

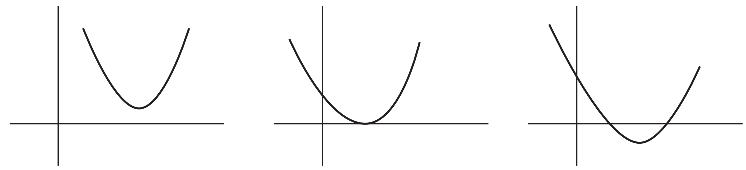


Fig. 1 Fig. 2 Fig. 3

Figure 1 is rejected as in this case the expression cannot take the value 0 and many more, so the function will not be onto.

Figure 3 is also rejected as in this case $9x^2 - 12x + \lambda$ will be negative for some values of x, which is not desired. Only acceptable case is Figure 2.

$\therefore$ Condition is $D = 0$

$$\therefore \quad 12^2 - 4 \cdot 9 \cdot \lambda = 0$$

$$\Rightarrow \quad \lambda = \frac{144}{36} = 4$$

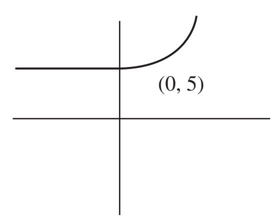
$\therefore$ Correct answer is (c). ■

EXAMPLE 13 Let $f: R \to R$ is defined as $f(x) \begin{cases} x^2 + ax + 5 & x \geq 0 \\ 2ax + 5 & x < 0 \end{cases}$. If $f(x)$ is injective then a can be equal to

(a) $(-\infty, 0)$ **(b)** $(-\infty, 0]$ **(c)** $[0, \infty)$ **(d)** $(0, \infty)$

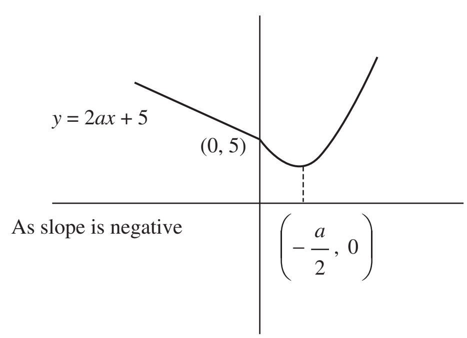
Solution We need the value of a such that any line parallel to x-axis intersects the graph of $y = f(x)$ at atmost one point

Case I: If $a = 0$, graph of $f(x)$ is



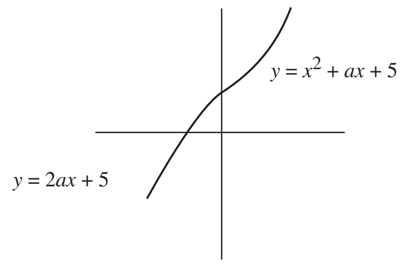
Clearly $f(x)$ is not injective.

Case II: If $a < 0$; graph of $y = f(x)$ is like



Clearly from the graph the function is not injective as vertex will be lying on the right-hand side of y-axis which makes it many one.

Case III: If $a > 0$, graph of $y = f(x)$ is



Clearly for all $a > 0$ $f(x)$ is injective.

$\therefore$ Correct answer is (d). ■

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x).$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 14 The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

■

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y-coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.23. The graphs of $f + g$ and $f \cdot g$ from Example 14 are shown in Figure 1.24.

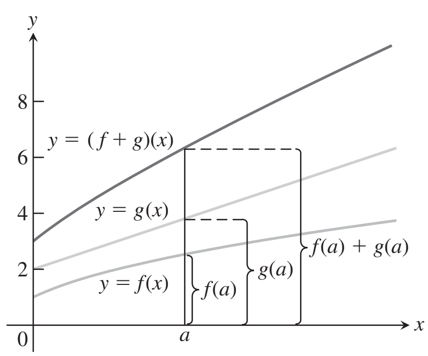


FIGURE 1.23 Graphical addition of two functions.

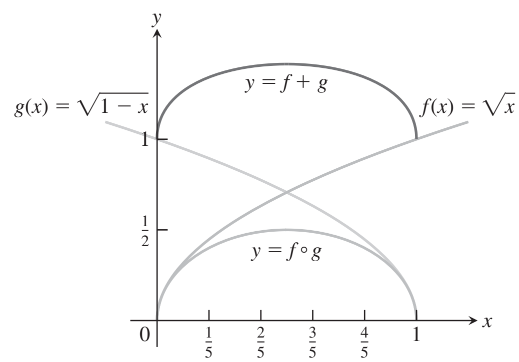


FIGURE 1.24 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 14).

Composite Functions

Composition is another method for combining functions.

> **DEFINITION** If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

The definition implies that $f \circ g$ can be formed when the range of g lies in the domain of f. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.25 pictures $f \circ g$ as a machine diagram, and Figure 1.26 shows the composite as an arrow diagram.

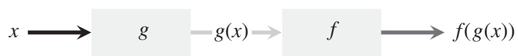


FIGURE 1.25 A composite function $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f.

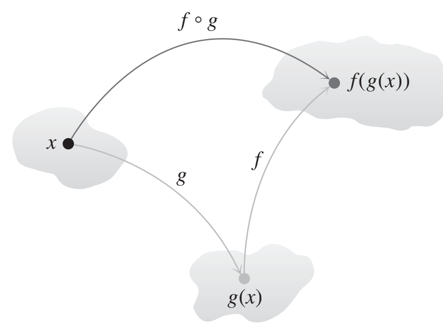


FIGURE 1.26 Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f, then the functions f and g can be composed to form $(f \circ g)(x)$.

To evaluate the composite function $g \circ f$ (when defined), we find $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 15 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	**Domain**
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

EXAMPLE 16 If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x}\,)^2$, then

(a) $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$ **(b)** $f(x) = \sin x$ and $g(x) = |x|$

(c) $f(x) = x^2$ and $g(x) = \sin \sqrt{x}$ **(d)** f and g cannot be determined

Solution $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $f(g(x)) = \sin^2 g(x) = \sin^2 \sqrt{x}$

Also, $g(f(x)) = \sqrt{f(x)} = \sqrt{\sin^2 x} = |\sin x|$ (As $\sqrt{x^2} = |x|$)

Clearly, this satisfies the given composition.

Option (b): $f(x) = \sin x$ and $g(x) = |x|$

$\therefore$ $f(g(x)) = \sin g((x)) = \sin |x|$

and $g|f(x)) = |f(x)) = |\sin x|$

$\therefore$ It does not satisfy the given composition.

Options (c): $f(x) = x^2$ and $g(x) = \sin\sqrt{x}$

$$g(f(x)) = \sin\sqrt{f(x)} = \sin\sqrt{x^2} = \sin|x|$$

$$f(g(x)) = (g(x))^2 = \sin^2\sqrt{x}$$

$\therefore$ It does not satisfy the given composition.

In such cases there can exist more than one combination $f(x)$ and $g(x)$, satisfying the given condition. ■

Properties of Composite Function

(i) The composite of two function is not commutative, i.e., $f(g(x)) \neq g(f(x))$ (in general).

(ii) The composite of function is associative, i.e., if f, g, h are three functions such that $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are defined then $f \circ (g \circ h) = (f \circ g) \circ h$.

(iii) The composite of two bijections is a bijection provided the composite is defined.

Example on Composite of a Function

EXAMPLE 17 Let $f_1(x) = x$, $f_2(x) = 1 - x$, $f_3(x) = \dfrac{1}{x}$, $f_4(x) = \dfrac{1}{1-x}$, $f_5(x) = \dfrac{x}{x-1}$ and $f_6(x) = \dfrac{x-1}{x}$ suppose that $f_4(f_m(x)) = f_5(x)$; $f_n(f_5(x)) = f_1(x)$ and $f_p(f_4(x)) = f_3(x)$, then (m, n, p) is

(a) (3, 5, 5) **(b)** (3, 5, 6) **(c)** (2, 6, 6) **(d)** (3, 6, 6)

Solution $f_4(f_m(x)) = f_5(x)$

$$\Rightarrow \quad \frac{1}{1-f_m(x)} = \frac{x}{x-1}$$

$$\Rightarrow \quad \frac{x-1}{x} = 1 - f_m(x)$$

$$\Rightarrow \quad 1 - \frac{1}{x} = 1 - f_m(x)$$

$$\Rightarrow \quad f_m(x) = \frac{1}{x} = f_3(x)$$

$$\Rightarrow \quad m = 3$$

$$f_n(f_5(x)) = f_1(x)$$

$$\Rightarrow \quad f_n\left(\frac{x}{x-1}\right) = x$$

Put $\quad \dfrac{x}{x-1} = t$

$$\Rightarrow \quad x = xt - t$$

$\Rightarrow \quad x = \dfrac{t}{t-1}$

$\therefore \quad f_n(t) = \dfrac{t}{t-1} = f_5(t)$

$\Rightarrow \quad n = 5$

Also $\quad f_p(f_4(x)) = f_3(x)$

$\Rightarrow \quad f_p\left(\dfrac{1}{1-x}\right) = \dfrac{1}{x}$

Put $\quad \dfrac{1}{1-x} = z$

$\Rightarrow \quad 1 - x = \dfrac{1}{z}$

$\Rightarrow \quad x = 1 - \dfrac{1}{z} = \dfrac{z-1}{z}$

$\therefore \quad f_p(z) = \dfrac{z-1}{z} = f_5(z)$

$\Rightarrow \quad P = 5$

$\therefore$ Correct answer is (a). ■

EXAMPLE 18 Let $f(x) = ax + b$, with a and b real numbers; $f_1(x) = f(x)$ and $f_{n+1}(x) = f(f_n(x))$; $n \in N$. If $f_7(x) = 128x + 381$, then $(a + b)$ is

(a) 4 **(b)** 5 **(c)** 7 **(d)** 9

Solution $f_2(x) = f(f(x)) = a(ax + b) + b = a^2x + ab + b$

$f_3(x) = f(f_2(x)) = f(a^2x + ab + b) = a(a^2x + ab + b) + b = a^3x + a^2b + ab + b$

$f_7(x) = a^7x + (a^6 + a^5 + \ldots + a + 1)b = 128x + 381$

$\therefore \quad a^7 = 128$

$\Rightarrow \quad a = 2$

and $\quad b(2^6 + 2^5 + \ldots + 2 + 1) = 381$

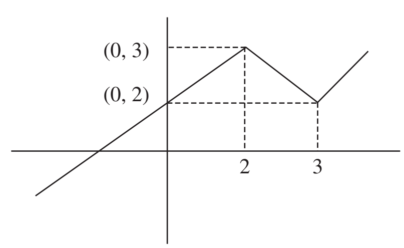
$\Rightarrow \quad b\left(\dfrac{2^7-1}{2-1}\right) = 381$

$\Rightarrow \quad b = 3$

$\therefore \quad a + b = 3 + 2 = 5$

$\therefore$ correct answer is (b). ■

EXAMPLE 19 The figure shows the graph of function $f(x)$. Then the number of solutions of $f(f(x)) = 2$ is



(a) 1 **(b)** 2 **(c)** 3 **(d)** 4

Solution We need the solution of $f(f(x)) = 2$

Now, suppose $f(x) = t$

$\therefore$ Solution of $f(t) = 2$

$\Rightarrow$ $t = 0, 3$

$\therefore$ $f(x) = 0$ or 3

$\Rightarrow$ $f(x) = 0$ has 1 solution

$f(x) = 3$ has 2 solutions

$\therefore$ Total no. of solutions = 3

$\therefore$ Correct answer is (c). ■

EXAMPLE 20 Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$ and $f_n(x) = f_1(f_{n-1}(x))$ (where $n \geq 2$). The value of x that satisfies $f_{1001}(x) = x - 3$ is

(a) $\frac{3}{5}$ **(b)** $\frac{7}{5}$ **(c)** $\frac{5}{3}$ **(d)** $\frac{5}{7}$

Solution $f_2(x) = f_1(f_1(x))$

$$= \frac{2}{3} - \frac{3}{3f_1(x)+1} = \frac{2}{3} - \frac{3}{3\left(\frac{2}{3} - \frac{3}{3x+1}\right)+1} = \frac{2}{3} - \frac{3}{3 - \frac{9}{3x+1}}$$

$$= \frac{2}{3} - \frac{3x+1}{3x-2}$$

$$f_3(x) = f_1(f_2(x)) = \frac{2}{3} - \frac{3}{3\left(\frac{2}{3} - \frac{3x+1}{3x-2}\right)+1} = \frac{2}{3} - \frac{1}{\left(1 - \frac{3x+1}{3x-2}\right)}$$

$$= \frac{2}{3} - \left(\frac{3x-2}{-3}\right) = \frac{2}{3} + x - \frac{2}{3} = x$$

$\therefore$ $f_{1001}(x) = f(x) = \frac{2}{3} - \frac{3x+1}{3x-2} = x - 3$

$\Rightarrow$ $\frac{6x - 4(9x+3)}{3(3x-2)} = (x-3)$

$\Rightarrow$ $-3x - 7 = 3(3x^2 - 11x + 6)$

$\Rightarrow$ $9x^2 - 3x + 25 = 0$

$\Rightarrow$ $x = \frac{5}{3}$

$\Rightarrow$ Correct answer is (c). ■

Examples on Composite of Non-Uniformly Defined Function

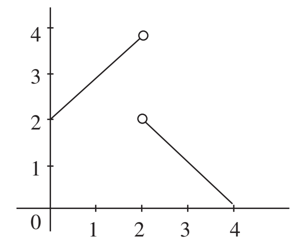
EXAMPLE 21 Let f is defined as $f(x) = \begin{cases} x+2 & 0 \leq x \leq 2 \\ 4-x & 2 < x \leq 4 \end{cases}$. Then find $f(f(x))$

Solution To complete $f(f(x))$, in first step we must replace x by $f(x)$ in the definition of $f(x)$.

$$\therefore \quad f(f(x)) = \begin{cases} f(x)+2 & 0 \leq f(x) \leq 2 \\ 4 - f(x) & 2 < f(x) \leq 4 \end{cases}$$

Further, we need to solve the inequality

$0 \le f(x) \le 2$ and $2 < f(x) \le 4$ for x. This can be easily solved by sketching the graph of $f(x)$.



Graph of $f(x)$

From the graph it is clear that solution of $0 \le f(x) \le 2$ is $x = 0$ and $x \in (2, 4]$ (as $f(x)$ is height, i.e., y so we have to look for the values of x for which $y \in [0, 2]$). $x = 0$ is included as $f(x) = 2$ for $x = 0$, $x = 2$ is not included as $f(2) = 4$. Also $x = 4$ is included as $f(4) = 0$.

Now, $\quad f(f(x)) = f(x) + 2 \qquad 0 \le f(x) \le 2$

$$\Rightarrow \qquad f(f(x) = \begin{cases} 4 & x = 0 \\ (4 - x) + 2 & x \in (2, 4] \end{cases}$$

$$\Rightarrow \qquad f(f(x)) = \begin{cases} 4 & x = 0 \\ 6 - x & x \in (2, 4] \end{cases}$$

We have used $f(x) = x + 2$ for $x = 0$ and $f(x) = 4 - x$ for $x \in (2, 4]$

Also from the graph it can be seen that $f(x) \in (2, 4] \Rightarrow x \in (0, 2]$

$$\therefore \qquad f(f(x)) = 4 - f(x) \qquad f(x) \in (2, 4]$$

$$\Rightarrow \qquad f(f(x)) = 4 - (x + 2) \qquad x \in (0, 2]$$

We have used $f(x) = x + 2 \ \forall \ x \in (0, 2]$

$$\therefore \qquad f(x)) = 2 - x \qquad x \in (0, 2]$$

$$\therefore \qquad f(f(x)) = \begin{cases} 4 & x = 0 \\ 2 - x & x \in (0, 2] \\ 6 - x & x \in (2, 4] \end{cases}$$

■

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

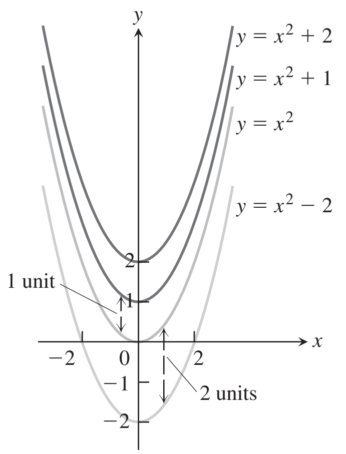


FIGURE 1.27 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 22).

EXAMPLE 22

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.27).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.27).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left, while adding -2 shifts the graph 2 units to the right (Figure 1.28).

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.29). ■

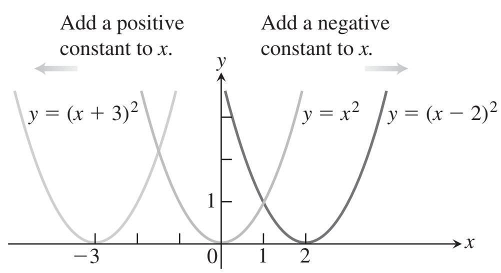


FIGURE 1.28 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x.

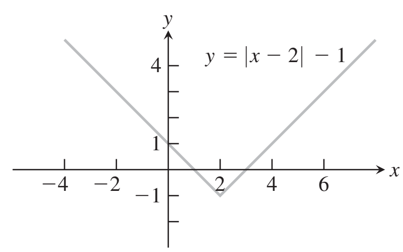


FIGURE 1.29 The graph of $y = |x|$ shifted 2 units to the right and 1 unit down (Example 3d).

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE 23 Here we scale and reflect the graph of $y = \sqrt{x}$.

(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph by a factor of 3 (Figure 1.30).

(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.31). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.32). ■

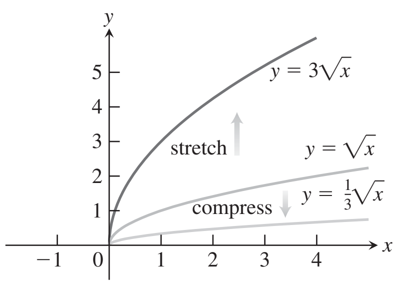


FIGURE 1.30 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 23).

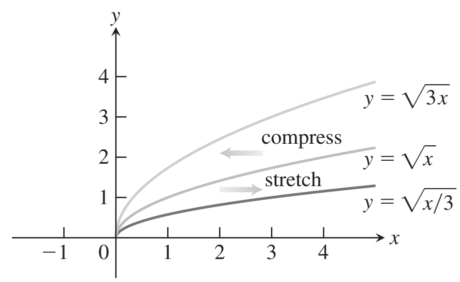


FIGURE 1.31 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 23).

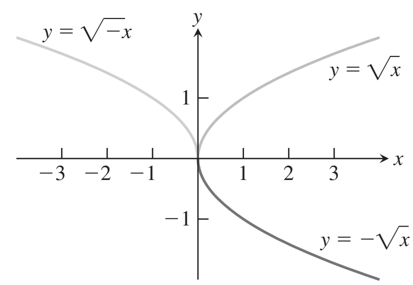


FIGURE 1.32 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 23).

EXAMPLE 24 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.33a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.33b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.33c).

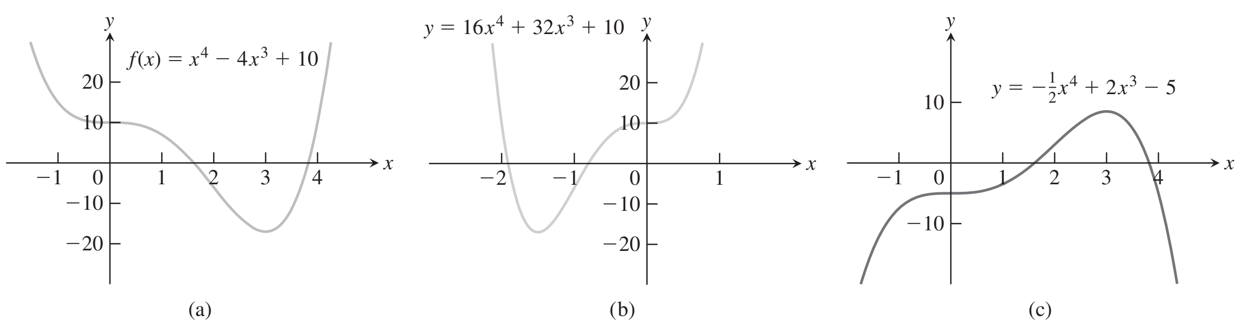


FIGURE 1.33 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 24).

Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$ ■

Exercises 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$

2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$

4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(-5))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a. $f(g(1/2))$ **b.** $g(f(1/2))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(2))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1, \quad g(x) = 3x, \quad h(x) = 4 - x$

8. $f(x) = 3x + 4, \quad g(x) = 2x - 1, \quad h(x) = x^2$

9. $f(x) = \sqrt{x + 1}, \quad g(x) = \dfrac{1}{x + 4}, \quad h(x) = \dfrac{1}{x}$

10. $f(x) = \dfrac{x + 2}{3 - x}, \quad g(x) = \dfrac{x^2}{x^2 + 1}, \quad h(x) = \sqrt{2 - x}$

Let $f(x) = x - 3, \quad g(x) = \sqrt{x}, \quad h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composite involving one or more of f, g, h, and j.

11. a. $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
c. $y = x^{1/4}$ **d.** $y = 4x$
e. $y = \sqrt{(x - 3)^3}$ **f.** $y = (2x - 6)^3$

12. a. $y = 2x - 3$ **b.** $y = x^{3/2}$
c. $y = x^9$ **d.** $y = x - 6$
e. $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	?
b.	$x + 2$	$3x$	?
c.	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\dfrac{x}{x - 1}$	$\dfrac{x}{x - 1}$	?
e.	?	$1 + \dfrac{1}{x}$	x
f.	$\dfrac{1}{x}$	?	x

14. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\dfrac{1}{x - 1}$	$\lvert x \rvert$	?
b.	?	$\dfrac{x - 1}{x}$	$\dfrac{x}{x + 1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

15. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a. $f(g(-1))$ **b.** $g(f(0))$ **c.** $f(f(-1))$
d. $g(g(2))$ **e.** $g(f(-2))$ **f.** $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2. \end{cases}$$

a. $f(g(0))$ **b.** $g(f(3))$ **c.** $g(g(-1))$
d. $f(f(2))$ **e.** $g(f(0))$ **f.** $f(g(1/2))$

In Exercises 17 and 18, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

17. $f(x) = \sqrt{x + 1}, \ g(x) = \dfrac{1}{x}$

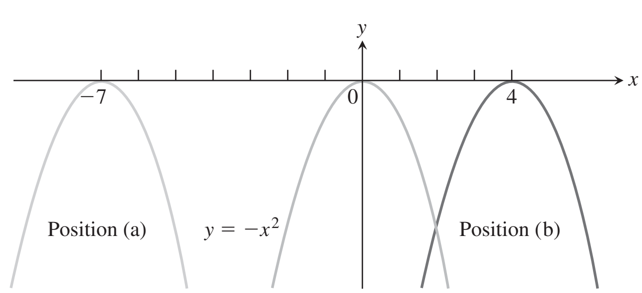
18. $f(x) = x^2, \ g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \dfrac{x}{x - 2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

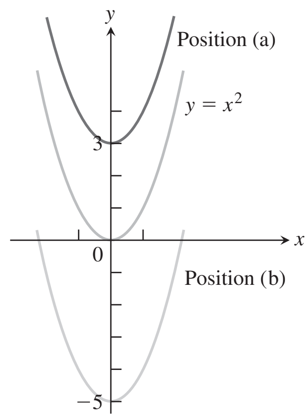
20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

Shifting Graphs

21. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



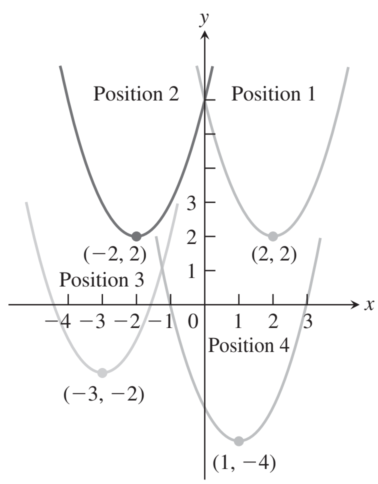
22. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



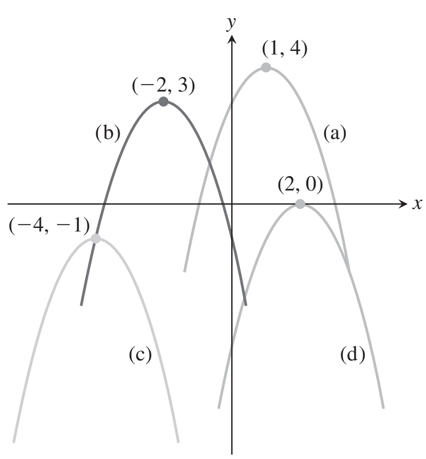
23. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



24. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 25–34 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

25. $x^2 + y^2 = 49$ Down 3, left 2

26. $x^2 + y^2 = 25$ Up 3, left 4

27. $y = x^3$ Left 1, down 1

28. $y = x^{2/3}$ Right 1, down 1

29. $y = \sqrt{x}$ Left 0.81

30. $y = -\sqrt{x}$ Right 3

31. $y = 2x - 7$ Up 7

32. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

33. $y = 1/x$ Up 1, right 1

34. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 35–54.

35. $y = \sqrt{x + 4}$ **36.** $y = \sqrt{9 - x}$

37. $y = |x - 2|$ **38.** $y = |1 - x| - 1$

39. $y = 1 + \sqrt{x - 1}$ **40.** $y = 1 - \sqrt{x}$

41. $y = (x + 1)^{2/3}$ **42.** $y = (x - 8)^{2/3}$

43. $y = 1 - x^{2/3}$ **44.** $y + 4 = x^{2/3}$

45. $y = \sqrt[3]{x - 1} - 1$ **46.** $y = (x + 2)^{3/2} + 1$

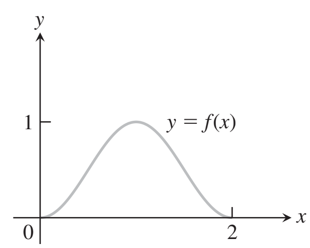
47. $y = \dfrac{1}{x - 2}$ **48.** $y = \dfrac{1}{x} - 2$

49. $y = \dfrac{1}{x} + 2$ **50.** $y = \dfrac{1}{x + 2}$

51. $y = \dfrac{1}{(x - 1)^2}$ **52.** $y = \dfrac{1}{x^2} - 1$

53. $y = \dfrac{1}{x^2} + 1$ **54.** $y = \dfrac{1}{(x + 1)^2}$

55. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



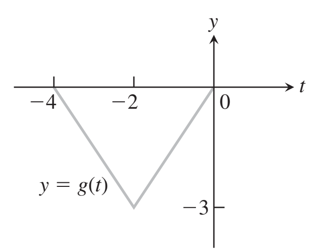
a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$

56. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a. $g(-t)$ **b.** $-g(t)$

c. $g(t) + 3$ **d.** $1 - g(t)$

e. $g(-t + 2)$ **f.** $g(t - 2)$

g. $g(1 - t)$ **h.** $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

57. $y = x^2 - 1$, stretched vertically by a factor of 3

58. $y = x^2 - 1$, compressed horizontally by a factor of 2

59. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

60. $y = 1 + \dfrac{1}{x^2}$, stretched horizontally by a factor of 3

61. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

62. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

63. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

64. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

65. $y = 1 - x^3$, compressed horizontally by a factor of 3

66. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 67–74, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.8–1.11 and applying an appropriate transformation.

67. $y = -\sqrt{2x + 1}$ **68.** $y = \sqrt{1 - \dfrac{x}{2}}$

69. $y = (x - 1)^3 + 2$ **70.** $y = (1 - x)^3 + 2$

71. $y = \dfrac{1}{2x} - 1$ **72.** $y = \dfrac{2}{x^2} + 1$

73. $y = -\sqrt[3]{x}$ **74.** $y = (-2x)^{2/3}$

75. Graph the function $y = |x^2 - 1|$.

76. Graph the function $y = \sqrt{|x|}$.

Combining Functions

77. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?

a. fg **b.** f/g **c.** g/f

d. $f^2 = ff$ **e.** $g^2 = gg$ **f.** $f \circ g$

g. $g \circ f$ **h.** $f \circ f$ **i.** $g \circ g$

78. Can a function be both even and odd? Give reasons for your answer.

T **79.** (*Continuation of Example 1.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T **80.** Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.34), or

$$s = r\theta \qquad (\theta \text{ in radians}). \tag{1}$$

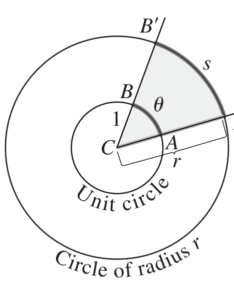


FIGURE 1.34 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

If the circle is a unit circle having radius $r = 1$, then from Figure 1.34 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180° \tag{2}$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.1 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.1 Angles measured in degrees and radians

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure 1.35). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

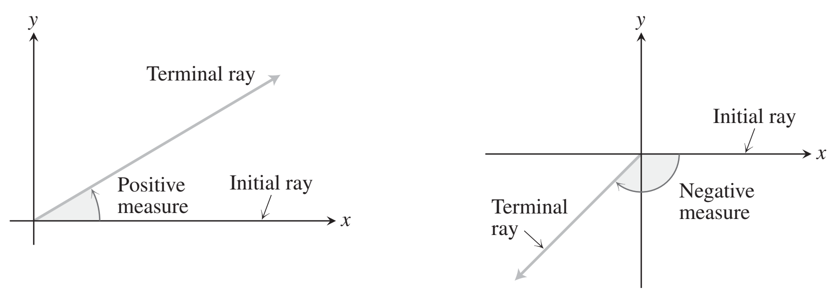


FIGURE 1.35 Angles in standard position in the xy-plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.36).

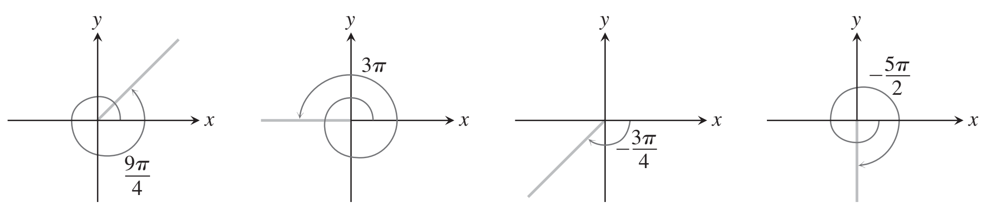
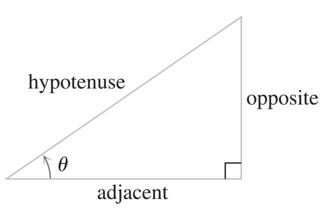


FIGURE 1.36 Nonzero radian measures can be positive or negative and can go beyond 2π.

Angle Convention: Use Radians From now on, in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. We use radians because it simplifies many of the operations in calculus, and some results we will obtain involving the trigonometric functions are not true when angles are measured in degrees.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.37 Trigonometric ratios of an acute angle.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.37). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.38).

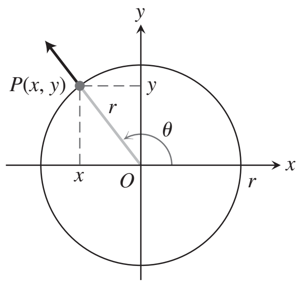


FIGURE 1.38 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

$$\textbf{sine:}\quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:}\quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:}\quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:}\quad \sec\theta = \frac{r}{x}$$

$$\textbf{tangent:}\quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:}\quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is acute.

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.39. For instance,

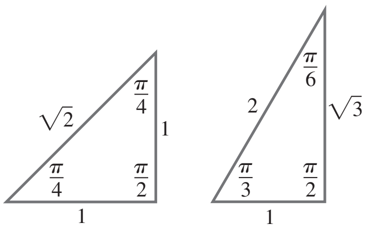


FIGURE 1.39 Radian angles and side lengths of two common triangles.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The CAST rule (Figure 1.40) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.41, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

Using a similar method we determined the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in Table 1.2.

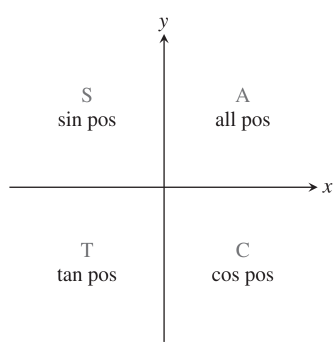


FIGURE 1.40 The CAST rule, remembered by the statement "Calculus Activates Student Thinking," tells which trigonometric functions are positive in each quadrant.

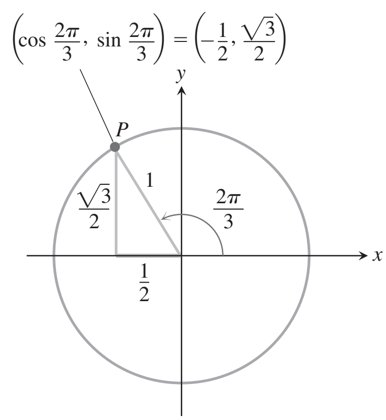


FIGURE 1.41 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ **(radians)**	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. Figure 1.42 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Even
$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

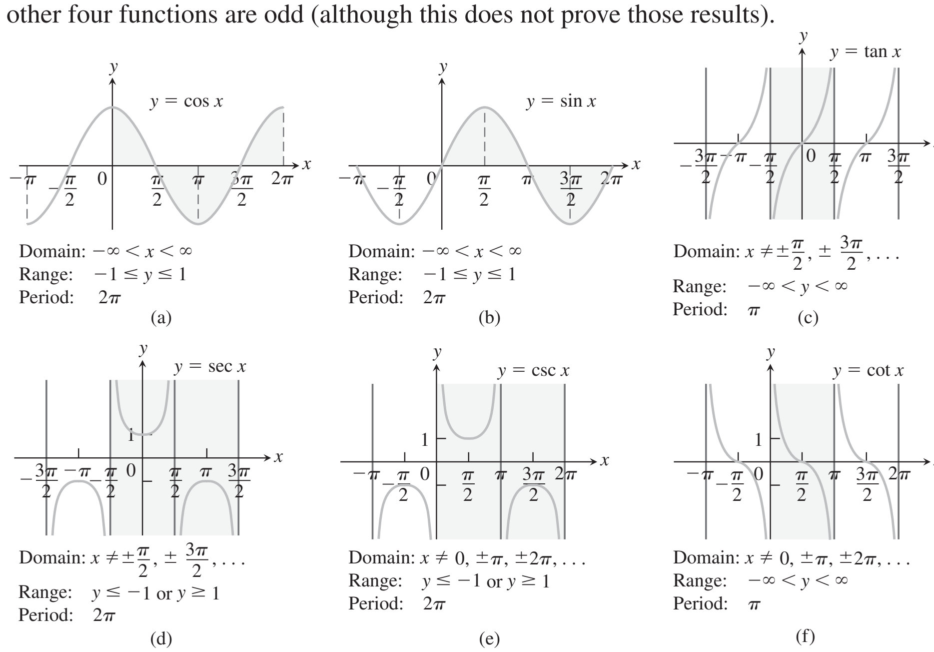


FIGURE 1.42 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x-axis (Figure 1.38). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

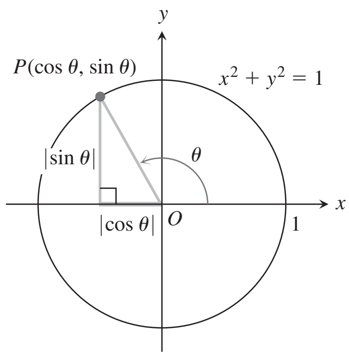


FIGURE 1.43 The reference triangle for a general angle θ.

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.43 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{3}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The following formulas hold for all angles A and B (Exercise 58).

Addition Formulas

$$\begin{aligned}\cos(A + B) &= \cos A\cos B - \sin A\sin B\\ \sin(A + B) &= \sin A\cos B + \cos A\sin B\end{aligned} \tag{4}$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned} \tag{5}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{6}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{7}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{8}$$

This equation is called the **law of cosines**.

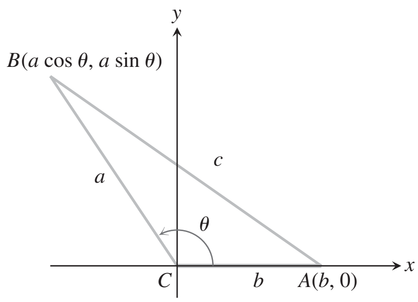


FIGURE 1.44 The square of the distance between A and B gives the law of cosines.

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.44. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Two Special Inequalities

For any angle θ measured in radians, the sine and cosine functions satisfy

$$\boxed{-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1 - \cos\theta \le |\theta|.}$$

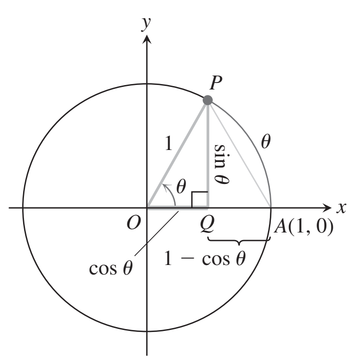


FIGURE 1.45 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 \le \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.45). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \le \theta^2 \quad \text{and} \quad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \quad \text{and} \quad |1 - \cos\theta| \le |\theta|,$$

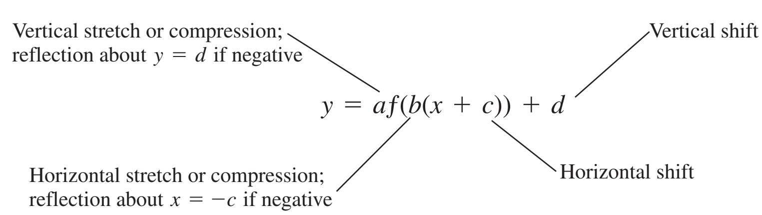
so

$$-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

These inequalities will be useful in the next chapter.

Transformations of Trigonometric Graphs

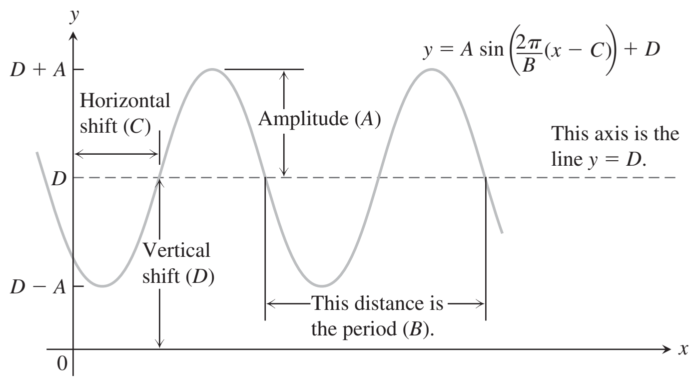
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is given below.



Exercises 1.3

Radians and Degrees

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (**a**) $4\pi/5$ radians? (**b**) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5},\quad x \in \left[\frac{\pi}{2}, \pi\right]$
8. $\tan x = 2,\quad x \in \left[0, \frac{\pi}{2}\right]$
9. $\cos x = \frac{1}{3},\quad x \in \left[-\frac{\pi}{2}, 0\right]$
10. $\cos x = -\frac{5}{13},\quad x \in \left[\frac{\pi}{2}, \pi\right]$
11. $\tan x = \frac{1}{2},\quad x \in \left[\pi, \frac{3\pi}{2}\right]$
12. $\sin x = -\frac{1}{2},\quad x \in \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$
14. $\sin(x/2)$
15. $\cos \pi x$
16. $\cos \frac{\pi x}{2}$
17. $-\sin \frac{\pi x}{3}$
18. $-\cos 2\pi x$
19. $\cos\left(x - \frac{\pi}{2}\right)$
20. $\sin\left(x + \frac{\pi}{6}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$ **22.** $\cos\left(x + \frac{2\pi}{3}\right) - 2$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$ **24.** $s = -\tan \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$ **26.** $s = \csc\left(\frac{t}{2}\right)$

T **27.** **a.** Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

T **28.** Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ **32.** $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ **34.** $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$ **40.** $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$ **42.** $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin \frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos \frac{11\pi}{12}$ as $\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$.

45. Evaluate $\cos \frac{\pi}{12}$. **46.** Evaluate $\sin \frac{5\pi}{12}$.

Using the Half-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \frac{\pi}{8}$ **48.** $\cos^2 \frac{5\pi}{12}$

49. $\sin^2 \frac{\pi}{12}$ **50.** $\sin^2 \frac{3\pi}{8}$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ, where $0 \le \theta \le 2\pi$.

51. $\sin^2 \theta = \frac{3}{4}$ **52.** $\sin^2 \theta = \cos^2 \theta$

53. $\sin 2\theta - \cos \theta = 0$ **54.** $\cos 2\theta + \cos \theta = 0$

Theory and Examples

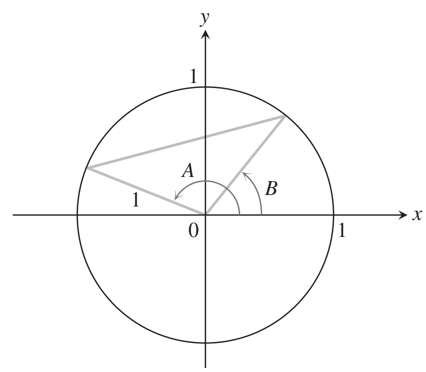
55. The tangent sum formula The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

56. (*Continuation of Exercise 55.*) Derive a formula for $\tan(A - B)$.

57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. **a.** Apply the formula for $\cos(A - B)$ to the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.

b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.

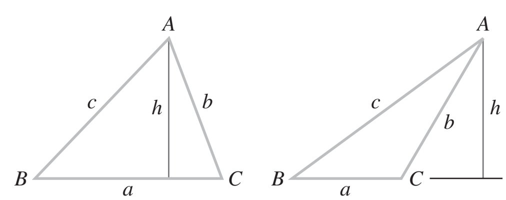
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

61. The law of sines *The law of sines* says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin \theta$, if required, to derive the law.



62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 59). Find the sine of angle B using the law of sines.

63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

T **64. The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.11, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a. With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b. With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A, B, C, and D for the sine functions in Exercises 65–68 and sketch their graphs.

65. $y = 2\sin(x + \pi) - 1$

66. $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

67. $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

68. $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$

1.4 Miscellaneous

Inverse of a Function

Let f: $A \to B$ be a one-one function and onto function, then there exists a unique function g: $B \to A$ such that

$$\boxed{f(x) = y \Leftrightarrow g(y) = x, \forall x \in A \text{ and } y \in B}$$

then g is said to be inverse of f and is denoted by $g = f^{-1}$

Note: For existence of inverse, a function must be one-one and onto suppose if f: $A \to B$ be a many one function then there must exists x, $x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ (where $f(x_1), f(x_2) \in B$. Let $f(x_1) = f(x_2) = \alpha$, so when we define an inverse function g: $B \to A$, then $g(\alpha) = x_1$ as well as $g(\alpha) = x_2$. Hence it violates the definition of function. Similarly if f: $A \to B$ is an into function then there exists at least one element $\delta \in B$ such that δ is not image of any element in A. So, when inverse function is defined g: $B \to A$; $g(\delta)$ is not defined. Hence it again violates the condition of function.

How to Find Inverse of a Function

If a function is bijective then the inverse rule can be found using the following rule.

Step-1: Write $y = f(x)$.
Step-2: Express x in terms of y.
Step-3: Interchange x and y to get $f^{-1}(x)$ in terms of x.

EXAMPLE 1 If f: $[1, \infty) \to [2, \infty)$ is given by $f(x) = x + 1/x$, then $f^{-1}(x)$ equals to

(a) $\dfrac{x+\sqrt{x^2-4}}{2}$ **(b)** $\dfrac{x}{x^2+1}$ **(c)** $\dfrac{x-\sqrt{x^2-4}}{2}$ **(d)** $1-\sqrt{x^2-4}$

Solution As $f[1, \infty) \to [2, \infty)$

$\therefore$ $D_f \in [1, \infty)$ and co-domain is $[2, \infty)$

Now, $f(x) = x + 1/x$

$\therefore$ $f'(x) = 1 - 1/x^2 = \dfrac{(x-1)(x+1)}{x^2} > 0 \ \forall\ x \in [1, \infty)$

$\therefore$ $f(x)$ is one-one as it is increasing on $[1, \infty)$

$\therefore$ $R_f \in [2, \infty)$

$\therefore$ co-domain = Range. Hence, function is onto

Now $y = x + 1/x$

$$\Rightarrow \qquad x^2 - yx + 1 = 0$$

$$\Rightarrow \qquad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

When $$x = \frac{y - \sqrt{y^2 - 4}}{2} = \frac{2}{y + \sqrt{y^2 - 4}}$$

When $$y \in [2, \infty)$$

$$\Rightarrow \qquad x \in (0, 1]$$

$$\therefore \qquad x = \frac{y - \sqrt{y^2 - 4}}{2} \text{ is rejected}$$

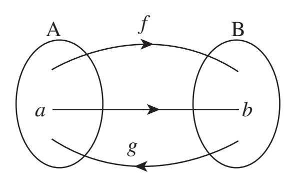
$$\therefore \quad f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2} \quad \text{(after interchanging } x \text{ and } y\text{)}$$

$\therefore$ (a) is correct answer. ■

Properties of Inverse Function

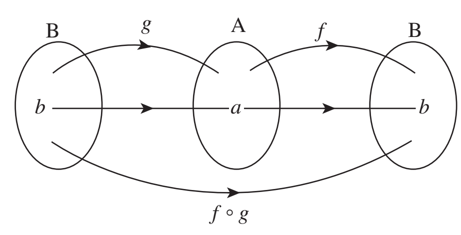
(i) The inverse of a bijection is unique:

(ii) If f: A $\to$ B is a bijection and g: $B \to A$ is inverse of f, then $f(g(x)) = I_B$ (Identity function on set B) and $g(f(x)) = I_A$
(Identity function on set A)



$f(a) = b$ and $g(b) = a$

Now, for the function $f(g(x))$

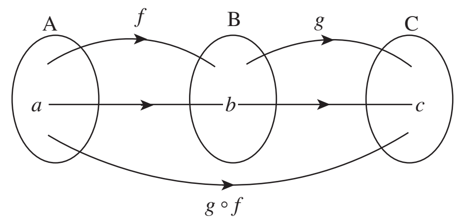


Clearly, $f(g(x))$ is identity function over set B. Similarly, $g(f(x)) = x$ can be proved.

(iii) The inverse of a bijection is also bijection.

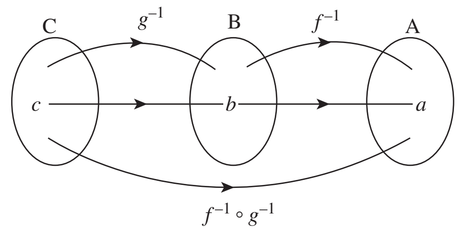
(iv) If $y = f(x)$ and $x = g(y)$ are inverse functions of each other, then domain of $f(x)$ = range of $g(x)$ and range of $f(x)$ = domain of $g(x)$.

(v) If f and g are two bijection such that f: $A \to B$ and g: $B \to C$, then the inverse $g \circ f$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. It can be extended for any number of function, i.e., $(f_1 \circ f_2 \circ \ldots \circ f_n)^{-1} = f_n^{-1} \circ \cdot \circ f_2^{-1} \circ f_1^{-1}$ (Law of reversal), provided both sides are defined.



$\therefore \qquad (f \circ g)^{-1}: C \to A$ and $(g \circ f)^{-1}\,(c) = a$

Now,



$$\therefore \qquad f^{-1} \circ g^{-1} : C \to A$$

$$\therefore \qquad (f^{-1} \circ g^{-1})\,(c) = a$$

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

(vi) The graphs of $f(x)$ and $g(x)$ are mirror images of each other about the line $y = x$, where $f(x)$ and $g(x)$ are inverse functions of each other. As if there is any point (α, β) on the graph of $y = f(x)$, then this point after reflection in the line $y = x$ will become (β, α) which lies on the graph of its inverse function $y = g(x)$.

(vii) $f(x)$ and $f^{-1}(x)$ if intersect, then the point of intersection should be on the line $y = x$ or $y = -x + k$ for some real value of k.

EXAMPLE 2 Suppose $f(x) = (x + 1)^2$ for $x \geq -1$. If is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

(a) $-\sqrt{x} - 1, x \geq 0$ **(b)** $\dfrac{1}{(1+x)^2}, x \geq -1$

(c) $\sqrt{x+1}, x \geq -1$ **(d)** $\sqrt{x} - 1, x \geq 0$

Solution Since $g(x)$ is reflection of $f(x)$ w.r.t. line $y = x$. Hence $g(x)$ is inverse of $f(x)$.

$$\therefore \qquad y = (x+1)^2\ x \geq -1$$

$$\Rightarrow \qquad y \geq 0$$

Now, $\qquad x + 1 = \pm\sqrt{y}$

$\Rightarrow \qquad x = \sqrt{y} - 1 \quad$ or $\quad \sqrt{y} - 1 \quad$ (rejected as for $y \geq 0 - \sqrt{y} = x \leq -1$)

$$\therefore \qquad g(x) = \sqrt{x} - 1, x \geq 0$$

■

Periodic Function

A function $y = f(x)$ is called periodic if there exists a positive number T such that $f(x + T) = f(x)\ \forall\ x \in$ domain of f. The functions which are not periodic are known as aperiodic functions.

EXAMPLE 3 Prove that $y = \sin x$ is a periodic function.

Solution If $f(x) = \sin x$ is a periodic function then it must satisfy the equation $f(x + T) = f(x)$ (where T is a positive real number)

$$\therefore \quad \sin(x + T) = \sin x$$

$$\Rightarrow \quad \sin(x + T) - \sin x = 0$$

$$\Rightarrow \quad 2\cos\left(\frac{2x+T}{2}\right)\sin\frac{T}{2} = 0$$

$$\therefore \quad \cos\left(\frac{2x+T}{2}\right) = 0$$

$$\Rightarrow \quad \frac{2x+T}{2} = (2n+1)\frac{\pi}{2} \qquad \text{(where } n \in I\text{)}$$

$\Rightarrow \quad T = (2n + 1)\pi - 2x$, which is not independent of x.

Now, $\sin\frac{T}{2} = 0$

$$\Rightarrow \quad \frac{T}{2} = n\pi \ (n \in I)$$

$$\Rightarrow \quad T = 2n\pi \ (n \in I)$$

Now if $T > 0$

$\therefore$ $T = 2\pi$ is the least value for which $f(x + T) = f(x)\ \forall\ x \in D_f$. Hence 2π is the period of function. ■

EXAMPLE 4 Prove that $f(x) = \sin x^2$ is a periodic.

Solution Let T be the positive number for which $f(x)$ is periodic.

$$\therefore \quad \sin(x + T)^2 = \sin x^2$$

$$\Rightarrow \quad \sin(x + T)^2 - \sin x^2 = 0$$

$$\Rightarrow \quad 2\cos\left(\frac{(x+T)^2 + x^2}{2}\right)\sin\left(\frac{(x+T)^2 - x^2}{2}\right) = 0$$

Now, if $\cos\left(\frac{(x+T)^2 + x^2}{2}\right) = 0$

or $\sin\left(\frac{(x+T)^2 - x^2}{2}\right) = 0$

Clearly from both the equations there cannot be a value of T which is independent of x. Hence $\sin x^2$ is not periodic. ■

Properties of Periodic Function

1. Inverse of a periodic function does not exist as the function is many-one when periodic.
2. Every constant function is periodic but its least period cannot be defined. For example, $f(x) = 2$ is periodic function as it satisfies $f(x + T) - f(x) = 0$ but least period cannot be defined as if we take any positive T claiming its least period there exists a smaller positive value $T_1 (T_1 < T)$ for which function is periodic.

3. If $f(x)$ has a period T and $g(x)$ has period T, then the function $f(x) + g(x)$ need not have a period T and it can be aperiodic also.

For example, suppose $h(x) = |\sin x| + |\cos x|$ clearly $|\sin x|$ has period π and $|\cos x|$ has period π but $h(x)$ has period $\frac{\pi}{2}$. Generally, if $f(x)$ has period T_1 and $g(x)$ has period T_2 then the function $f(x) + g(x)$ is periodic with period L.C.M. of T_1 and T_2 provided L.C.M. is defined but there can be cases where least period can be less than L.C.M. of T_1 and T_2. It happens in those cases where the functions $f(x)$ and $g(x)$ are interchangeable before L.C.M. of T_1 and T_2.

4. If $f(x)$ has period T_1 and $g(x)$ has period T_2, then $f(x)$ + g(x) need not be periodic, for example, suppose $h(x) = \sin x + \{x\}$ clearly $\sin x$ has period 2π and $\{x\}$ (fractional part of x) has period 1, but the L.C.M. of 2π and 1 is not defined. Hence $h(x)$ is aperiodic function.

5. If $f(x)$ has a period T, then $f(ax + b)$ has period $\frac{T}{|a|}$

Proof: If $f(x)$ has period T, then $f(x + T) = f(x)$. Now suppose T' is the period of $f(ax + b)$

Therefore, $\quad f(a(x + T') + b) = f(ax + b)$

$\Rightarrow \quad f(ax + b + aT') = f(ax + b)$

$\Rightarrow \quad f(z + aT') = f(z)$ (Let $z = ax + b$)

Now, if f has period T, then $aT' = T$

$$\Rightarrow \quad T' = \frac{T}{a} \quad \text{or} \quad T' = \frac{T}{|a|}$$

($|a|$ is taken so that T' is positive)

EXAMPLE 5 The period of the function $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$ is:

(a) 3π **(b)** 4π **(c)** 5π **(d)** 15π

Solution Given $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$

We know if $f(x)$ has period T then $f(ax + \text{b})$ has period $\frac{T}{|a|}$. Now period of $\cos\left(\frac{2x}{3}\right)$ is 3π and period of $\sin\frac{4x}{5}$ is $\left(\frac{2\pi}{\left(\frac{4}{5}\right)}\right) = \frac{5\pi}{2}$. Now, L.C.M. of $\frac{3\pi}{1}$ and $\frac{5\pi}{2}$ is 15π.

As we can use the fact L.C.M. of $\left\{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, ...\right\} = \frac{\text{L.C.M. of } [a, c, e, ...]}{\text{H.C.F. of } [b, d, f, ...]}$

$\therefore$ Correct answer is (d). ■

EXAMPLE 6 The period of function $f(x) = \cos(\sin x)$ is

(a) $\frac{\pi}{2}$ **(b)** π **(c)** $\frac{3\pi}{2}$ **(d)** None of these

Solution Given $f(x) = \cos(\sin x)$; clearly $\sin x$ is periodic with period 2π. Hence the period is 2π or less than it. It is clear that $f(x + \pi) = \cos(\sin(x + \pi) = \cos(-\sin x) = \cos(\sin x) = f(x)$. Hence period of function is π.

$\therefore$ Correct answer is (b). ■

EXAMPLE 7 Let f be a real-valued function defined for all real numbers x such that for some positive constant a, the equation

$$f(x + \mathrm{a}) = \frac{1}{2} + \sqrt{f(x) - f^2(x)} \text{ holds for all } x.$$

Prove that f is periodic.

Solution For any function to be periodic, we need to establish a relation

$$f(x + T) = f(x)\ \forall\ x \in D_f \quad \text{(where } T \text{ is a positive number).}$$

Clearly $\quad f(x) - f^2(x) \geq 0$

$\Rightarrow \quad f(x)\,(f(x) - 1) \leq 0$

$\Rightarrow \quad f(x) \in [0, 1].$

Also $\quad f(x + a) = \dfrac{1}{2} + \sqrt{f(x) - f^2(x)} \geq \dfrac{1}{2}$

$\therefore \quad f(x) \in \left[\dfrac{1}{2}, 1\right]$ (As range of $f(x)$ and $f(x + \mathrm{a})$ is same)

Now, putting x as $x + a$ in the equation, we get

$$\begin{aligned}
f(x + 2a) &= \frac{1}{2} + \sqrt{f(x+a) - f^2(x+a)} \\
&= \frac{1}{2} + \sqrt{f(x+a)(1 - f(x+a))} \\
&= \frac{1}{2} + \sqrt{\left(\frac{1}{2} + \sqrt{f(x) - f^2(x)}\right)\left(1 - \left(\frac{1}{2} + \sqrt{f(x) - f^2(x)}\right)\right)} \\
&= \frac{1}{2} + \sqrt{\left(\frac{1}{2} + \sqrt{f(x) - f^2(x)}\right)\left(\frac{1}{2} - \sqrt{f(x) - f^2(x)}\right)} \\
&= \frac{1}{2} + \sqrt{\frac{1}{4} - f(x) + f^2(x)} = \frac{1}{2} + \sqrt{\left(f(x) - \frac{1}{2}\right)^2}
\end{aligned}$$

$\Rightarrow \quad f(x + 2a) = \dfrac{1}{2} + \left|f(x) - \dfrac{1}{2}\right| = \dfrac{1}{2} + f(x) - \dfrac{1}{2} \quad$ (as $f(x) \geq \dfrac{1}{2}$)

$\Rightarrow \quad f(x + 2a) = f(x)$

which suggest $f(x)$ is a periodic function ■

EXAMPLE 8 The period of the function $f(x) = e^{\sin^2\left(\frac{\pi x}{4}\right)} + e^{\cos^2\left(\frac{\pi x}{4}\right)} + \dfrac{x}{2} - \left[\dfrac{x}{2}\right]$ (where [·] denotes greatest integer function.)

(a) 1 **(b)** 2 **(c)** 3 **(d)** None of these

Solution Given $f(x) = e^{\sin^2\left(\frac{\pi x}{4}\right)} + e^{\cos^2\left(\frac{\pi x}{4}\right)} + \dfrac{x}{2} - \left[\dfrac{x}{2}\right]$

Clearly $\sin^2 \dfrac{\pi x}{4}$ has period $\dfrac{\pi}{\left(\dfrac{\pi}{4}\right)} = 4$ (as $\sin^2 x$ has period π) similarly $\cos^2 \dfrac{\pi x}{4}$ has period 4, but $\sin^2 \dfrac{\pi x}{4}$ and $\cos^2 \dfrac{\pi x}{4}$ interchange after $x = 2$. As $\sin^2\left(\dfrac{\pi}{4} + \dfrac{\pi x}{4}\right) = \cos^2\left(\dfrac{\pi x}{4}\right)$

Also $\frac{x}{2}-\left[\frac{x}{2}\right]=\left[\frac{x}{2}\right]$. Now $\{x\}$ has period 1. Hence has period $\left(\frac{1}{\frac{1}{2}}\right) = 2$. Therefore, $f(x)$ has period 2.

$\therefore$ Hence correct answer is (b). ■

EXAMPLE 9 Let $f(x)$ be a function such that $f(x-2)+f(x+2)=\sqrt{3}\,f(x)$ and let T be a positive number such that $f(x+T)=f(x)\ \forall\ x \in R$, then the value of T is

(a) 36 **(b)** 48 **(c)** 60 **(d)** 24

Solution Given

$$f(x-2)+f(x+2)=\sqrt{3}\,f(x)$$

Put $x \to x+2$

$$f(x)+f(x+4)=\sqrt{3}\,f(x+2) \tag{1}$$

$x \to x-2$

$$f(x-4)+f(x)=\sqrt{3}\,f(x-2) \tag{2}$$

Adding (1) and (2), we get

$$2f(x)+f(x-4)+f(x+4)=\sqrt{3}\,(f(x+2)+f(x-2))$$

$$=\sqrt{3}\cdot\sqrt{3}\ f(x)=3f(x)$$

$\Rightarrow$

$$f(x-4)+f(x+4)=f(x) \tag{3}$$

Now, replace $x \to x+4$

$\Rightarrow$

$$f(x)+f(x+8)=f(x+4) \tag{4}$$

Adding (3) and (4), we get

$$f(x-4)+f(x+8)=0 \tag{5}$$

Put $x \to x+12$

$$f(x+8)+f(x+20)=0 \tag{6}$$

From (5) and (6), we get

$$f(x-4)=f(x+20)$$

Replace $x \to x+4$

$\Rightarrow$

$$f(x)=f(x+24)$$

Hence possible value of T is 24.

Correct answer is (d). ■

EXAMPLE 10 If P be the period of the function that satisfies the equation $f(x+\cos\alpha)+f(x+2\cos\alpha)+\ldots+f(x+150\cos\alpha)=2015\ \forall\ x \in R$ (where $\alpha \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$) then number of integral values in the range of P as $\alpha \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is

(a) 145 **(b)** 150 **(c)** 151 **(d)** 155

Solution Given $f(x+\cos\alpha)+f(x+2\cos\alpha)+\ldots+f(x+150\cos\alpha)=2015$ (1)

Put $x \to x+\cos\alpha$

$$f(x+2\cos\alpha)+f(x+3\cos\alpha)+\ldots+f(x+151\cos\alpha)=2015 \tag{2}$$

From (1) and (2), we get

$$f(x + \cos\alpha) = f(x + 151\cos\alpha)$$

$\Rightarrow$ $$f(x) = f(x + 150\cos\alpha)$$

$\therefore$ $$P = 150\cos\alpha$$

Now $$\cos\alpha \in (0, 1]$$

$\therefore$ $$P \in (0, 150]$$

$\therefore$ No. of integral value of P is 150.

$\therefore$ Correct answer is (b). ■

EXAMPLE 11 Least period of $f(x) = \{x\} + \left\{x+\frac{1}{4}\right\}+\left\{x+\frac{1}{2}\right\}+\left\{x+\frac{3}{4}\right\}$ is equal to (where $\{\cdot\}$ denotes fractional part function)

(a) 1 **(b)** 1/2 **(c)** 1/3 **(d)** 1/4

Solution Since $f(x) = \{x\}+\left\{x+\frac{1}{4}\right\}+\left\{x+\frac{1}{2}\right\}+\left\{x+\frac{3}{4}\right\}$

$$f\left(x+\frac{1}{4}\right)=\left\{x+\frac{1}{4}\right\}+\left\{x+\frac{2}{4}\right\}+\left\{x+\frac{3}{4}\right\}+\{x+1\}$$
$$= f(x) \quad (\because \{x+1\}=\{x\})$$

$\therefore$ Least period is 1/4.

$\therefore$ Correct answer is (d). ■

EXAMPLE 12 Let f: $R \to R$ be a real-valued function such that $f(10 + x) = f(10 - x)$ $\forall\, x \in R$ and $f(20 + x) = -f(20 - x)$ $\forall\, x \in R$. Then which of the following statement is true.

(a) $f(x)$ is odd and periodic **(b)** $f(x)$ is odd and periodic

(c) $f(x)$ is even and periodic **(d)** $f(x)$ even and periodic

Solution Given $f(10 + x) = f(10 - x)$ and $f(20 + x) = -f(20 - x)$

Put $$x \to x + 10$$

$\therefore$ $$f(20 + x) = f(-x) = -f(20 - x)$$

Put $$-x = t$$

$\therefore$ $$f(t) = -f(20 + t)$$

Put $$t \to t + 20$$

$\therefore$ $$f(20 + t) = -f(40 + t)$$

$\therefore$ $$f(t) = f(40 + t)$$

Hence $f(x)$ is periodic.
Put $20 + t = z$, we get

$$f(z - 20) = f(20 + z)$$

$\Rightarrow$ $$f(z - 20) = -f(20 - z)$$

$\Rightarrow$ $$f(-x) = -f(x)$$

Hence, function is odd
$\therefore$ Correct answer is (a). ■

Equal or Identical Functions

Two functions $f(x)$ and $g(x)$ are said to be equal if

1. Domain of $f(x)$ = domain of $g(x)$
2. Range of $f(x)$ = Range of $g(x)$
3. $f(x) = g(x)\ \forall\ x \in$ domain of $f(x)$ or domain of $g(x)$, i.e., graph of the two functions must be same.

EXAMPLE 13 Check $f(x) = \operatorname{cosec} x$ and $g(x) = \dfrac{1}{\sin x}$ are identical or not?

Solution To see whether two functions are identical or not, we need to just check whether domains of two functions are same or not and if they are then simply check whether the two functions satisfy the third condition or not. So, checking range of function is not important at all.

On the above discussion, we can see.

$$D_f \in R - \{n\pi\}\ (n \in I)$$

$$D_g \in R - \{n\pi\}\ (n \in \mathrm{I})$$

and $$f(x) = g(x)\ \forall\ x \in D_f \text{ or } D_g$$

$\therefore$ functions are identical. ■

EXAMPLE 14 Which of the following function is identical?

(a) $f(x) = \log_e x$ and $g(x) = \dfrac{1}{\log_x e}$

(b) $f(x) = \dfrac{1}{1+\dfrac{1}{x}}$ and $g(x) = \dfrac{x}{1+x}$

(c) $f(x) = \sqrt{x^2-1}$ and $\mathrm{g}(x) = \sqrt{x-1}\cdot\sqrt{x+1}$

(d) $f(x) = [\{x\}]$ and $g(x) = \{[x]\}$ (where $[\cdot]$ and $\{\cdot\}$ denote greatest integer function and fractional part function, respectively)

Solution **Option A:** $f(x) = \log_e x$ $\quad D_f \in (0, \infty)$

and $g(x) = \dfrac{1}{\log_x e}$ $\quad D_g \in (0, \infty) - \{1\}$

Hence $f(x)$ and $g(x)$ are not identical.

Option B: $f(x) = \dfrac{1}{1+\dfrac{1}{x}}$ $\quad D_f \in R - \{0, -1\}$

and $g(x) = \dfrac{x}{1+x}$ $\quad D_g \in R - \{-1\}$

Clearly $f(x)$ and $g(x)$ are not identical.

Option C: $f(x) = \sqrt{x^2-1} \Rightarrow$ for domain of $f(x) = x^2 - 1 \geq 0 \Rightarrow x \geq 1$ or $x \leq -1$

$\therefore$ $$D_f \in (-\infty, -1] \cup [1, \infty)$$

Also $$g(x) = \sqrt{x-1}\cdot\sqrt{x+1}$$

$\therefore$ For domain of $g(x)$

$$x - 1 \geq 0 \quad \text{and} \quad x + 1 \geq 0$$

$\Rightarrow$ $$x \geq 1 \text{ and } x \geq -1$$

$\therefore$ $$D_g \in [1, \infty)$$

$\therefore$ Domains are not same. Hence functions are not identical.

Option (D): $f(x) = [\{x\}]$

$\Rightarrow$ $$D_f \in R$$

Now, $$\forall\, x \in R\, ;\, \{x\} \in [0, 1)$$

$\therefore$ $$[\{x\}] = 0$$

Also $$g(x) = \{[x]\}$$

$\Rightarrow$ $$D_g \in R$$

Now, $[x]$ denotes integer values.

$\therefore$ $$\{[x]\} = 0$$

$\therefore$ $$f(x) = g(x) = 0 \ \forall\, x \in R.$$

Hence they are identical.

$\therefore$ Correct answer is (d). ■

Homogeneous Function

A function is said to be homogeneous with respect to any set of variable when each of its terms is of the same degree with respect to those variable.

Symbolically if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then $f(x, y)$ is a homogeneous function of degree n.

EXAMPLE 15 Which of the following function(s) are homogeneous?

(a) $f(x, y) = \dfrac{x + y\cos x}{y\sin x - x}$ **(b)** $f(x, y) = \dfrac{x^2}{y^2}\ln\left(\dfrac{x}{y}\right) + \dfrac{y^2}{x^2}\ln\left(\dfrac{y}{x}\right)$

(c) $f(x, y) = x + y\cos\dfrac{y}{x}$ **(d)** $f(x, y) = \sqrt[3]{x^6 - y^6} + x^2 + y^2$

Solution

(a) Given $$f(x, y) = \frac{x + y\cos x}{y\sin x - x}$$

Now $$f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y\cos(\lambda x)}{\lambda y\sin(\lambda x) - \lambda x}$$

$\therefore$ $$f(\lambda x, \lambda y) = \frac{x + y\cos(\lambda x)}{y\sin(\lambda x) - x} \neq f(x, y)$$

Hence not homogeneous.

(b) Given $$f(x, y) = \frac{x^2}{y^2}\ln\left(\frac{x}{y}\right) + \frac{y^2}{x^2}\ln\left(\frac{y}{x}\right)$$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2}{\lambda^2 y^2}\ln\left(\frac{\lambda x}{\lambda y}\right) + \frac{\lambda^2 y^2}{\lambda^2 x^2}\ln\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \frac{x^2}{y^2}\ln\left(\frac{x}{y}\right) + \frac{y^2}{x^2}\ln\left(\frac{y}{x}\right) = f(x, y)$$

Hence homogeneous of degree 0.

(c) Given $$f(x, y) = x + y \cos \frac{y}{x}$$

Now $$f(\lambda x, \lambda y) = \lambda x + \lambda y \cos \frac{\lambda y}{\lambda x}$$

$$\Rightarrow \quad \lambda\left(x + y\cos\left(\frac{y}{x}\right)\right)$$

$$= \lambda f(x, y)$$

Hence homogeneous of degree 1.

(d) Given $$f(x, y) = \sqrt[3]{x^6 - y^6} + x^2 + y^2$$

$$\Rightarrow \quad f(\lambda x, \lambda y) = \sqrt[3]{\lambda^6 x^6 - \lambda^6 y^6} + \lambda^2 x^2 + \lambda^2 y^2$$

$$= \lambda^2(\sqrt[3]{x^6 - y^6} + x^2 + y^2) = \lambda^2 f(x, y)$$

Hence, homogeneous function of degree 2. ■

Bounded Function

A function is said to be bounded if the function lies between two finite values.

EXAMPLE 16 Which of the following function(s) is/are bounded?

(a) $f(x) = 2^{\frac{1}{x-2}}$ on (1, 2)

(b) $g(x) = \sin x$ on $(-\infty, \infty)$

(c) $h(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $h(x) = \sqrt{x^2+1}$ on $[0, 5] \cup [7, 10]$

Solution **Option A:**

$f(x) = 2^{\frac{1}{x-2}}$; clearly this function for $x \in (1, 2)$ we can say is decreasing

$$\therefore \quad f(1^+) = 2^{-1} = \frac{1}{2} \text{ and } f(2^-) = 2^{-\infty} = 0$$

$\therefore$ Range is $\left(0, \frac{1}{2}\right)$, which function is bounded

Now, **Option B:**
Clearly $g(x) = \sin x \; \forall \; x \in R$ is bounded as range of the function is $[-1, 1]$.

Option C:
$h(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has range $(-\infty, \infty)$ which suggest function is unbounded.

Option D:

$$k(x) = \sqrt{x^2+1} \text{ on } [0, 5] \; [7, 10]$$

Clearly on indicated interval $K(x)$ will increase as x increases.

$\therefore$ Range of $K(x)$ is $[1, \sqrt{26}] \cup [5\sqrt{2}, \sqrt{101}]$. Here also we can say the function lies between two finite values for given values of x. Hence $K(x)$ is also bounded.

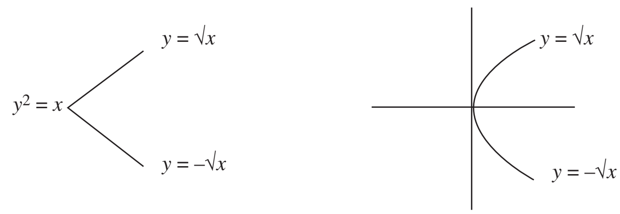
$\therefore$ Correct answers are (a), (b), and (d). ■

Implicit and Explicit Function A function defined by an equation not solved for the dependent variable is called an implicit function. For the equation $x^3 + y^3 = 1$, define y as an implicit function. If y has been expressed in terms of x alone then it is called an explicit function.

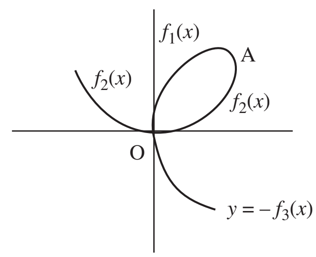
Examples of implicit and explicit function f: $(x, y) = 0$

1. $x\sqrt{1+y} + y\sqrt{1+x} = 0$; explicit $y = -\dfrac{x}{1+x}$ or $y = x$ (rejected)

2. $y^2 = x$ represent two separate branches.



3. $x^3 + y^3 - 3xy = 0$



folium of Descartes

Functional Equation

EXAMPLE 17 Prove that there can be two nonconstant polynomial functions which satisfy the functional equation $f(x) + f\left(\dfrac{1}{x}\right) = f(x)\cdot f\left(\dfrac{1}{x}\right)$ $\forall\, x \in R - \{0\}$.

Solution Let $f(x)$ be a n degree polynomial which satisfies the equation $f(x) + f\left(\dfrac{1}{x}\right) = f(x)\cdot f\left(\dfrac{1}{x}\right) \forall\, x \in R - \{0\}$.

So, we can assume $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ $(a_n \neq 0)$. Now putting this into equation we get

$$(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0) + \left(\frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \ldots . + a_0\right)$$

$$= (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0)\left(\frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \ldots . + a_0\right)$$

Taking L.C.M., we get

$$(a_n x^{2n} + a_{n-1}x^{2n-1} + \ldots + a_0 x^n + a_n + a_{n-1}x + \ldots + a_0 x^n)$$

$$= (a_n x^n + a_{n-1}x^{n-1} + \ldots a_0)\,(a_n + a_{n-1}x + \ldots a_0 x^n)$$

Equation the coefficient of $x^{2n}, x^{2n-1}, \ldots, x^n$ on both sides, we get
Coefficient of x^{2n} on L.H.S. $= a_n =$ coefficient of x^{2n} on R.H.S. $= a_n a_0$

$$\Rightarrow \qquad a_n = a_n a_0$$

$$\Rightarrow \qquad a_0 = 1 \quad (\text{as } a_n \neq 0)$$

Similarly, for coefficient of x^{2n-1}, we get

$$a_{n-1} = a_n a_1 + a_0\, a_{n-1}$$

$\Rightarrow \quad a_{n-1} = a_n a_1 + a_{n-1} \quad (\text{as } a_0 = 1)$

$\Rightarrow \quad a_n a_1 = 0$

$\Rightarrow \quad a_1 = 0 \quad (\text{as } a_n \neq 1)$

for x^{2n-2} we get

$$a_{n-2} = a_n a_2 + a_{n-1} a_1 + a_{n-2} a_0$$

Now, $a_0 = 1$ and $a_1 \neq 0$

$\Rightarrow \quad a_{n-2} = a_n a_2 + a_{n-2}$

$\Rightarrow \quad a_n a_2 = 0$

$\Rightarrow \quad a_2 = 0$

Similarly, we can show $a_3 = a_4 = \ldots = a_{n-1} = 0$

Equating coefficient of x^n, we get

$$2a_0 = a_n^2 + a_{n-1}^2 + \cdots + a_0^2$$

$\Rightarrow \quad 2a_0 = a_n^2 + a_0^2$

$\Rightarrow \quad a_n^2 = 1$

$\Rightarrow \quad a_n = \pm 1$

$\therefore \quad f(x) = 1 + x^n$ or $1 - x^n$

Hence, the two polynomial functions satisfying the given rule are $f(x) = 1 \pm x^n$

Aliter:

$$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$$

$\Rightarrow \quad f(x) \cdot f\left(\frac{1}{x}\right) - f(x) - f\left(\frac{1}{x}\right) + 1 = 1$

$\Rightarrow \quad (f(x) - 1)\left(f\left(\frac{1}{x}\right) - 1\right) = 1$

Now, suppose $g(x) = f(x) - 1$

$\Rightarrow \quad g\left(\frac{1}{x}\right) = f\left(\frac{1}{x}\right) - 1$

$\therefore \quad g(x) \times g\left(\frac{1}{x}\right) = 1$

Clearly we need the product to be 1 $\forall\, x \in R - \{0\}$ which is possible when $g(x)$ is of from $\pm x^n$.

$\therefore \quad g(x) = \pm x^n = f(x) - 1$

$\Rightarrow \quad f(x) = 1 \pm x^n$ ■

EXAMPLE 18 If an injective function $f(x)$ satisfies the rule $f(x) + f(y) + f(xy) = f(x) \cdot f(y) + 2\ \forall\, x, y \in R$. $f(1) \neq 1$ and $f(2) = 9$, then the value of $f(4)$ is

Solution Given $f(x) + f(y) + f(xy) = f(x) \cdot f(y) + 2$

Clearly x and y in this functional equation can be treated as independent variables.

Now, put $x = 1$ and $y = 1$; we get

$$3f(1) = f((1))^2 + 2 = f((1))^2 - 3f(1) + 2 = 0$$

$\Rightarrow \quad f(1) = 1$ or 2

But given $f(1) \neq 1$

$\Rightarrow$ $f(1) = 2$

Also, if we put $y = \dfrac{1}{x}$, we get

$$f(x) + f\left(\frac{1}{x}\right) + f(1) = f(x) \cdot f\left(\frac{1}{x}\right) + 2$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = f(x) \times f\left(\frac{1}{x}\right)$$

Hence, $f(x) = 1 \pm x^n$

Now it given $f(2) = 9$

If $f(x) = 1 - x^n$

$\Rightarrow$ $f(2) = 1 - 2^n = 9$

$\Rightarrow$ $2^n = -8$

As n is non-negative integer, so 2^n is always positive hence no solutions if $f(x) = 1 - x^n$

$\Rightarrow$ $f(2) = 1 + 2^n = 9$

$\Rightarrow$ $2^n = 8 = 2^3$

$\Rightarrow$ $n = 3$

$\therefore$ $f(x) = 1 + x^3$

$\Rightarrow$ $f(4) = 65$

EXAMPLE 19 Suppose $f(x)$ is a real function satisfying $f\left(\dfrac{x+1}{x-1}\right) = 2f(x) + \dfrac{1}{x-1}$ $\forall\, x \in R - \{1\}$; then $f(0)$ is equal to

(a) 0 **(b)** $\dfrac{2}{3}$ **(c)** $\dfrac{3}{4}$ **(d)** $\dfrac{5}{6}$

Solution Given $f\left(\dfrac{x+1}{x-1}\right) - 2f(x) = \dfrac{1}{x-1}\ \forall\, x \in \mathrm{R} - \{1\}$ (1)

In such cases we try to form more equation in $f\left(\dfrac{x+1}{x-1}\right)$ and $f(x)$ and for that purpose replace $x \rightarrow \dfrac{x+1}{x-1}$

Now,
$$f\left(\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}\right) - 2f\left(\frac{x+1}{x-1}\right) = \frac{1}{\left(\frac{x+1}{x-1}\right)-1}$$

$$\Rightarrow \quad f\left(\frac{x+1+x-1}{x+1-x+1}\right) - 2f\left(\frac{x+1}{x-1}\right) = \frac{x-1}{(x+1)-(x-1)}$$

$$\Rightarrow \quad f(x) - 2f\left(\frac{x+1}{x-1}\right) = \frac{x-1}{2} \qquad (2)$$

Now, multiply first equation by 2 and add with second equation, we get

$$2f\left(\frac{x+1}{x-1}\right) - 4f(x) + f(x) - 2f\left(\frac{x+1}{x-1}\right) = \frac{2}{x-1} + \frac{x-1}{2}$$

$$f(x) = -\frac{1}{3}\left(\frac{2}{x-1} + \frac{x-1}{2}\right)$$

$$f(0) = -\frac{1}{3}\left(-2 - \frac{1}{2}\right) = \frac{5}{6}$$

$\therefore$ Correct answer is (d). ■

EXAMPLE 20 Given that $f(x)$ is a function defined on R satisfying $f(1) = 1$ and for any $x \in R$, $f(x + 5) \geq f(x) + 5$ and $f(x + 1) \leq f(x) + 1$. If $g(x) = f(x) + 5 - x$, then $g(2015)$ is greater than or equal to

(a) 3 **(b)** 4 **(c)** 5 **(d)** 6

Solution Given $f(x + 5) \geq f(x) + 5$ and $f(x + 1) \leq f(x) + 1$

Now,
$$\begin{aligned} f(x+5) &\leq f(x+4) + 1 \\ &\leq f(x+3) + 2 \\ &\leq f(x+2) + 3 \\ &\leq f(x+1) + 4 \\ &\leq f(x) + 5 \end{aligned}$$

$\therefore$ $f(x + 5) = f(x) + 5$ (As $f(x + 5) \geq f(x) + 5$ and $f(x + 5) \leq f(x) + 5$)

Put $x = 1, 2, 3, \ldots, 2010$, we get

$$f(6) = f(1) + 5 = 6$$

$$f(7) = f(6) + 1 = 7$$

$$f(2015) = 2015$$

Now, $$g(2015) = f(2015) + 5 - 2015 = 5$$

$\therefore$ Correct answers are (a), (b) and (c). ■

EXAMPLE 21 Let $f(x)$ be a polynomial of degree 5 with leading coefficient unity such that $f(1) = 5$; $f(2) = 4$; $f(3) = 3$; $f(4) = 2$; $f(5) = 1$, then $f(6)$ is equal to

(a) 0 **(b)** 24 **(c)** 120 **(d)** 720

Solution Given: $f(1) = 5$; $f(2) = 4$; $f(3) = 3$; $f(4) = 2$ and $f(5) = 1$

Now we can observe

$$f(x) = 6 - x \ \forall \ x \in \{1, 2, 3, 4, 5\}$$

Hence, in such cases we assume a new function

$$g(x) = f(x) + x - 6$$

Clearly $f(x)$ is a polynomial of degree 5, hence $g(x)$ must be a polynomial of degree 5 with leading coefficient 1. Also $g(1) = f(1) + 1 - 6 = 5 - 5 = 0$

Similarly $$g(2) = g(3) = g(4) = g(5) = 0$$

Hence the polynomial equation $g(x) = 0$ has 5 roots

$\therefore$ $$g(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$$

$\therefore$ $$f(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5) - x + 6$$

$\therefore$ $$f(6) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 - 6 + 6 = 120$$

$\therefore$ Correct answer is (c). ■

EXAMPLE 22 $P(x)$ is a polynomial function of least degree such that $P(i) = \frac{1}{i}$, $i = 1, 2, 3, \ldots 9$. The value of $P(10)$ is

(a) $\frac{1}{5}$ **(b)** $\frac{2}{5}$ **(c)** $\frac{3}{5}$ **(d)** $\frac{4}{5}$

Solution Given $P(i) = \frac{1}{i} \; \forall \; i \in [1, 2, 3, \ldots, 9]$

Here also we can assume a new function $g(x)$ must be a polynomial whose degree is one more than $P(x)$

Also $$g(1) = 0 = g(2) = \ldots = g(9)$$

Now $x = 1, 2, 3, \ldots 9$ are the roots $g(x) = 0$ and the leading coefficient of $g(x)$ is not given; so, assume the leading coefficient $g(x)$ to be a.

$$\therefore \quad g(x) = xP(x) - 1 = a(x-1)(x-2) \ldots (x-9)$$

Now, "a" can be evaluated by putting $x = 0$

$$\therefore \quad -1 = a \times (-1)(-2) \ldots (-9)$$

$$\Rightarrow \quad a = \frac{1}{9!}$$

$$\therefore \quad xP(x) - 1 = \frac{1}{9!}(x-1)(x-2) \ldots (x-9)$$

Now, put $x = 10$; we get

$$10 \cdot p(10) - 1 = \frac{1}{9!} \cdot (9) \cdot (8) \ldots (1) = 1$$

$$\Rightarrow \quad P(10) = \frac{2}{10} = \frac{1}{5}$$

Hence correct answer is (a). ■

EXAMPLE 23 If $f(x) = \alpha x^5 + \beta \sin x + 5\gamma \ln\left(\frac{1+x}{1-x}\right) + 7$, where α, β, γ are non-zero real numbers and $f\left(-\frac{1}{2}\right) = 4$, then $f\left(\frac{\operatorname{sgn}(\pi)}{2}\right)$ will be (where sgn(x) represents signum function)

(a) 0 **(b)** 4 **(c)** 7 **(d)** 10

Solution $$f(x) = \alpha x^5 + \beta \sin x + \gamma \ln\left(\frac{1+x}{1-x}\right) + 7 \quad (1)$$

Clearly, $f\left(-\frac{1}{2}\right)$ is given and we have to compute the value of $f\left(\frac{1}{2}\right)$ as sgn $(\pi) = 1$

$$\therefore \quad f(-x) = -\alpha x^5 - \beta \sin x + 5\gamma \ln\left(\frac{1-x}{1+x}\right) + 7 \quad (2)$$

Adding (1) and (2), we get

$$f(x) + f(-x) = 14$$

Put $x = \frac{1}{2}$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) + f\left(-\frac{1}{2}\right) = 14 \qquad f\left(\frac{1}{2}\right) = 10$$

$\therefore$ Correct answer is (d). ■

EXAMPLE 24 Function $f(x)$ = satisfies the relation $f(xy) = f(x) \times f(y)$ for all positive number x and y and $f(2) = 10$; then

(a) $\sum_{\gamma=0}^{4} f(2^\gamma) = 10000$ (b) $\sum_{\gamma=0}^{4} f(2^\gamma) = 11111$

(c) $\sum_{\gamma=0}^{4} f^{-1}(10^\gamma) = 31$ (d) $\sum_{\gamma=0}^{4} f^{-1}(10^\gamma) = 32$

Solution Given $f(xy) = f(x) \cdot f(y)$

Here it can be seen that we need only the function values like $f(1), f(4), f(8)$ and $f(16)$. Here we need not find function in the given functional relation; put $x = 2$ and $y = 1$

$$f(2) = f(2) \times f(1)$$

$$\Rightarrow \qquad f(1) = 1$$

Put $x = 2$ and $y = 2$, we get

$$f(4) = f(2) \times f(12) = 10^2$$

Similarly $f(8) = 10^3$ and $f(16) = 10^4$

$$\sum_{r=0}^{4} f(2^\gamma) = f(1) + f(2) + f(4) + f(8) + f(16)$$

$$= 1 + 10 + 10^2 + 10^3 + 10^4 = 11111$$

Also, $\sum_{r=0}^{4} f^{-1}(10^\gamma) = f^{-1}(1) + f^{-1}(10) + f^{-1}(10^3) + f^{-1}(10^4) = 1 + 2 + 4 + 8 + 16 = 31$

Hence correct answers are (b) and (c). ■

Chapter 1 Questions to Guide Your Review

1. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
2. What is the graph of a real-valued function of a real variable? What is the vertical line test?
3. What is a piecewise-defined function? Give examples.
4. What are the important types of functions frequently encountered in calculus? Give an example of each type.
5. What is meant by an increasing function? A decreasing function? Give an example of each.
6. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
7. If f and g are real-valued functions, how are the domains of $f + g, f - g, fg$, and f/g related to the domains of f and g? Give examples.
8. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
9. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
10. How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
11. What is radian measure? How do you convert from radians to degrees? Degrees to radians?
12. Graph the six basic trigonometric functions. What symmetries do the graphs have?
13. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
14. Starting with the identity $\sin^2\theta + \cos^2\theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
15. How does the formula for the general sine function $f(x) = A\sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A, B, C, and D.

Chapter 1 Practice Exercises

Functions and Graphs

1. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
2. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
3. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
4. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

5. $y = x^{1/5}$
6. $y = x^{2/5}$
7. $y = x^2 - 2x - 1$
8. $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

9. $y = x^2 + 1$
10. $y = x^5 - x^3 - x$
11. $y = 1 - \cos x$
12. $y = \sec x \tan x$
13. $y = \dfrac{x^4 + 1}{x^3 - 2x}$
14. $y = x - \sin x$
15. $y = x + \cos x$
16. $y = x \cos x$
17. Suppose that f and g are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?

    a. fg **b.** f^3 **c.** $f(\sin x)$ **d.** $g(\sec x)$ **e.** $|g|$
18. If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–28, find the **(a)** domain and **(b)** range.

19. $y = |x| - 2$
20. $y = -2 + \sqrt{1 - x}$
21. $y = \sqrt{16 - x^2}$
22. $y = 3^{2-x} + 1$
23. $y = 2e^{-x} - 3$
24. $y = \tan(2x - \pi)$
25. $y = 2\sin(3x + \pi) - 1$
26. $y = x^{2/5}$
27. $y = \ln(x - 3) + 1$
28. $y = -1 + \sqrt[3]{2 - x}$
29. State whether each function is increasing, decreasing, or neither.

    a. Volume of a sphere as a function of its radius

    b. Greatest integer function

    c. Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)

    d. Kinetic energy as a function of a particle's velocity
30. Find the largest interval on which the given function is increasing.

    a. $f(x) = |x - 2| + 1$ **b.** $f(x) = (x + 1)^4$

    c. $g(x) = (3x - 1)^{1/3}$ **d.** $R(x) = \sqrt{2x - 1}$

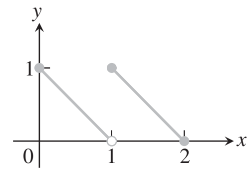
Piecewise-Defined Functions

In Exercises 31 and 32, find the **(a)** domain and **(b)** range.

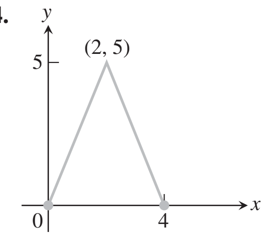
31. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$
32. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 33 and 34, write a piecewise formula for the function.

33.



34.



Composition of Functions

In Exercises 35 and 36, find

a. $(f \circ g)(-1)$. **b.** $(g \circ f)(2)$.

c. $(f \circ f)(x)$. **d.** $(g \circ g)(x)$.

35. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{\sqrt{x + 2}}$
36. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x + 1}$

In Exercises 37 and 38, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

37. $f(x) = 2 - x^2$, $g(x) = \sqrt{x + 2}$
38. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1 - x}$

For Exercises 39 and 40, sketch the graphs of f and $f \circ f$.

39. $f(x) = \begin{cases} -x - 2, & -4 \le x \le -1 \\ -1, & -1 < x \le 1 \\ x - 2, & 1 < x \le 2 \end{cases}$
40. $f(x) = \begin{cases} x + 1, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2 \end{cases}$

Composition with absolute values In Exercises 41–48, graph f_1 and f_2 together. Then describe how applying the absolute value function in f_2 affects the graph of f_1.

	$f_1(x)$	$f_2(x)$
41.	x	$\lvert x \rvert$
42.	x^2	$\lvert x \rvert^2$
43.	x^3	$\lvert x^3 \rvert$
44.	$x^2 + x$	$\lvert x^2 + x \rvert$
45.	$4 - x^2$	$\lvert 4 - x^2 \rvert$
46.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x \rvert}$
47.	$\sqrt{x}$	$\sqrt{\lvert x \rvert}$
48.	$\sin x$	$\sin \lvert x \rvert$

Shifting and Scaling Graphs

49. Suppose the graph of g is given. Write equations for the graphs that are obtained from the graph of g by shifting, scaling, or reflecting, as indicated.

a. Up $\frac{1}{2}$ unit, right 3

b. Down 2 units, left $\frac{2}{3}$

c. Reflect about the y-axis

d. Reflect about the x-axis

e. Stretch vertically by a factor of 5

f. Compress horizontally by a factor of 5

50. Describe how each graph is obtained from the graph of $y = f(x)$.

a. $y = f(x - 5)$ **b.** $y = f(4x)$

c. $y = f(-3x)$ **d.** $y = f(2x + 1)$

e. $y = f\left(\frac{x}{3}\right) - 4$ **f.** $y = -3f(x) + \frac{1}{4}$

In Exercises 51–54, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.15–1.17, and applying an appropriate transformation.

51. $y = -\sqrt{1 + \frac{x}{2}}$ **52.** $y = 1 - \frac{x}{3}$

53. $y = \frac{1}{2x^2} + 1$ **54.** $y = (-5x)^{1/3}$

Trigonometry

In Exercises 55–58, sketch the graph of the given function. What is the period of the function?

55. $y = \cos 2x$ **56.** $y = \sin \frac{x}{2}$

57. $y = \sin \pi x$ **58.** $y = \cos \frac{\pi x}{2}$

59. Sketch the graph $y = 2\cos\left(x - \frac{\pi}{3}\right)$.

60. Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 61–64, ABC is a right triangle with the right angle at C. The sides opposite angles A, B, and C are a, b, and c, respectively.

61. a. Find a and b if $c = 2$, $B = \pi/3$.

b. Find a and c if $b = 2$, $B = \pi/3$.

62. a. Express a in terms of A and c.

b. Express a in terms of A and b.

63. a. Express a in terms of B and b.

b. Express c in terms of A and a.

64. a. Express $\sin A$ in terms of a and c.

b. Express $\sin A$ in terms of b and c.

65. Height of a pole Two wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal and wire CT makes an angle of 50° with the horizontal, how high is the pole?

66. Height of a weather balloon Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

T **67. a.** Graph the function $f(x) = \sin x + \cos(x/2)$.

b. What appears to be the period of this function?

c. Confirm your finding in part (b) algebraically.

T **68. a.** Graph $f(x) = \sin(1/x)$.

b. What are the domain and range of f?

c. Is f periodic? Give reasons for your answer.

Chapter 1 Single Choice Questions

1. The value of $[\sqrt[3]{1}] + [\sqrt[3]{2}] + \ldots + [\sqrt[3]{124}]$ is (where $[\cdot]$ denotes greatest integer function)

a. 400 **b.** 401 **c.** 402 **d.** 403

2. The number of solutions of equation $\{x^2\} = \left\{\frac{1}{x}\right\}$ (where $\{\cdot\}$ denotes fractional part function) for $x \in (\sqrt{2}, \sqrt{3})$ is :

a. 0 **b.** 1 **c.** 2 **d.** More than two

3. The range of function $f(x) = \frac{2x-1}{x-3}$ is:

a. R **b.** $R - \{3\}$ **c.** $R - \{2\}$ **d.** $R - \left\{\frac{1}{2}\right\}$

4. Range of the function $f(x) = [x^2 - x + 1]$ (where $[\cdot]$ denotes greatest integer function) is:

a. $[0, \infty)$ **b.** All positive integer

c. All negative integers **d.** All non-negative integers.

5. If $f(x)$ be a polynomial function satisfying $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ $\forall x \in R - \{0\}$ and $f(5) = 26$, then $f(2)$ is:

a. 5 **b.** 9

c. 3 **d.** –5

6. If $f(x) \frac{2^x}{2^x + 2^{1-x}}$ = then $f\left(\frac{1}{2015}\right) + f\left(\frac{2}{2015}\right) + \ldots + f\left(\frac{2014}{2015}\right)$ is

a. 1007 **b.** 1007.5

c. 2015 **d.** cannot be determined

7. If $f(x)$ has domain of definition $[-3, 5]$, then sum of integer values of x in domain of $f(|x| - 5)$ is:

a. 0 **b.** 44

c. – 44 **c.** 45

8. The fundamental period of the function $f(x) = \sin^4 x + \cos^4 x$ is (where fundamental period is least positive value of T for which $f(x+T) = f(x)$)

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{2}$ **c.** π **d.** $\frac{3\pi}{2}$

9. The range of the function $f(x) = \cos\sqrt{[x]+[-x]} + \sqrt{4-x^2} + \frac{1}{|x|}$ is (where [·] denotes greatest integer function)

a. $[0, \infty)$ **b.** $\left\{\frac{3}{2}\right\}$

c. $\left\{\frac{3}{2}, 2+\sqrt{3}\right\}$ **d.** $\left[\frac{3}{2}, 2+\sqrt{3}\right]$

10. If $\begin{vmatrix} f(x-3) & f(x+4) & f((x+1)(x-2)-(x-1)^2) \\ 5 & 4 & -5 \\ 5 & 6 & 15 \end{vmatrix} = 0 \ \forall x \in R$, then:

a. $f(x)$ is not periodic

b. $f(x)$ is periodic with period 1

c. $f(x)$ is periodic with period 7

d. $f(x)$ must be an odd function

11. Let [·] denotes greatest integer function. Then the value of is $\sum_{r=1}^{150}\left[\frac{3}{5}+\frac{r}{100}\right]$

a. 120 **b.** 121 **c.** 122 **d.** 123

12. The number of solutions of the equation $5x = 7[x]$ (where [·] denotes greatest integer function) is :

a. 1 **b.** 2 **c.** 3 **d.** 4

13. A real-valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x) - f(a-x)\,f(a+y)$, where a is a given constant and $f(0) = 1$, then $f(2a-x)$ is equal to

a. $-f(x)$ **b.** $f(x)$ **c.** $f(a) + f(a-x)$ **d.** $f(-x)$

14. Number of elements in the range of $f(x) = \left[\frac{x}{15}\right]\left[-\frac{15}{x}\right] \ \forall x \in (0, 90)$ (where [·] denotes greatest integer function)

a. 5 **b.** 6 **c.** 7 **d.** 8

15. Sum of the absolute values of all distinct real solution(s) of $x^2 - 3x + 4 = \frac{3}{2} + \sqrt{x - \frac{7}{4}}$ is:

a. 1 **b.** 2 **c.** 3 **d.** 4

16. A function $f: R \to R$ is denoted by $f(x) = x^3 + x^2 + 100x + 4\sin x$. the function $f(x)$ is

a. Injective but not surjective

b. Bijective but not periodic

c. Surjective but not injective

d. Bijective and periodic

17. Number of solution(s) of the equation $[\cos x] + [\sin x] = \frac{1}{3}$ where $x \in [0, 2\pi]$ is (where [·] and {·} denote greatest integer and fractional part function, respectively)

a. 0 **b.** 1 **c.** 2 **d.** 4

18. The function $f(x) = \sin(\log(x + \sqrt{x^2+1}))$ is:

a. even function **b.** odd function

c. constant function **d.** neither even nor odd

19. If $\{x\}$ denotes the fractional part function and $N = \dfrac{\{\sqrt{3}\} - 2\{\sqrt{2}\}}{\{\sqrt{3}\}^2 - 2\{\sqrt{2}\}^2}$, then $25N^2$ is:

a. 1 **b.** 3 **c.** 4 **d.** 7

20. Which of the following statements is true?

a. $f(x) = [\log\{x\}]$ and $g(x) = 0$ are identical functions (where [·] and {·} denote greatest integer function and fractional part function, respectively)

b. Function $y = f(x)$ given by $4x + 4^{-y} = 2$ is unbounded.

c. If $f(x) = \frac{x^2+1}{x^2-1}$ and $g(x) = \frac{x^4+2x^2+1}{x^4-1}$, then $\frac{f(x)}{g(x)}$ is a constant function for all values of x.

d. If $f(x) = xe^{-|x|} + e^{|x|} + x^2$, then $f(x) + f(-x) = 0$.

21. Let $f: (0, 1) \to R$ be defined by $f(x) = \frac{b-x}{1+bx}$; b is a constant $b \in (0, 1)$, then

a. f is invertible on (0, 1)

b. f is not invertible on (0, 1) because f is one-one but not onto

c. f is not invertible on (0, 1) because f is onto but not one-one

d. f is not invertible on (0, 1) because f is neither one-one nor onto

22. Domain of $f(x) = \sqrt{x^{10} - x^5 + x^4 - x + 1}$ is

a. $(-\infty, 1]$ **b.** $(-\infty, \infty)$

c. $[1, \infty]$ **d.** $[-\infty, 0]$

23. Let $f: \left[\frac{1}{2}, 1\right] \to [-1, 1]$ is given by $f(x) = 4x^3 - 3x$, then $f^{-1}(1)$ is given by

a. $\frac{1}{2}$ **b.** -1 **c.** 1 **d.** $-\frac{1}{2}$

24. Let $f(x) = \sqrt{[\sin 2x] - [\cos 2x]}$ (where [·] denotes greatest integer function), then range of $f(x)$ will be

a. $\{0\}$ **b.** $\{1\}$

c. $\{0, 1\}$ **d.** $\{0, 1, \sqrt{2}\}$

25. If $g^3(x) - 3g^2(x) + 3g(x) - 1 = x^6$, then $g(0)$ is (where $g^n(x) = (g(x))^n$)

a. -1 **b.** 1 **c.** 2 **d.** 3

26. The range of the function is

$$f(x) = \frac{\sin x(3\cos^2 x + \cos^4 x + 3\sin^2 x + \sin^2 x \cdot \cos^2 x)}{\tan x(\sec x - \sin x \cdot \tan x)}$$

a. $[3, 4]$ **b.** $(3, 4)$

c. $[3, 4)$ **d.** $(3, 4]$

27. If $[x] + 3\{y\} = 3.9$ and $\{x\} + 3[y] = 3.4$, then the number of possible values of $(x + y)$ is (where [·] denotes greatest integer function and {·} denotes fractional part function):

a. 0 **b.** 1 **c.** 2 **d.** 3

28. Number of integers in the range of $g(x)=\dfrac{(13-3\sin^2 x-6\,|\cos x|)}{(4-\sin^2 x-2\,|\cos x|)}$

a. 0 **b.** 3 **c.** 6 **d.** 9

29. The function where graph is reflection about the line $x+y=0$ of the inverse function $g^{-1}(x)$ of a function $g(x)$ is:

a. $-g(x)$ **b.** $-g(-x)$

c. $-g^{-1}(x)$ **d.** $-g^{-1}(-x)$

30. If f: $R^+ \to R$ is a function such that $(f(x^2+1))^{\frac{1}{x^{1/4}}}=8$, then the value of $\left(f\left(\dfrac{256}{x^2}+1\right)\right)^{x^{1/4}}$ is

a. 4 **b.** 8 **c.** 32 **d.** 64

31. If $f(x)=\begin{cases} x+1 & -1\le x\le 0 \\ x^2+1 & 0<x\le 1 \end{cases}$; then the value of $\dfrac{f(-1)+f(0)+f(1)}{f^{-1}(0)+f^{-1}(1)+f^{-1}(2)+1}$ is

a. 0 **b.** 1 **c.** 3 **d.** Non existent

32. If the domain of $f(x)$ is $[-1, 2]$ then domain of function $f([x]-x^2+4)$ is (where $[\cdot]$ denotes greatest integer function)

a. $[-1, \sqrt{8}]$ **b.** $[-\sqrt{3}, -1]\cup[\sqrt{3}, \sqrt{7}]$

c. $[-\sqrt{3}, \sqrt{8}]-(-1, 2)$ **d.** $[\sqrt{13}, \sqrt{14}]$

33. If the function f: $(-\infty, 0] \to [3, \infty)$ is defined by $f(x)=e^x+e^{-x}+1$, then $f^{-1}(x)$ is

a. $\ln\left(\dfrac{(x-1)+\sqrt{x^2-2x-5}}{2}\right)$

b. $\ln(e^x+e^{-y}+1)$

c. $\ln\left(\dfrac{x-1-\sqrt{x-2}}{3}\right)$

d. $\ln\left(\dfrac{(x-1)-\sqrt{x^2-2x-3}}{2}\right)$

34. Consider $f(x)=\begin{cases} x^2-2 & x<0 \\ x & 0\le x<2 \\ (x-2)^2 & 2\le x<4 \\ x-4 & x\ge 4 \end{cases}$. If the equation $f(x)=K$ has four roots, then k can be equal to:

a. -2 **b.** $\{0, 1\}$

c. 2 **d.** No value of K

35. If $\alpha, \beta \in R$, then period of $f(x)=x-[x+\alpha]-\beta$ (where $[\cdot]$ denotes greatest integer function) is

a. 1 **b.** $|\alpha-b|$

c. $|\alpha+b|$ **d.** Non-periodic

36. The domain of the function $f(x)=\sqrt{\dfrac{(\sin x-e^x)(\cos x-3)(\ln x+2)}{\left(x^4-x+\dfrac{1}{2}\right)(2x-7)^{30}}}$ belongs to:

a. $\left[\dfrac{1}{e^2}, \dfrac{7}{2}\right)\cup\left(\dfrac{7}{2}, \infty\right)$ **b.** $\left(-\dfrac{7}{2}, 0\right)$

c. $\left(0, \dfrac{1}{e^5}\right)$ **d.** ϕ

37. Let $A=\left\{x:\dfrac{|x(x-1)|\,(x+1)^{3/2}\ln(x+2)}{(x^2-2x)^2(x-1)(e^x-2)}\ge 0\right\}$ and $B=\{a: a>0,$ $2+\sin\theta=x^2+\dfrac{a}{x^2}\ \forall\, x\in R\ \{0\}, \theta\in R\}$ are two sets, then (given $\log_{10}2=0.3010$):

a. $A\subseteq B$ **b.** $B\subseteq A$

c. $A\cap B=\phi$ **d.** $A\cap B=(0, \ln 2)$

38. Let f: $R\to R$ be a function such that $f(0)=1$ and $f(xy+1)=f(x)\cdot f(y)-f(y)-x+2\ \forall\, x, y\ \forall\, x\in R$, then $f(x)$ is:

a. x^3+1 **b.** $2x+1$ **c.** $x+1$ **d.** None of these

39. The range of $f(x)=2^x+3^x-4^x+6^x-9^x\ \forall\, x\in R$ is:

a. $(-\infty, 1]$ **b.** $(-\infty, 2]$ **c.** R **d.** $(-\infty, 3]$

40. The value of $\displaystyle\sum_{K=1}^{2004}\frac{1}{1+\tan^4\left(\dfrac{K\pi}{4010}\right)}$ is:

a. 1001 **b.** 1002 **c.** 2002 **d.** 2004

41. The sum of all solutions of equation $[x]=3\{5x\}$ (where $[\cdot]$ and $\{\cdot\}$ denote greatest integer function and fractional part function, respectively) is:

a. 18 **b.** 20 **c.** 21 **d.** 22

42. The number of quadratic polynomials $f(x)$, $g(x)$ and $h(x)$ such that the equation $f(g(h(x)))=0$ has eight roots 1, 2, 3, 4, 5, 6, 7, 8 is

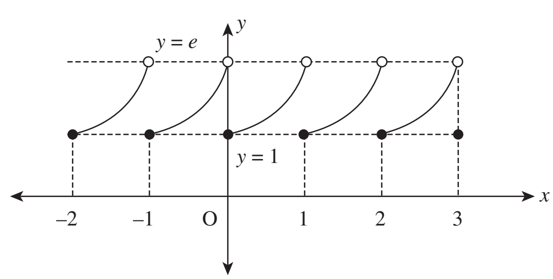
a. 0 **b.** 1 **c.** 2 **d.** 3

43. The number of solution of the equation is $\left[\dfrac{25x-2}{4}\right]=\dfrac{13x+4}{3}$ is (where $[\cdot]$ denotes greatest integer function.)

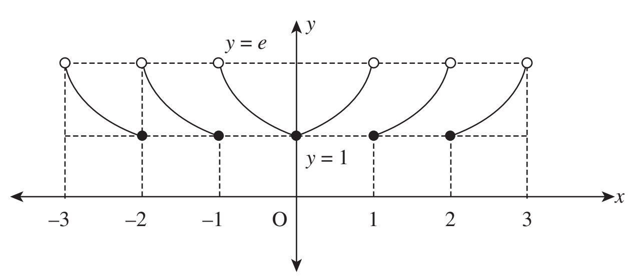
a. 0 **b.** 1 **c.** 2 **d.** more than two

44. Which one of the following best represent the graph of function $f(x)=e^{\{|x|\}}$ [Note : $\{\alpha\}$ denotes the fractional part of α.]

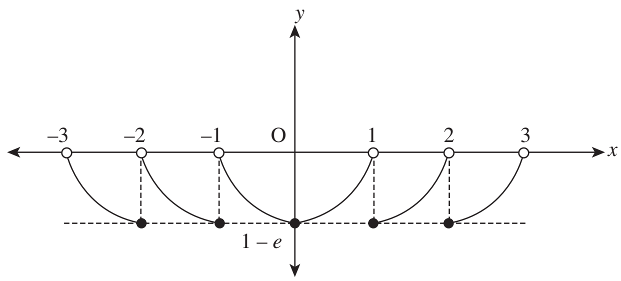
a.



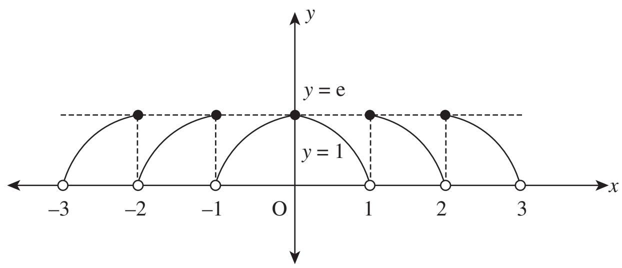
b.



c.



d.



45. Let $f(x) = \frac{3}{2} + \sqrt{x - \frac{7}{4}}$ and $g(x)$ be the inverse function of $f(x)$, then the value of $(f^{-1} \circ g^{-1})(17)$ is equation to

a. $\frac{3+\sqrt{61}}{2}$ **b.** 242

c. 17 **d.** $\frac{3-\sqrt{61}}{2}$

46. The set of values of K ($K \in R$) for which the equation $x^2 - 4|x| + 3 - |K - 1| = 0$ will have exactly four real roots is

a. $(-2, 4)$ **b.** $(-4, 4)$

c. $(-4, 2)$ **d.** $(-1, 0)$

47. The values of a and b for which $|e^{|x-b|} - a| = 2$, has four distinct solutions, are

a. $a \in (-3, \infty)$ and $b = 0$

b. $a \in (2, \infty)$ and $b = 0$

c. $a \in (-4, \infty)$ and $b = 0$

d. $a \in (1, \infty)$ and $b = 0$

48. Let $f(x) = x^2 - 4|x|$ and $g(x) = \begin{cases} \min\{f(t) : -6 \le t \le x\}, & x \in [-6, 0] \\ \max\{f(t) : 0 < t \le x\}, & x \in (0, 6] \end{cases}$

Then number of solutions of $g(x) = \{x\}$ in $[-6, 6]$ is (where $\{\cdot\}$ denotes functional part function)

a. 6 **b.** 7 **c.** 8 **d.** 9

Chapter 1 Multiple Choice Questions

1. Let α be a solution of the equation $2[x + 32] = 3[x - 64]$ where $[x]$ is the greatest integer less than or equal to x and let $\beta = \prod_{r=1}^{9} \sin\left(\frac{2r-1}{18}\right)\pi$, then

a. $[\alpha] = [\beta]$ **b.** $[\alpha] = \left[\frac{1}{\beta}\right]$

c. $[\alpha]\left[\frac{1}{\beta}\right] = 1$ **d.** $\left[\frac{1}{\alpha}\right] + \left[\frac{1}{\beta}\right] = 2^8$

2. The equation $||x - 1| - 2| = a$, $a \in R$, has

a. three distinct real roots for unique value of a

b. four distinct real roots for $a \in (0, 2)$

c. two distinct real roots for $|a| < 4$

d. no real roots for $a > 4$

3. For the equation $\frac{e^{-x}}{1+x} = \lambda$, which of the following statement(s) is/are correct?

a. when $\lambda \in (0, \infty)$ equation has 2 real and distinct roots

b. when $\lambda \in (-\infty, -e^2)$ equation has 2 real and distinct roots

c. when $\lambda \in (0, \infty)$ equation has 1 real root

d. when $\lambda \in (-e, 0)$ equation has no real root

4. Let $f(x) = \begin{cases} x^2 & 0 < x < 2 \\ 2x - 3 & 2 \le x < 3 \\ x + 2 & x \ge 3 \end{cases}$, then the true equations

a. $f\left(f\left(f\left(\frac{3}{2}\right)\right)\right) = f\left(\frac{3}{2}\right)$

b. $1 + f\left(f\left(f\left(\frac{5}{2}\right)\right)\right) = f\left(\frac{5}{2}\right)$

c. $f(f(f(2))) = f(1)$

d. $\underbrace{f(f(f(\ldots\ldots f(4))\ldots\ldots)}_{1004 \text{ times}} = 2012$

5. For the quadratic polynomial $f(x) = 4x^2 - ax + a$, the statement(s) which holds good is/are

a. There is only one integral a for which $f(x)$ is non-negative $\forall x \in R$

b. For $a < 0$, the number zero lies between the zeroes of the polynomial

c. $f(x) = 0$ has two distinct solutions in $(0, 1)$ for $a \in \left(\frac{1}{7}, \frac{4}{7}\right)$

d. The minimum value of a for which $f(x)$ is non-negative $\forall x \in R$ is 0

6. Let $f: \left[\frac{2\pi}{3}, \frac{5\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sin x - \cos x + 2$, then $\left(\frac{1}{7}, \frac{4}{7}\right)$
   a. $f^{-1}(1) = \frac{4\pi}{3}$ b. $f^{-1}(1) = \pi$
   c. $f^{-1}(2) = \frac{5\pi}{4}$ d. $f^{-1}(2) = \frac{7\pi}{4}$

7. Let $f: I \to I$, defined as $f(x) = 2\sin(2\pi x) - 10\tan(5\pi x) + 7\cos(4\pi x) + 3$, then which of following statement(s) is/are true?
   a. $f(x)$ is periodic function.
   b. $f(x)$ is an even function
   c. $f(x)$ is an odd function and its inverse exists
   d. $f(f(f(x))) = f(f(x))$ for all $x \in I$.
   [**Note:** I denotes the set of all integers.]

8. Let f, g and h be three functions defined as follows:
   $f(x) = \frac{32}{4 + x^2 + x^4}$. $g(x) = 9 + x^2$ and $h(x) = -x^2 - 3x + k$
   Identify which of the following statement(s) is/are correct
   a. Number of integers in the range of $f(x)$ is 8.
   b. Number of integral value of k for which $h(f(x)) > 0$ and $h(g(x)) < 0\ \forall\ x \in R$ is 20.
   c. Number of integral values of k for which $h(f(x)) > 0$ and $h(g(x)) < 0\ \forall\ x \in R$ is 19.
   d. Maximum value of $f(g(x))$ is 73.

9. If $f(x) = \left([\{x\}]\tan^{-1}\left(\frac{x^2 - 3x + 1}{x^2 - 3x + 5}\right) + 3 - x^7\right)^{\frac{1}{7}}$, where $[k]$ and $\{k\}$ denote greatest integer and fractional part functions of k, respectively, then the value of $f^{-1}(50) - f(50) + f(f(100))$, is greater than
   a. 0 b. 25 c. 50 d. 100

10. Which option(s) is/are true?
    a. $f: R \to R, f(x) = e^{|x|} - e^{-x}$ is many one into function
    b. $f: R \to R, f(x) = 2x + |\sin x|$ is one-one onto
    c. $f: R \to R, f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is many one onto
    d. $f: R \to R, f(x) = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$ is many one into

11. If $h(x) = \left[\ln\frac{x}{e}\right] + \left[\ln\frac{e}{x}\right]$, where $[\cdot]$ denotes greatest integer function, then which of the following are true?
    a. range of $h(x)$ is $\{-1, 0\}$
    b. If $h(x) = 0$, then x must be irrational
    c. If $h(x) = -1$, then x can be rational as well as irrational
    d. $h(x)$ is periodic function

12. If $f(x) = \begin{cases} x^3; & x \in Q \\ -x^3; & x \notin Q \end{cases}$, then
    a. $f(x)$ is periodic b. $f(x)$ is many-one
    c. $f(x)$ is one-one d. range of the function is R

13. If $||\ln|x|| = |k - 1| - 3$ has two distinct real roots for $x \in (-e^2, e^2) - \{0\}$, then k may take value(s)
    a. -5 b. -2 c. 4 d. 7

14. For $x \in R^+$, if x, $[x]$, $\{x\}$ are in harmonic progression then the value of x cannot be equal to (where $[\cdot]$ greatest integer function, $\{\cdot\}$ fractional part function)
    a. $\frac{1}{\sqrt{2}}\tan\frac{\pi}{8}$ b. $\frac{1}{\sqrt{2}}\cot\frac{\pi}{8}$
    c. $\frac{1}{\sqrt{2}}\tan\frac{\pi}{12}$ d. $\frac{1}{\sqrt{2}}\cot\frac{\pi}{12}$

15. Consider $f(x) = x^2 - 3x + a + \frac{1}{a}$, $a \in R - \{0\}$, such that $f(3) > 0$ and $f(2) \le 0$. If α and β are the roots of equation $f(x) = 0$, then the value of $\alpha^2 + \beta^2$ is
    a. greater than 11
    b. less than 11
    c. 5
    d. depends upon a and cannot be determined

16. If $f(x)$ is an even periodic function with period 10. $f(x) = \begin{cases} 2x & 0 \le x < 2 \\ 3x^2 - 8 & 2 \le x < 4 \\ 10x & 4 \le x \le 5 \end{cases}$. Then
    a. $f(-4) = 40$ b. $\frac{f(-13) - f(11)}{f(13) + f(-11)} = \frac{17}{21}$
    c. $f(5)$ is not defined d. Range of $f(x)$, is $[0, 50]$

17. Let $f: [-1, 1]$ onto $[3, 5]$ be a linear polynomial. Which of the following can be true?
    a. $f\left(\frac{-1}{2}\right) = \frac{7}{2}$ b. $f^{-1}\left(\frac{15}{4}\right) = \frac{1}{4}$
    c. $f(0) \ne 4$ d. $f\left(\frac{1}{2}\right) + f\left(\frac{-1}{2}\right) = 8$

18. If the rational number $\frac{p}{q}$, $q \ne 0$, p and q are relatively prime, is a root of the equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, where a_0, a_1, $\ldots a_n$ are integers and $a_n \ne 0$, then which of the following is necessarily true:
    a. p is divisor of a_0 b. q is divisor of a_n
    c. q is divisor of a_0 d. p, q are both divisor of a_n

19. Identify the correct statement(s)
    a. $f(x) = \sqrt[5]{(1 - x^3)^4} + \sqrt[5]{(1 + x^3)^4}$ is an even function
    b. $g(x) = \frac{\sqrt{1 - x + 2x^2 - 3x^3} - \sqrt{1 + x + 2x^2 + 3x^3}}{(5^x + 5^{-x})}$ is an odd function
    c. $(\sqrt{\sin x} - \sqrt{\tan x})(\sqrt{\sin x} + \sqrt{\tan x})$ is an odd function
    d. Any function can be expressed as sum of an even and an odd function

20. If $f(g(x)) = x$ and $g(f(x)) = x$, then which of the following may be the function $f(x)$ and $g(x)$?

a. $f(x) = g(x) = (7 - x^{1/3})^3$

b. $f(x) = \dfrac{8x-7}{5x-8}\ x \neq -\dfrac{8}{5}$; $g(x) = \dfrac{8x+7}{8-5x}\ x \neq \dfrac{8}{5}$

c. $g(x) = f(x) = \begin{cases} x, & x \in Q \\ -x, & x \in Q \end{cases}$

d. $f(x) = \log(x-2), x \geq 2$; $g(x) = e^x + 2, x \in R$

21. If $f(x)$ is an onto function defined from $[-2, 2]$ to $[1, 5]$, then:

a. Range of $f(x)$ and $f(x+1)$ is same

b. Domain of $f(x)$ and $7f(x) + 9$ is same

c. Range of $\dfrac{4}{f(x-1)+2}$ is $\left[\dfrac{4}{7}, \dfrac{4}{3}\right]$

d. Domain of $f(\sin x - \cos x)$ is R

22. Identify surjective function:

a. $f: (0, \pi/2) \to [6, \infty), f(x) = 4\tan^2 x + 9\cot^2 x$

b. $f: [0, 2\pi] \to [0, 50], f(x) = 24\cos x - 7\sin x + 25$

c. $f: R \to R\ f(x) = (2x+5)(3x+7)(4x+9)$

d. $f: R \to [0, \infty)\ f(x) = \dfrac{|x|}{|x|+1}$

23. If domain and range of $f(x)$ are $[0, 2]$ and $[1, 3]$, respectively, then identify the correct statement:

a. domain of $\sqrt{3f(x)+5}$ is $[0, 2]$

b. range of $\dfrac{1}{f(x)+2}$ is $\left[\dfrac{1}{5}, \dfrac{1}{3}\right]$

c. domain of $f(2x+1)$ is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

d. range of $f(7x) + 5$ is $[6, 8]$

24. Let $f: A \to B$ and $g: B \to C$ be two functions and $g \circ f: A \to C$ is defined. Then which of the following statement(s) is true?

a. If $g \circ f$ is onto then f must be onto

b. If f is into and g is onto then $g \circ f$ must be onto function

c. If $g \circ f$ is one-one then g is not necessarily one-one

d. If f is injective and g is surjective, then $g \circ f$ must be bijective mapping.

25. Let us consider the function $f(x) = \sqrt{p\{x\}^2 + q\{x\}}$ (where $\{\cdot\}$ denotes fractional part function), then which of the following statements is/are correct?

a. If $p > 0 > q$ and $|q| > |p|$, then domain of $f(x)$ is an integer.

b. If $p < 0 < q$ and $|q| > |p|$, then domain of $f(x)$ is R.

c. If $p > 0$ and $q > 0$ then domain of $f(x)$ is R.

d. If $p < 0$ and $q < 0$, then $f(x)$ is not defined for any real value of x.

Chapter 1 Passage Type Questions

Passage 1

Let three positive reals a, b, and c satisfy the equations.

$[a]\,bc = 3$

$a[b]\,c = 4$

$ab\,[c] = 5$

where $[\cdot]$ denotes greatest integer function, then

1. A possible value of a is

a. $\dfrac{\sqrt{30}}{3}$ **b.** $\dfrac{\sqrt{30}}{7}$ **c.** $\dfrac{\sqrt{30}}{5}$ **d.** $\dfrac{2\sqrt{30}}{5}$

2. A possible value of c is

a. $\dfrac{\sqrt{30}}{3}$ **b.** $\dfrac{\sqrt{30}}{7}$ **c.** $\dfrac{\sqrt{30}}{5}$ **d.** $\dfrac{2\sqrt{30}}{5}$

Passage 2

Consider functions $f: R \to R$ defined as $f(x) = \begin{cases} x+3, & x \in \text{rational} \\ 4x, & x \in \text{irrational} \end{cases}$

and $g: R \to R$ defined as $g(x) = \begin{cases} x+\sqrt{5}, & x \in \text{irrational} \\ -x, & x \in \text{rational} \end{cases}$ Also

$h: [1, \infty) \to [3, \infty)$ be defined as $h(x) = (\log_2 x)^2 + 2(\log_2 x) + 3$.

1. The function $(f(x) - g(x))$ is

a. injective but no surjective

b. surjective but not injective

c. injective as well as surjective

d. neither injective nor surjective

2. The function $h^{-1}(x)$ is equal to

a. $2^{\sqrt{x-3}}$ **b.** $4^{\sqrt{x-3}}$

c. $2^{-1-\sqrt{x-2}}$ **d.** $2^{-1+\sqrt{x-2}}$

Passage 3

An even periodic function $f: R \to R$ with period 4 is such that $f(x) = \begin{cases} \max.(|x|, x^2); & 0 \leq x < 1 \\ x; & 1 \leq x \leq 2 \end{cases}$

1. The value of $\{f(5.12)\}$ (where $\{\cdot\}$ denotes fractional part function), is

a. $\{f(3.26)\}$ **b.** $\{f(7.88)\}$

c. $\{f(2.12)\}$ **d.** $\{f(5.88)\}$

2. The number of solution of $f(x) = |3\sin x|$ for $x \in (-6, 6)$

a. 5 **b.** 4 **c.** 7 **d.** 9

Passage 4

Let $f(x) = 2 - |x - 3|$, $1 \le x \le 5$ and for rest of the values $f(x)$ can be obtained by using the relation $f(5x) = \alpha f(x)\ \forall\ x \in R$

1. The maximum value of $f(x)$ in $[5^4, 5^5]$ for $\alpha = 2$ is
   a. 16 **b.** 32
   c. 64 **d.** 8
2. The value of $f(2007)$, taking $\alpha = 5$, is
   a. 1118 **b.** 2007
   c. 1250 **d.** 132

Passage 5

Let the function f be defined in (0, 1). Two more functions are given as $g(x) = e^x$ and $h(x) = \ln |x|$.

1. The number of solutions of the equation $g(x) \cdot h(g(x)) = 1$ is
   a. 0 **b.** 1
   c. 2 **d.** 3
2. Domain of $f(g(x)) + f(h(x))$ is
   a. $(-e, -1)$ **b.** $(-e, -1) \cup (1, e)$
   c. $[-e, -1]$ **d.** $(-\infty, 0)$
3. Let f be one-one function with range (1, 2) and $g \circ f(x)$ is defined then domain of $f^{-1} \circ g^{-1}(x)$ is
   a. (0, 1) **b.** (e, e^2)
   c. $(\sqrt{e}, e)$ **d.** (1, e)

Passage 6

The function whose value at any number x is the smallest integer greater than or equal to x is called the least integer function or the ceiling function. It is denoted by $[x]$.

For example, $[1 \cdot 1] = 2$, $[\cdot 2] = 1$, $[-1 \cdot 2] = -1$, $[2] = 2$

Answer the following questions.

1. $[x]^2 - 3[x] + 2 = 0$, then x belongs to
   a. [1, 3] **b.** $(0, 1) \cup (1, 2)$
   c. {1, 2} **d.** (0, 2]
2. $[1] + [\sqrt{2}] + [\sqrt{3}] + \ldots + [\sqrt{10}] =$
   a. 26 **b.** 19
   c. 10 **d.** None of these

Passage 7

There are six trigonometric and six inverse trigonometric functions. Similarly there are six hyperbolic function defined as follows:

The quantity $\dfrac{e^x - e^{-x}}{2}$, where $x \in R$ is called hyperbolic sine of x and is written as sin hx, cos hx = $\dfrac{e^x + e^{-x}}{2}$ called as hyperbolic cos x and hyperbolic tan of x is written as tan hx = $\dfrac{\sinh x}{\cosh x}$. Similarly sec hx = $\dfrac{1}{\cosh x}$, cosec hx = $\dfrac{1}{\sinh x}$, cot hx = $\dfrac{1}{\tanh x}$. Now answer the following questions.

1. Which out of following is many one?
   a. tan hx **b.** cot hx
   c. cos hx **d.** sin hx
2. Which out of following is an even function?
   a. tan hx
   b. cosec hx
   c. sec hx + cos hx
   d. tan hx + cot hx
3. The functions whose domain is R
   a. sin hx, cos hx, tan hx
   b. cosec hx, sec hx, cot hx
   c. sin hx, cos hx, cot hx
   d. sin hx, cosec hx, tan hx

Passage 8

Consider the function $f(x) = \begin{cases} x^2 - 1, & -1 \le x \le 1 \\ \ln x, & 1 < x \le e \end{cases}$

Let $f_1(x) = f(|x|)$

$f_2(x) = |f(|x|)|$

$f_3(x) = f(-x)$

Now answer the following questions:

1. Number of positive solutions of the equation $2f_2(x) - 1 = 0$ is (are)
   a. 4 **b.** 3
   c. 2 **d.** 1
2. Number of integer solutions of the equation $f_1(x) = f_2(x)$ is (are)
   a. 1 **b.** 2
   c. 3 **d.** 4
3. If $f_4(x) = \log_{27}(f_3(x) + 2)$, then range of $f_4(x)$ is:
   a. [1, 9] **b.** $\left[\frac{1}{3}, \infty\right)$
   c. $\left[0, \frac{1}{3}\right]$ **d.** [1, 27]

Chapter 1 Matrix Match Type Questions

1. Consider the equation in real number x; $|x-1|-|x-2|+|x-4|=\lambda$

Column-A	Column-B
(a) For any $\lambda \in R$, the maximum possible number of solutions is	(p) 4
(b) For the equation to have exactly two solutions, the parameter λ can take the value(s)	(q) 3
(c) For the equation to have exactly three solution, the parameter λ can take the value(s)	(r) 5
(d) For $\lambda = 1+\sqrt{2}$, the number of solution that the equation has is	(s) 2

2.

Column-A	Column-B
(a) The function $f(x)=\dfrac{3x-2}{x+4}$ has an inverse that can be written in the form $f^{-1}(x)=\dfrac{x+b}{cx+d}$. The value of $(b+c+d)$ is	(p) an even number
(b) Let $f(x)$ $\dfrac{x}{1+x}$ and let $g(x)=\dfrac{rx}{1-x}$. Let S be the set of all real numbers r such that $f(g(x))=g(f(x))$, then S contains	(q) an odd number
(c) If $f(x)=2x+1$ then the value of x satisfying the equation $f(x)+f(f(x))+f(f(f(x)))+f(f(f(f(x))))=116$, is	(r) a prime number
	(s) neither prime nor composite number

3.

Column-A	Column-B
(a) Let $f:\left[\dfrac{5}{2},3\right]\to R$ defined by $f(x)=|x-2|+|3x-2|+|x-2|$, then $f(x)$ is	(p) One-one
(b) Let $g: R\to R$ defined by $g(x)=\sin\pi[\pi^x+e^x+1]$ where $[k]$ denotes greatest integer function less than or equal to k, then $g(x)$ is	(q) Many-one
(c) Let $h: R\to\{1\}$ defined by $h(x)=\operatorname{sgn}(\pi^x+\pi^{-x}+2)$, then $h(x)$ is [where sgn k denotes signum function k]	(r) Into
(d) Let $f: [-1, 1]\to R$ be defined by $f(x)=\sqrt[5]{x}+\sin^{-1}x$, then $f(x)$ is	(s) Onto

(*Continued*)

4. If $x, y, z \in R$ satisfies the system of equations $x+[y]+\{z\}=12.7$, $[x]+\{y\}+z=4.1$ and $\{x\}+y+[z]=2$ (where $\{\cdot\}$ and $[\cdot]$ denote the fractional and integral parts, respectively), then match the following:

Column-A	Column-B
(a) $\{x\}+\{y\}=$	(p) 7.7
(b) $[z]+[x]=$	(q) 1.1
(c) $x+\{z\}=$	(r) 1
(d) $z+[y]-\{x\}=$	(s) 3

5. Match the range of functions given in Column-A to intervals given in Column-B

Column-A	Column-B
(a) $\dfrac{2\sin^2 x+2\sin x+3}{\sin^2 x+\sin x+1}$	(p) $\left[\dfrac{7}{3},\dfrac{10}{3}\right]$
(b) $\dfrac{1}{9}(\sqrt{8-x}+2\sqrt{x-2})^2$	(q) $\left[0,\dfrac{3}{2}\right]$
(c) $\dfrac{x^3}{3}-\dfrac{3x^2}{2}+2x$ where $x\in[0,3]$	(r) $\left[\dfrac{2}{3},\dfrac{10}{3}\right]$
(d) $\dfrac{3-x}{3-\sqrt{x+6}}$	(s) $[3,6]\cup(6,\infty)$

Chapter 1 Integer Type Questions

1. A function $f(x)$ is a polynomial of degree 5 with leading coefficient 1 such that $f(1) = 4, f(2) = 9, f(3) = 16, f(4) = 25, f(5) = 36$, then the value of $(\sqrt{f(6)} - 5)$ is equal to
2. Suppose a quadratic function $f(x) = ax^2 + bx + c$ ($a, b, c \in R$ and $a \neq 0$) satisfies the following conditions:
   a. When $x \in R, f(x-4) = f(2-x)$ and $f(x) \geq x$
   b. When $x \in (0, 2), f(x) \leq \left(\frac{x+1}{2}\right)^2$
   c. The minimum value of $f(x)$ on R is 0.
   Then the maximum value of m ($m > 1$) such that there exists $t \in R$ $f(x+t) \leq x \; \forall \; x \in R \; [1, m]$
3. Given p and q are relatively prime positive integers. Prove that
$$\left[\frac{p}{q}\right]+\left[\frac{2p}{q}\right]+.....+\left[\frac{(q-1)p}{q}\right]=\left[\frac{q}{p}\right]+\left[\frac{2q}{p}\right]+\ldots+\left[\frac{(p-1)q}{p}\right]$$
   (where $[\cdot]$ denotes greatest integer function.)
4. Let the solution set of the equation $\sqrt{\left[x+\left[\frac{x}{2}\right]\right]}+\left[\sqrt{\{x\}}+\left[\frac{x}{3}\right]\right]=3$ is $[a, b)$. Find the product ab. (where $[\cdot]$ and $\{\cdot\}$ denote greatest integer and fractional part function, respectively).
5. Find the sum of all the solutions of the equation $\cot \frac{\pi x}{2} = \log_2\{x\}$ in $x \in (0, 100)$. [**Note:** $\{k\}$ denotes the fractional part function of k.]
6. Let f be a function satisfying the functional equation $f(x) + 2f\left(\frac{2x+1}{x-2}\right) = 3x, x \neq 2$ then find the value of $\left|\frac{f(3)}{f(7)}\right|$.
7. Let $f(x) = \begin{cases} 2x^2 - 10x, & -\infty < x \leq -5 \\ x^2 - 5 & -5 < x < 3 \\ x^2 + 1, & 3 \leq x < \infty \end{cases}$

   Number of negative integers in the range of the function $f(x)$ is
8. Let $f(x)$ be an odd function defined on R such that $f(x) = x^2$ when $x \geq 9$. If $f(x+a) \geq 2f(x)$ holds$\forall x \in [a, a+2]$ then the least integer value of a is
9. If $f: N \to R$ is such that $f(1) = 1$ and $f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n+1)\,f(n)$ ($n \geq 2$), then the value of $\frac{1}{f(4)}$ is
10. Let $f(x)$ be a polynomial of degree 2005 with leading coefficient 2006 such that $f(n) = n$ for $n = 1, 2, 3, \ldots, 2005$. The number of zeros at the ends of $f(2006) - 2006$ is

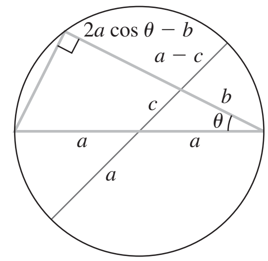
Chapter 1 Additional and Advanced Exercises

Functions and Graphs

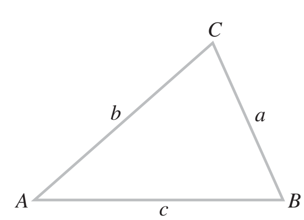
1. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.
2. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.
3. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.
4. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.
5. Graph the equation $|x| + |y| = 1 + x$.
6. Graph the equation $y + |y| = x + |x|$.

Derivations and Proofs

7. Prove the following identities.
   a. $\dfrac{1-\cos x}{\sin x} = \dfrac{\sin x}{1+\cos x}$
   b. $\dfrac{1-\cos x}{1+\cos x} = \tan^2 \dfrac{x}{2}$
8. Explain the following "proof without words" of the law of cosines. (*Source:* Kung, Sidney H., "Proof Without Words: The Law of Cosines," *Mathematics Magazine*, Vol. 63, no. 5, Dec. 1990, p. 342.)



9. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



10. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.
11. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

12. a. Even-odd decompositions Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

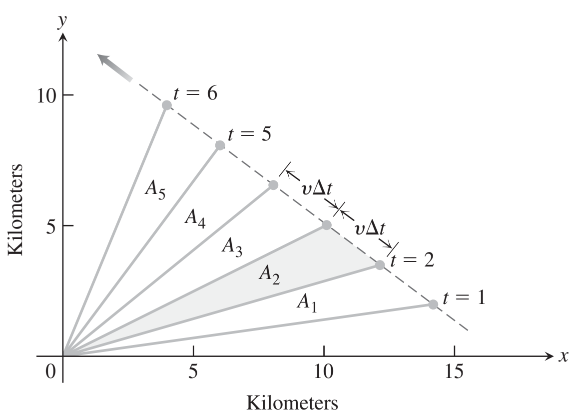
$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

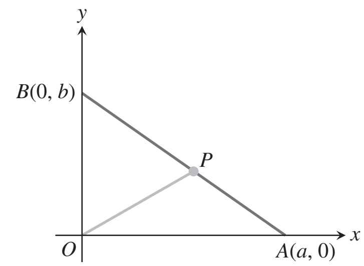
b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

Geometry

13. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



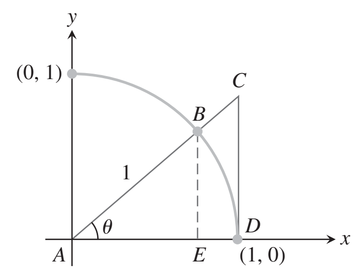
14. a. Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure ($a, b > 0$).



b. When is OP perpendicular to AB?

15. Consider the quarter-circle of radius 1 and right triangles ABE and ACD given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2}\sin\theta\cos\theta < \frac{\theta}{2} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}.$$



16. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x?

2

Limits and Continuity

OVERVIEW Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars. This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction at a given instant was along a line tangent to the path of motion. The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

2.1 Rates of Change and Tangents to Curves

Calculus is a tool that helps us understand how a change in one quantity is related to a change in another. How does the speed of a falling object change as a function of time? How does the level of water in a barrel change as a function of the amount of liquid poured into it? We see change occurring in nearly everything we observe in the world and universe, and powerful modern instruments help us see more and more. In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve. We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of this chapter, the *limit*. The idea of a limit plays a foundational role throughout calculus.

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY*
Galileo Galilei
(1564–1642)

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving object's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

*To learn more about the historical figures mentioned in the text and the development of many major elements and topics of calculus, visit **www.aw.com/thomas**.

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (Increments like Δy and Δt are pronounced "delta y" and "delta t.") Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$ ■

We want a way to determine the speed of a falling object at a single instant t_0, instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0. The next example illustrates this process. Our discussion is informal here, but it will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16{t_0}^2}{h}. \tag{1}$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, by taking smaller and smaller values of h, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0 + h)^2 - 16{t_0}^2}{h}$

Length of time interval h	**Average speed over interval of length h starting at $t_0 = 1$**	**Average speed over interval of length h starting at $t_0 = 2$**
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1+2h+h^2) - 16}{h}$$
$$= \frac{32h + 16h^2}{h} = 32 + 16h.$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $32 + 16h$ ft/sec. We can now see why the average speed has the limiting value $32 + 16(0) = 32$ ft/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h$$

for values of h different from 0. As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea which we discuss next.

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

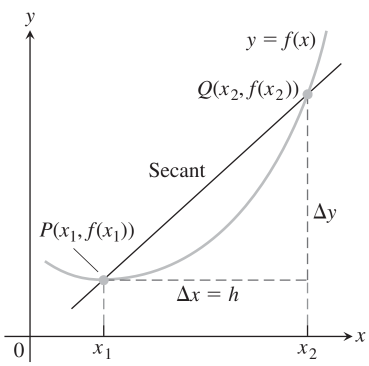


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ. Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to defining the slope of a curve at a point.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent* line to the curve at P—a line that just touches the curve like the tangent to a circle—it would be reasonable to identify *the slope of the tangent* as the slope of the curve at P. So we need a precise meaning for the tangent at a point on a curve.

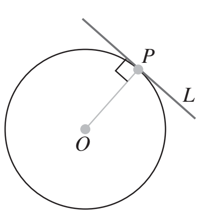


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Figure 2.2). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P?

To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). Here is the idea:

1. Start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limiting value of the secant slope as Q approaches P along the curve. (We clarify the *limit* idea in the next section.)
3. If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve at P to be the line through P with this slope.

HISTORICAL BIOGRAPHY
Pierre de Fermat
(1601–1665)

This procedure is what we were doing in the falling-rock problem discussed in Example 2. The next example illustrates the geometric idea for the tangent to a curve.

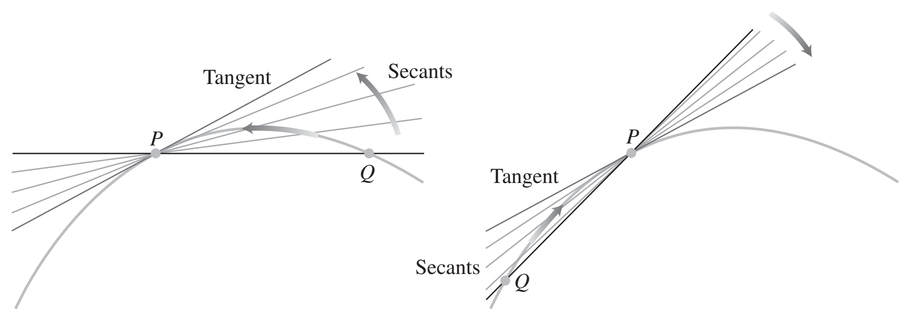


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

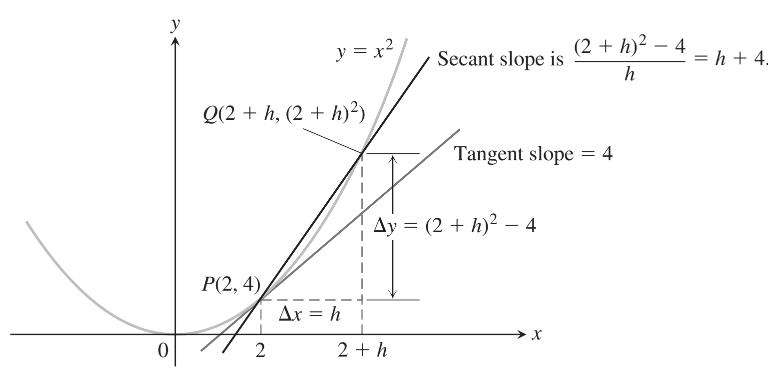


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$
$$y = 4x - 4.$$ ■

Instantaneous Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates and slopes of tangent lines are closely connected, as we see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time t, and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

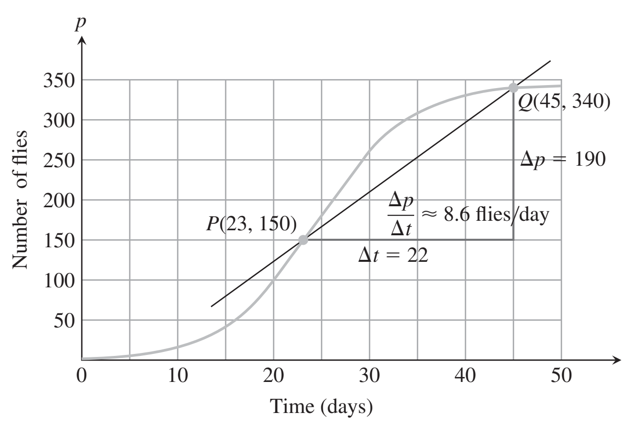


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant through the points P and Q on the graph in Figure 2.5. ■

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

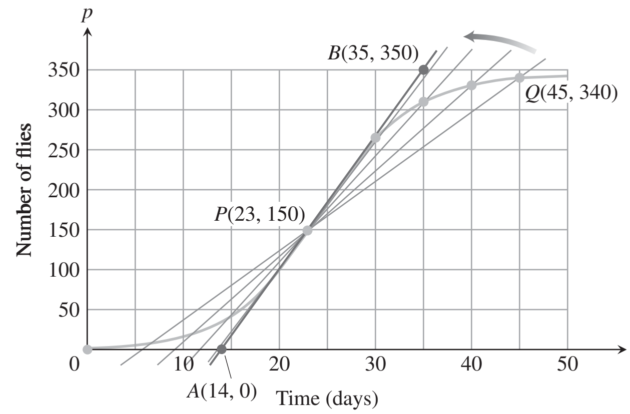


FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate counterclockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day (approximately).}$$

On day 23 the population was increasing at a rate of about 16.7 flies/day. ■

The instantaneous rates in Example 2 were found to be the values of the average speeds, or average rates of change, as the time interval of length h approached 0. That is, the instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line as the independent variable approaches a fixed value. In Example 2, the independent variable t approached the values $t = 1$ and $t = 2$. In Example 3, the independent variable x approached the value $x = 2$. So we see that instantaneous rates and slopes of tangent lines are closely connected. We investigate this connection thoroughly in the next chapter, but to do so we need the concept of a *limit*.

Exercises 2.1

Average Rates of Change

In Exercises 1, find the average rate of change of the function over the given interval or intervals.

1. $g(t) = 2 + \cos t$

a. $[0, \pi]$ **b.** $[-\pi, \pi]$

Slope of a Curve at a Point

In Exercises 2–3, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

2. $y = x^2 - 5, \quad P(2, -1)$

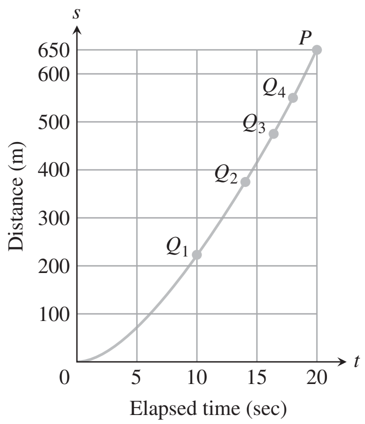
3. $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

Instantaneous Rates of Change

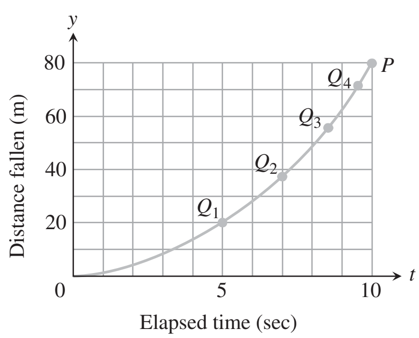
4. Speed of a car The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.

a. Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

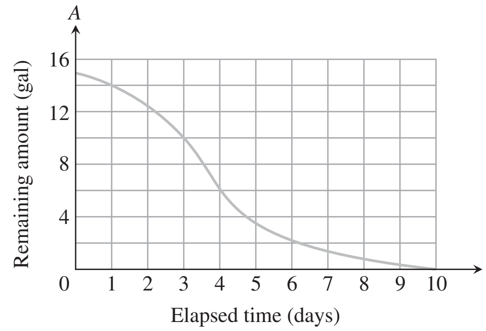
b. Then estimate the car's speed at time $t = 20$ sec.



5. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.
   a. Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.
   b. About how fast was the object going when it hit the surface?



6. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



   a. Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.
   b. Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.
   c. Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw that limits arise when finding the instantaneous rate of change of a function or the tangent to a curve. Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition is presented in the next section.

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c, but not *at* c. This might be the case, for instance, if c is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero.

Similarly, we have 7 indeterminate forms in which we should be careful while evaluating limits. If an expression acquires an indeterminate form while $x \to c$, then we have to first remove the indeterminate form using any of algebraic operations like rationalization, double rationalization, use of algebraic identities, or use of standard limits. Following are seven indeterminate forms.

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \cdot \infty$ (where the first expression is not exact zero but approaching zero)
4. $\infty - \infty$
5. 1^{∞} (where base approaches 1)
6. 0° (where base and exponents are approaching 0)
7. ∞° (where the exponent approaches 0)

Here's a specific example in which we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

HISTORICAL ESSAY
Limits

EXAMPLE 1 How does the function

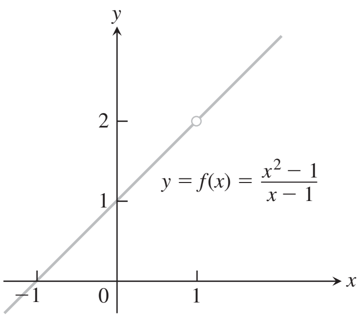
$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point (1, 2) *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing *x close enough* to 1 (Table 2.2). ■



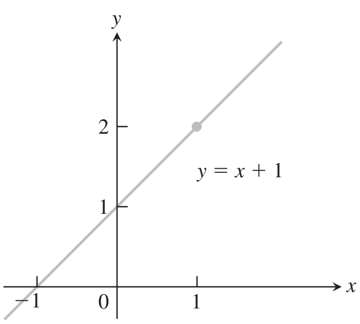


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Generalizing the idea illustrated in Example 1, suppose $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

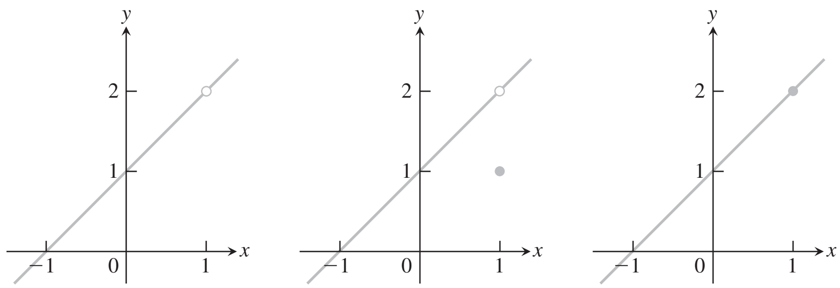
which is read "the limit of $f(x)$ as x approaches c is L." For instance, in Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c (on either side of c).

Our definition here is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions. We will need the precise definition given in Section 2.3, however, when we set out to prove theorems about limits or study complicated functions. Here are several more examples exploring the idea of limits.

EXAMPLE 2 The limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x\to 1} h(x) = h(1)$. This equality of limit and function value is of special importance, and we return to it in Section 2.5. ■



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$

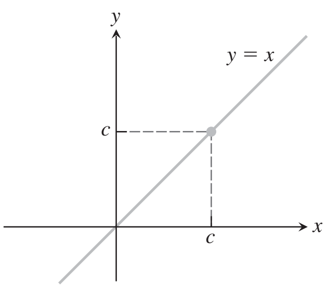
(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$

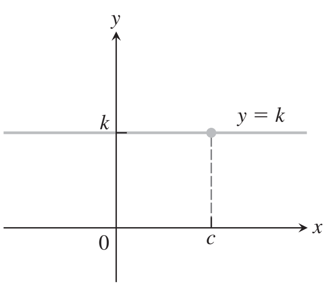
For instances of each of these rules we have

$$\lim_{x\to 3} x = 3 \qquad \text{and} \qquad \lim_{x\to -7} (4) = \lim_{x\to 2} (4) = 4.$$

We prove these rules in Example 3 in Section 2.3. ■



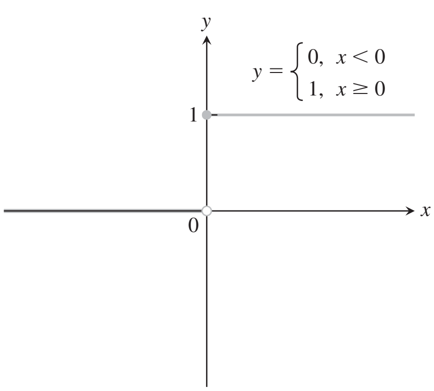
(a) Identity function



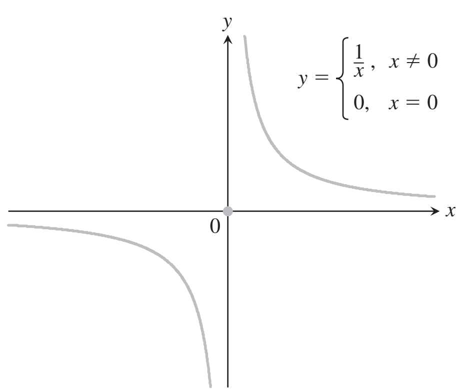
(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

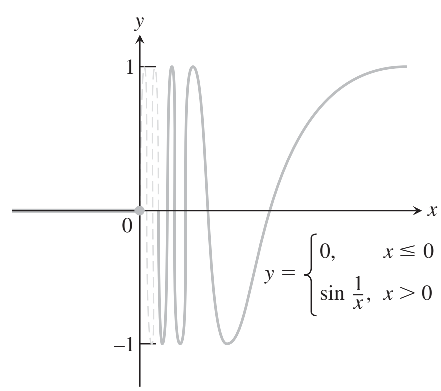
A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

Solution

(a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Figure 2.10c). ■

The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

In words, the Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

It is reasonable that the properties in Theorem 1 are true (although these intuitive arguments do not constitute proofs). If x is sufficiently close to c, then $f(x)$ is close to L and $g(x)$ is close to M, from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL; $f(x)g(x)$ is close to LM; and $f(x)/g(x)$ is close to L/M if M is not zero. The Sum, Difference, and Product Rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the fundamental rules of limits to find the following limits.

(a) $\displaystyle\lim_{x\to c}(x^3 + 4x^2 - 3)$

(b) $\displaystyle\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\displaystyle\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x\to -2}\sqrt{4x^2 - 3} &= \sqrt{\lim_{x\to -2}(4x^2 - 3)} && \text{Root Rule with } n = 2\\ &= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} && \text{Difference Rule}\\ &= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules}\\ &= \sqrt{16 - 3}\\ &= \sqrt{13}\end{aligned}$$

■

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

> **THEOREM 2—Limits of Polynomials**
> If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then
> $$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

> **THEOREM 3—Limits of Rational Functions**
> If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then
> $$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$ ■

Identifying Common Factors
It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by Theorem 3:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■

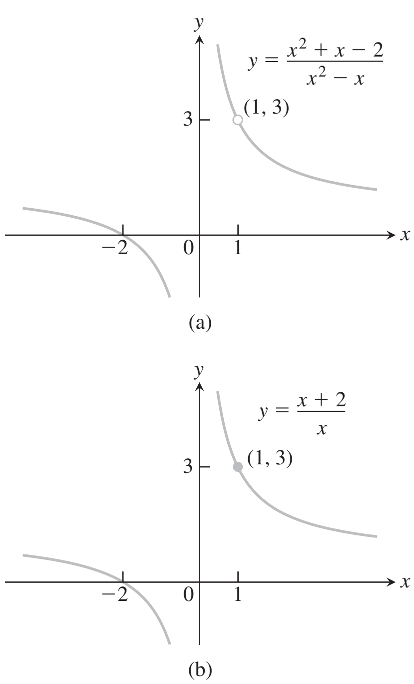


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Figure 2.12).

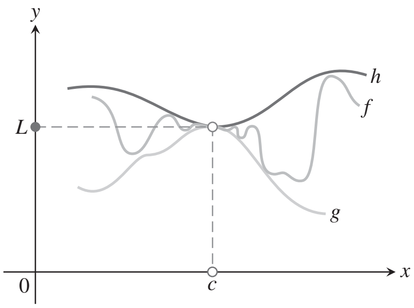


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

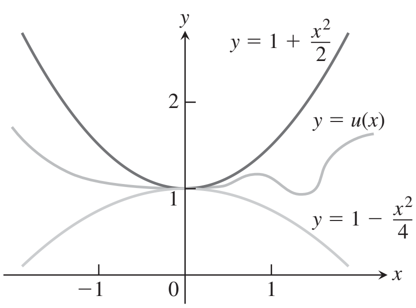


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 8).

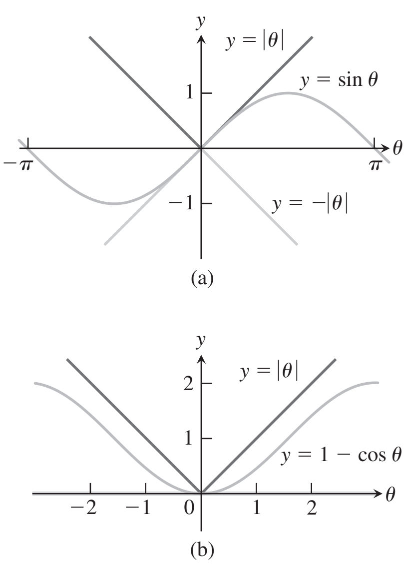


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 9.

> **THEOREM 4—The Sandwich Theorem** Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that
>
> $$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$
>
> Then $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 8 Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \to 0}\left(1 - (x^2/4)\right) = 1 \qquad \text{and} \qquad \lim_{x \to 0}\left(1 + (x^2/2)\right) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.13). ■

EXAMPLE 9 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim\limits_{\theta \to 0} \sin\theta = 0$ **(b)** $\lim\limits_{\theta \to 0} \cos\theta = 1$

(c) For any function f, $\lim\limits_{x \to c} |f(x)| = 0$ implies $\lim\limits_{x \to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin\theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta \to 0}(1 - \cos\theta) = 0$ or

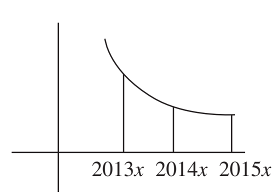
$$\lim_{\theta \to 0} \cos\theta = 1.$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$. ■

EXAMPLE 10 Let f: $(1, \infty) \to (0, \infty)$ be a continuous decreasing function such that $\lim\limits_{x \to \infty} \dfrac{f(2013x)}{f(2015x)} = 1$, then find the value of $\lim\limits_{x \to \infty} \dfrac{f(2014x)}{f(2015x)}$.

Solution Above example is based on sandwich theorem.

If $f(x)$ is a decreasing function $\forall\, x \in (1, \infty)$, then graph of $f(x)$ looks like



Clearly the order of function values is $f(2015x) < f(2014x) < f(2013x)$

$$\therefore \qquad 1 < \frac{f(2014x)}{f(2015x)} < \frac{f(2013x)}{f(2015x)}$$

Clearly the function $\dfrac{f(2014x)}{f(2015x)}$ is sandwiched

between 1 and $\dfrac{f(2013x)}{f(2015x)}$ and it is given that $\underset{x\to\infty}{\text{lt}} \dfrac{f(2013x)}{f(2015x)} = 1$;

Hence, $$\lim_{x\to\infty} \frac{f(2014x)}{f(2015x)} = 1.$$ ■

EXAMPLE 11 Evaluate: $\displaystyle \lim_{n\to\infty}\left(\frac{1^3}{1+n^4}+\frac{2^3}{2+n^4}+\frac{3^3}{3+n^4}+\ldots+\frac{n^3}{n+n^4}\right)$

Solution Here also, we can use the concept of sandwich theorem.

$$\frac{1^3}{n+n^4} < \frac{1^3}{1+n^4} = \frac{1^3}{1+n^4}$$

$$\frac{2^3}{n+n^4} < \frac{2^3}{2+n^4} = \frac{2^3}{1+n^4}$$

$$\vdots$$

$$\frac{n^3}{n+n^4} = \frac{n^3}{n+n^4} < \frac{n^3}{1+n^4}$$

Adding all the inequalities, we get

$$\frac{1^3+2^3+\ldots+n^3}{n+n^4} < \frac{1^3}{1+n^4}+\frac{2^3}{2+n^4}+\frac{3^3}{3+n^4}+\cdots+\frac{n^3}{n+n^4} < \frac{1^3+2^3+\ldots+n^3}{1+n^4}$$

$$\Rightarrow \qquad \frac{n^2(n+1)^2}{4(n+n^4)} < \sum_{r=1}^{n}\frac{r^3}{r+n^4} < \frac{n^2(n+1)^2}{4(1+n^4)}$$

Clearly $$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^3}{r+n^4} = \frac{1}{4}$$

As $$\lim_{n\to\infty}\frac{n^2(n+1)^2}{4(n+n^4)} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{4\left(\frac{1}{n^3}+1\right)} = \frac{1}{4}$$

Also $$= \lim_{n\to\infty}\frac{n^2(n+1)}{4(1+n^4)} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{4\left(\frac{1}{n^4}+1\right)} = \frac{1}{4}$$

$$\therefore \qquad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^3}{r+n^4} = \frac{1}{4}$$ ■

EXAMPLE 12 Find the value of $\lim\limits_{x\to 0}\dfrac{x}{c}\left[\dfrac{c}{x}\right]$ (where $c \neq 0$ and $[\cdot]$ denotes greatest integer function).

Solution This problem can be solved using sandwich theorem by generating two functions. We have already come across the inequality $x - 1 < [x] \leq x$ (where $[\cdot]$ denotes greatest integer function)

Now, $$\frac{c}{x} - 1 < \left[\frac{c}{x}\right] \leq \frac{c}{x}$$

$\Rightarrow$ $$\frac{x}{c}\left(\frac{c}{x}-1\right) < \frac{x}{c}\left[\frac{c}{x}\right] \leq \frac{x}{c}\frac{c}{x}\cdot$$

Clearly, $$\lim_{x\to 0}\left(\frac{x}{c}\left(\frac{c}{x}-1\right)\right) = \frac{c}{c} = 1$$

and $$\lim_{x\to 0}\frac{x}{c}\cdot\frac{c}{x} = 1$$

Hence, $$\operatorname*{lt}_{x\to 0}\frac{x}{c}\left[\frac{c}{x}\right] = 1$$

Aliter: The above problem can be tackled as follows.

$\therefore$ $[x] + \{x\} = x$, (where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ denotes fractional part function).

Now, $$[x] = x - \{x\}$$

$\therefore$ $$\frac{x}{c}\left[\frac{c}{x}\right] = \frac{x}{c}\left(\frac{c}{x} - \left\{\frac{c}{x}\right\}\right) = 1 - \frac{x}{c}\left\{\frac{c}{x}\right\}$$

Now, $$\lim_{x\to 0}\frac{x}{c}\left[\frac{c}{x}\right] = \operatorname*{lt}_{x\to 0}\left(1 - \frac{x}{c}\left\{\frac{c}{x}\right\}\right)$$

Also, $\left\{\dfrac{c}{x}\right\}$ is a bounded function. Hence, when $x \to 0$, we can say $\dfrac{c}{x}$ is very large or very small but $\left\{\dfrac{c}{x}\right\}$ will have a finite value in the interval $[0, 1)$. Therefore, $\lim\limits_{x\to 0}\dfrac{x}{c}\left\{\dfrac{c}{x}\right\}$ vanishes.

$\therefore$ $$\operatorname*{lt}_{x\to 0}\frac{x}{c}\left[\frac{c}{x}\right] = 1$$ ■

Another important property of limits is given by the next theorem. A proof is given in the next section.

> **THEOREM 5** If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then
> $$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

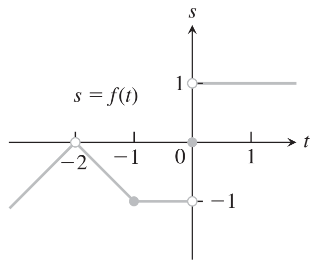
Caution The assertion resulting from replacing the less than or equal to ($\leq$) inequality by the strict less than ($<$) inequality in Theorem 5 is false. Figure 2.14a shows that for $\theta \neq 0$, $-|\theta| < \sin\theta < |\theta|$. So $\lim_{\theta\to 0}\sin\theta = 0 = \lim_{\theta\to 0}|\theta|$, not $\lim_{\theta\to 0}\sin\theta < \lim_{\theta\to 0}|\theta|$.

Exercises 2.2

Limits from Graphs

1. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t\to -2} f(t)$ **b.** $\lim_{t\to -1} f(t)$ **c.** $\lim_{t\to 0} f(t)$ **d.** $\lim_{t\to -0.5} f(t)$



2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 0} f(x)$ exists.

   b. $\lim_{x\to 0} f(x) = 0$

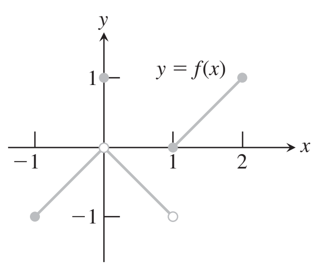
   c. $\lim_{x\to 0} f(x) = 1$

   d. $\lim_{x\to 1} f(x) = 1$

   e. $\lim_{x\to 1} f(x) = 0$

   f. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.

   g. $\lim_{x\to 1} f(x)$ does not exist.



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

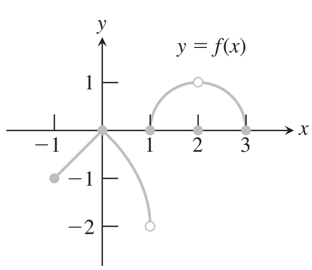
   a. $\lim_{x\to 2} f(x)$ does not exist.

   b. $\lim_{x\to 2} f(x) = 2$

   c. $\lim_{x\to 1} f(x)$ does not exist.

   d. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.

   e. $\lim_{x\to c} f(x)$ exists at every point c in $(1, 3)$.



Existence of Limits

In Exercises 4 and 5, explain why the limits do not exist.

4. $\lim_{x\to 0} \dfrac{x}{|x|}$ **5.** $\lim_{x\to 1} \dfrac{1}{x-1}$

6. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

7. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

8. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

9. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Limits of quotients Find the limits in Exercises 10–16.

10. $\lim_{x\to 4} \dfrac{4x - x^2}{2 - \sqrt{x}}$ **11.** $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3} - 2}$

12. $\lim_{x\to -1} \dfrac{\sqrt{x^2+8} - 3}{x+1}$ **13.** $\lim_{x\to 2} \dfrac{\sqrt{x^2+12} - 4}{x-2}$

14. $\lim_{x\to -2} \dfrac{x+2}{\sqrt{x^2+5} - 3}$ **15.** $\lim_{x\to -3} \dfrac{2 - \sqrt{x^2-5}}{x+3}$

16. $\lim_{x\to 4} \dfrac{4-x}{5 - \sqrt{x^2+9}}$

Limits with trigonometric functions Find the limits in Exercises 17–24.

17. $\lim_{x\to 0} (2\sin x - 1)$ **18.** $\lim_{x\to \pi/4} \sin^2 x$

19. $\lim_{x\to 0} \sec x$ **20.** $\lim_{x\to \pi/3} \tan x$

21. $\lim_{x\to 0} \dfrac{1 + x + \sin x}{3\cos x}$ **22.** $\lim_{x\to 0} (x^2 - 1)(2 - \cos x)$

23. $\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$ **24.** $\lim_{x\to 0} \sqrt{7 + \sec^2 x}$

Using Limit Rules

25. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x) - g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x) - g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1) - (-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

26. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4 - r(x)))} \qquad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4 - r(x))\right)} \qquad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \qquad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2}$$

27. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim_{x\to c} f(x)g(x)$ **b.** $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c}(f(x) + 3g(x))$ **d.** $\lim_{x\to c} \dfrac{f(x)}{f(x) - g(x)}$

28. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

a. $\lim_{x\to 4}(g(x) + 3)$ **b.** $\lim_{x\to 4} xf(x)$

c. $\lim_{x\to 4}(g(x))^2$ **d.** $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

29. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim_{x\to b}(f(x) + g(x))$ **b.** $\lim_{x\to b} f(x)\cdot g(x)$

c. $\lim_{x\to b} 4g(x)$ **d.** $\lim_{x\to b} f(x)/g(x)$

30. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

a. $\lim_{x\to -2}(p(x) + r(x) + s(x))$

b. $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to -2}(-4p(x) + 5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 31–36, evaluate this limit for the given value of x and function f.

31. $f(x) = x^2, \quad x = 1$

32. $f(x) = x^2, \quad x = -2$

33. $f(x) = 3x - 4, \quad x = 2$

34. $f(x) = 1/x, \quad x = -2$

35. $f(x) = \sqrt{x}, \quad x = 7$

36. $f(x) = \sqrt{3x + 1}, \quad x = 0$

Using the Sandwich Theorem

37. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

38. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

39. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x}?$$

Give reasons for your answer.

40. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

Theory and Examples

41. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

42. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

43. If $\lim_{x\to 4} \dfrac{f(x) - 5}{x - 2} = 1$, find $\lim_{x\to 4} f(x)$.

44. If $\lim_{x\to -2} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to -2} f(x)$ **b.** $\lim_{x\to -2} \dfrac{f(x)}{x}$

45. a. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 3$, find $\lim_{x\to 2} f(x)$.

b. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 4$, find $\lim_{x\to 2} f(x)$.

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c. Let us see what this would require if we specified the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x \to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

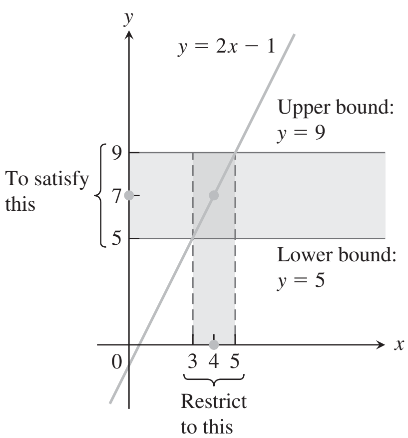


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 && \text{Solve for } x. \\ -1 < x - 4 &< 1. && \text{Solve for } x - 4. \end{aligned}$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.15). ■

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L. To show that the limit of $f(x)$ as $x \to c$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how small, by holding x close enough to c.

Definition of Limit

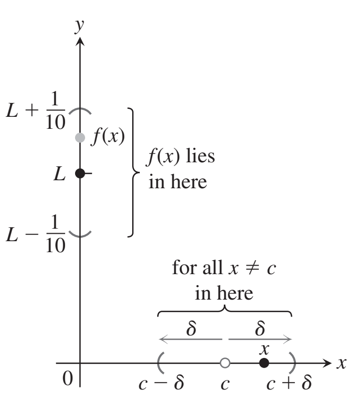


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value of c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of c (Figure 2.16). But that in itself is not enough, because as x continues on its course toward c, what is to prevent $f(x)$ from jittering about within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L?

We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100{,}000$. Each time, we find a new δ-interval about c so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ jitters away from L at some stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ϵ-challenges to prove that the limit does not exist or, more precisely, that there is room for doubt. The scholar answers every challenge with a δ-interval around c that keeps the function values within ϵ of L.

How do we stop this seemingly endless series of challenges and responses? We can do so by proving that for every error tolerance ϵ that the challenger can produce, we can present a matching distance δ that keeps x "close enough" to c to keep $f(x)$ within that ϵ-tolerance of L (Figure 2.17). This leads us to the precise definition of a limit.

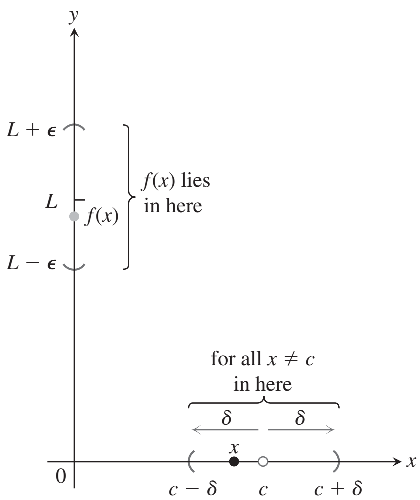


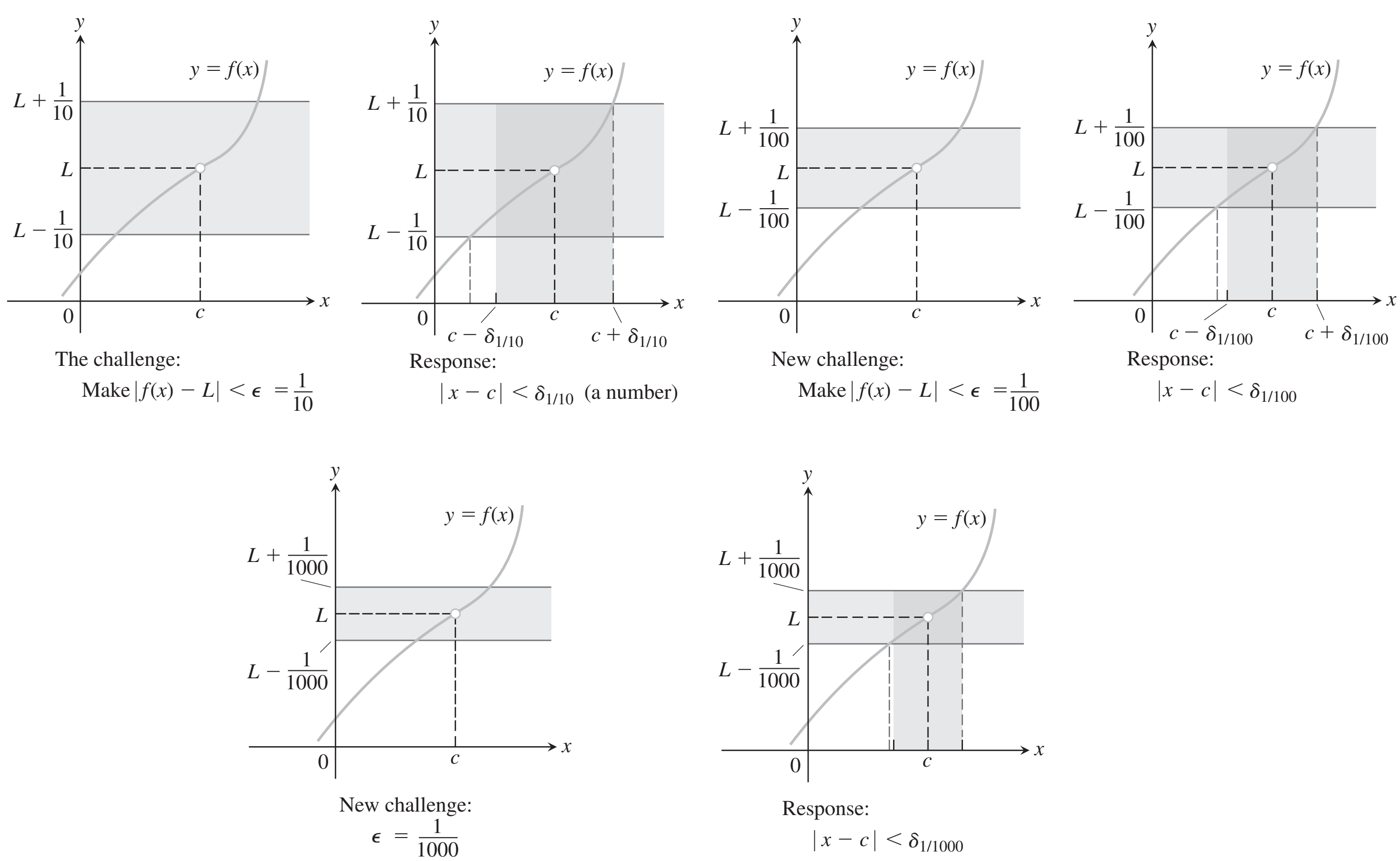
FIGURE 2.17 The relation of δ and ϵ in the definition of limit.

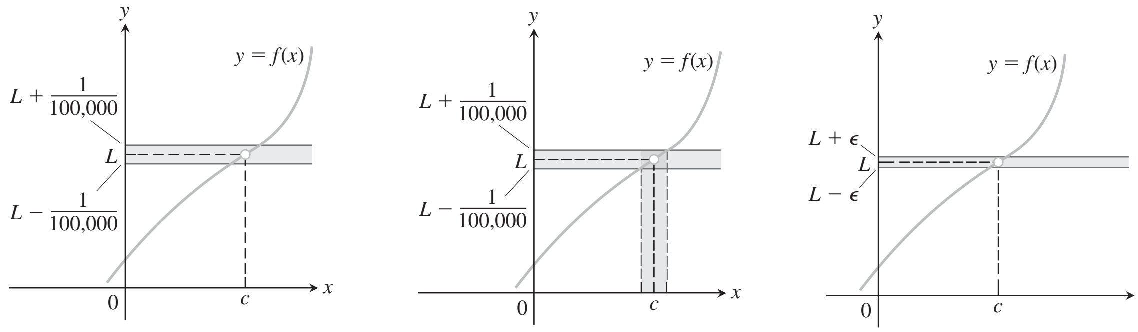
> **DEFINITION** Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write
>
> $$\lim_{x \to c} f(x) = L,$$
>
> if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,
>
> $$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L, but since nothing is perfect, we must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust δ. That is, the value of δ, how tight our control setting must be, depends on the value of ϵ, the error tolerance.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified, such as the theorems stated in the previous section.





New challenge: $\epsilon = \dfrac{1}{100{,}000}$

Response: $|x - c| < \delta_{1/100{,}000}$

New challenge: $\epsilon = \cdots$

EXAMPLE 2 Show that

$$\lim_{x\to 1}(5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5\,|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x\to 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ■

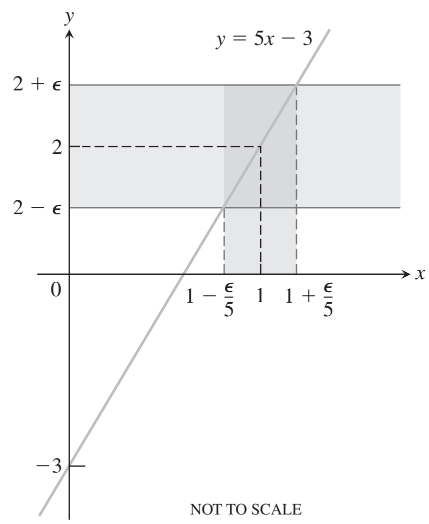


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

EXAMPLE 3 Prove the following results presented graphically in Section 2.2.

(a) $\lim_{x\to c} x = c$

(b) $\lim_{x\to c} k = k$ (k constant)

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \qquad \text{implies} \qquad |x - c| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Figure 2.19). This proves that $\lim_{x\to c} x = c$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.20). This proves that $\lim_{x\to c} k = k$. ■

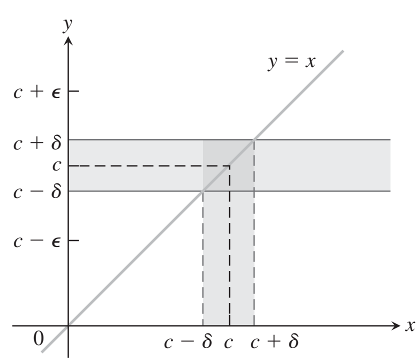


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

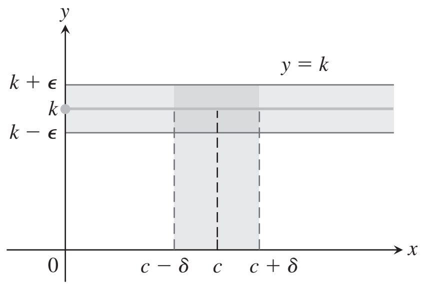


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

FIGURE 2.21 An open interval of radius 3 about $x = 5$ will lie inside the open interval (2, 10).

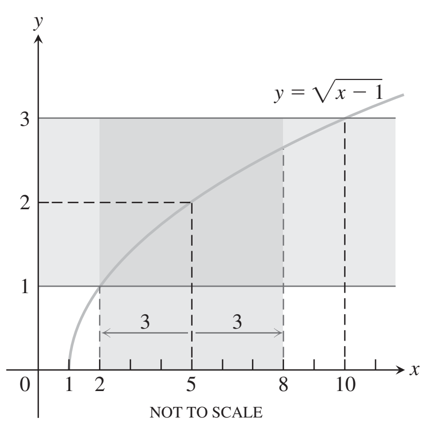


FIGURE 2.22 The function and intervals in Example 4.

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about c for which $|f(x) - L|$ was less than ϵ was symmetric about c and we could take δ to be half the length of that interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from c to the interval's *nearer* endpoint.

EXAMPLE 4 For the limit $\lim_{x\to 5}\sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \qquad \Rightarrow \qquad |\sqrt{x-1} - 2| < 1.$$

Solution We organize the search into two steps.

1. *Solve the inequality* $|\sqrt{x-1} - 2| < 1$ *to find an interval containing* $x = 5$ *on which the inequality holds for all* $x \neq 5$.

$$\begin{gathered} |\sqrt{x-1} - 2| < 1 \\ -1 < \sqrt{x-1} - 2 < 1 \\ 1 < \sqrt{x-1} < 3 \\ 1 < x - 1 < 9 \\ 2 < x < 10 \end{gathered}$$

   The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1} - 2| < 1$ (Figure 2.22):

$$0 < |x - 5| < 3 \qquad \Rightarrow \qquad |\sqrt{x-1} - 2| < 1. \qquad \blacksquare$$

> **How to Find Algebraically a δ for a Given f, L, c, and $\epsilon > 0$**
>
> The process of finding a $\delta > 0$ such that for all x
>
> $$0 < |x - c| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$
>
> can be accomplished in two steps.
>
> 1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$.
> 2. *Find a value of* $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq c$ in this δ-interval.

EXAMPLE 5 Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \qquad \Rightarrow \qquad |f(x) - 4| < \epsilon.$$

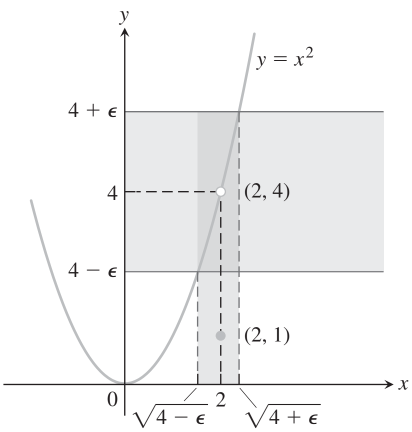


FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

1. *Solve the inequality* $|f(x) - 4| < \epsilon$ *to find an open interval containing* $x = 2$ *on which the inequality holds for all* $x \neq 2$.

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2 - 4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$
$$4 - \epsilon < x^2 < 4 + \epsilon$$
$$\sqrt{4 - \epsilon} < |x| < \sqrt{4 + \epsilon} \qquad \text{Assumes } \epsilon < 4\text{; see below.}$$
$$\sqrt{4 - \epsilon} < x < \sqrt{4 + \epsilon}. \qquad \text{An open interval about } x = 2 \text{ that solves the inequality}$$

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$ (Figure 2.23).

2. *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$.

Take δ to be the distance from $x = 2$ to the nearer endpoint of $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$. In other words, take $\delta = \min\left\{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\right\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \qquad \Rightarrow \qquad |f(x) - 4| < \epsilon.$$

This completes the proof for $\epsilon < 4$.

If $\epsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $\left(0, \sqrt{4 + \epsilon}\right)$. In other words, take $\delta = \min\left\{2, \sqrt{4 + \epsilon} - 2\right\}$. (See Figure 2.23.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems. As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, prove that

$$\lim_{x \to c} (f(x) + g(x)) = L + M.$$

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that for all x

$$0 < |x - c| < \delta \qquad \Rightarrow \qquad |f(x) + g(x) - (L + M)| < \epsilon.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Triangle Inequality: $|a + b| \leq |a| + |b|$

Since $\lim_{x \to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x \to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \qquad \Rightarrow \qquad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x\to c}(f(x) + g(x)) = L + M$. ■

Next we prove Theorem 5 of Section 2.2.

EXAMPLE 7 Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, and that $f(x) \le g(x)$ for all x in an open interval containing c (except possibly c itself), prove that $L \le M$.

Solution We use the method of proof by contradiction. Suppose, on the contrary, that $L > M$. Then by the limit of a difference property in Theorem 1,

$$\lim_{x\to c}(g(x) - f(x)) = M - L.$$

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < \epsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $L - M > 0$ by hypothesis, we take $\epsilon = L - M$ in particular and we have a number $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < L - M \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $a \le |a|$ for any number a, we have

$$(g(x) - f(x)) - (M - L) < L - M \qquad \text{whenever} \quad 0 < |x - c| < \delta$$

which simplifies to

$$g(x) < f(x) \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

But this contradicts $f(x) \le g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \le M$. ■

2.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

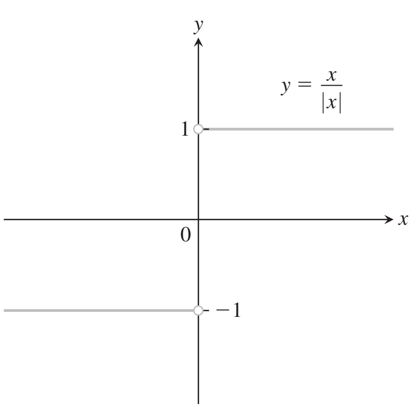


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Approaching a Limit from One Side

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. That is, f must be defined in some open interval about c, but not necessarily at c. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

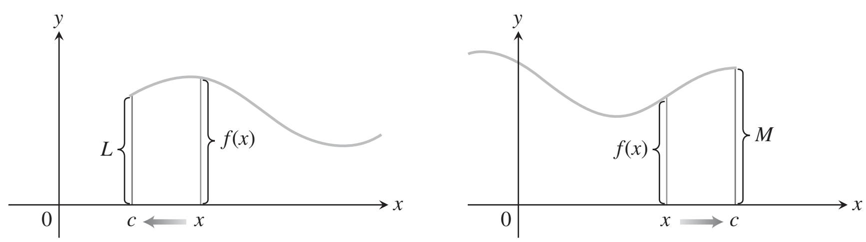
Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x-values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

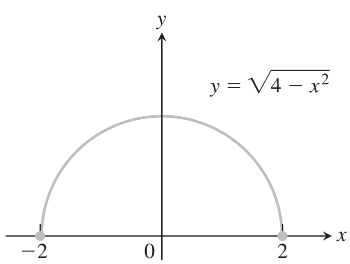


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$ (Example 1).

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because each point does not belong to an open interval over which f is defined. ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

> **THEOREM 6** A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:
>
> $$\lim_{x \to c} f(x) = L \qquad \Leftrightarrow \qquad \lim_{x \to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = L.$$

EXAMPLE 2 For the function graphed in Figure 2.27,

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,
$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

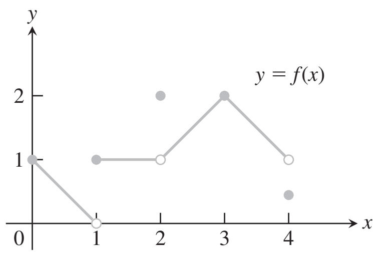


FIGURE 2.27 Graph of the function in Example 2.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,
$\lim_{x \to 1^+} f(x) = 1$,
$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,
$\lim_{x \to 2^+} f(x) = 1$,
$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,
$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

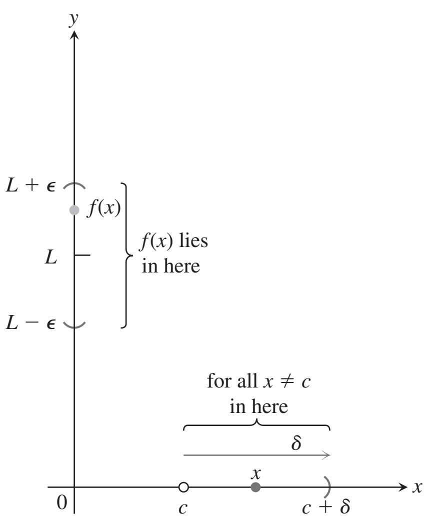


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

DEFINITIONS We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

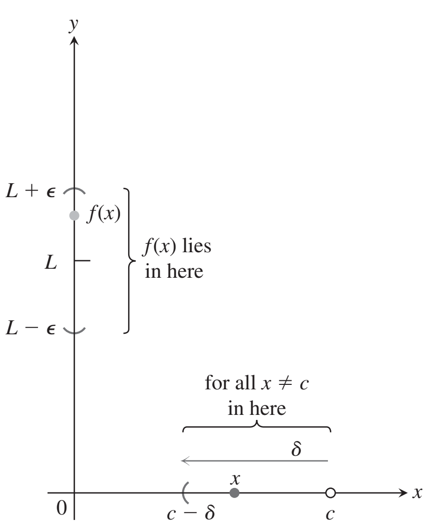


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

EXAMPLE 3 Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \qquad \Rightarrow \qquad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \qquad \Rightarrow \qquad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \qquad \Rightarrow \qquad \sqrt{x} < \epsilon,$$

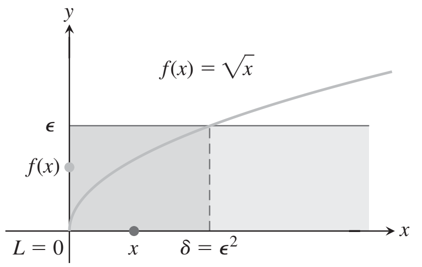


FIGURE 2.30 $\lim_{x\to 0^+} \sqrt{x} = 0$ in Example 3.

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x} - 0\right| < \epsilon.$$

According to the definition, this shows that $\lim_{x\to 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

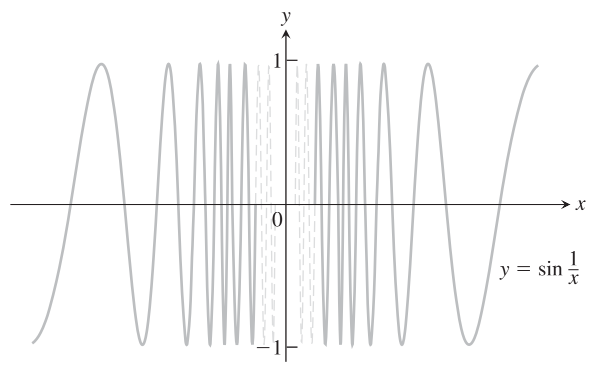


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem.

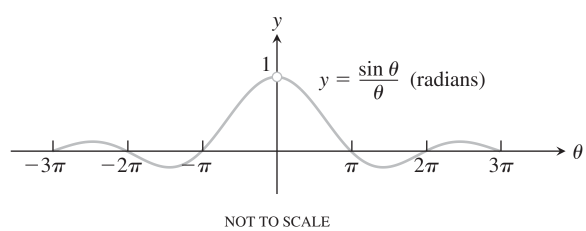


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

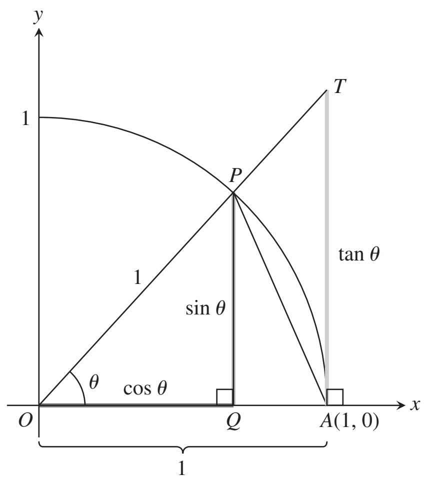


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

Equation (2) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to0^+}\cos\theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to0}(\sin\theta)/\theta = 1$ by Theorem 6. ■

EXAMPLE 5 Show that **(a)** $\displaystyle\lim_{h\to0}\frac{\cos h - 1}{h} = 0$ and **(b)** $\displaystyle\lim_{x\to0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to0}\frac{\cos h - 1}{h} &= \lim_{h\to0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = h/2. \\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\ \sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Now, Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

■

EXAMPLE 6 Find $\lim_{t\to 0}\dfrac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\tan t\sec 2t}{3t} &= \lim_{t\to 0}\frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. && \text{Eq. (1) and Example 11b in Section 2.2}\end{aligned}$$

More trigonometric limits:

We have already seen $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$. Similarly we can say

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1;\ \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1;\ \lim_{\theta\to 0}\frac{\theta}{\tan\theta}=1;\ \lim_{\theta\to 0}\frac{\sin^{-1}\theta}{\theta}=1;$$

$$\lim_{\theta\to 0}\frac{\theta}{\sin^{-1}\theta}=1;\ \lim_{\theta\to 0}\frac{\tan^{-1}\theta}{\theta}=1;\ \lim_{\theta\to 0}\frac{\theta}{\tan^{-1}\theta}=1$$

■

EXAMPLE 7 Evaluate $\lim_{x\to 0}\left[\dfrac{3\sin x}{x}\right]+\left[\dfrac{4x}{\tan x}\right]$ (where [·] denotes greatest integer function)

Solution In the proof of $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$, we had established an inequality for θ (θ > 0)

$\sin\theta < \theta < \tan\theta$

Hence $\dfrac{\sin\theta}{\theta} < 1$, i.e., we can say $\dfrac{\sin\theta}{\theta}$ approaches 1 from the left-hand side of 1. Also $\dfrac{\sin\theta}{\theta}$ is an even function. So we can say the same thing for θ < 0. Hence $\lim_{\theta\to 0}\left[\dfrac{\sin\theta}{\theta}\right]=0$ (where [·] denotes greatest integer function). Similarly it can be established from the right inequality $\left[\dfrac{\tan\theta}{\theta}\right]> 1$. Hence, $\lim_{\theta\to 0}\left[\dfrac{\tan\theta}{\theta}\right] = 1$ (where [·] denotes greatest integer function). Coming back to our original question we can say.

$$\lim_{x\to 0}\left[\frac{3\sin x}{x}\right]=2 \text{ (As } \frac{3\sin x}{x} < 3\text{) and } \lim_{x\to 0}\left[\frac{4x}{\tan x}\right]=3$$

As $$\frac{\tan x}{x} > 1$$

$\Rightarrow$ $$\frac{x}{\tan x} < 1;$$

Hence $$\frac{4x}{\tan x} < 4$$

$\therefore$ $$\lim_{x\to 0}\left[\frac{3\sin x}{x}\right] + \left[\frac{4x}{\tan x}\right] = 5$$ ■

EXAMPLE 8 Find the value of $\lim_{x\to 0}\dfrac{1-\cos x.\sqrt{\cos 2x}}{x^2}$

Solution Clearly when x approaches 0, the above expression has form $\dfrac{0}{0}$.

Now, $$\lim_{x\to 0}\frac{(1-\cos x.\sqrt{\cos 2x})}{x^2}\times\frac{(1+\cos x.\sqrt{\cos 2x})}{(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x\cdot\cos 2x)}{x^2(1+\cos x\cdot\sqrt{\cos 2x})} = \lim_{x\to 0}\frac{1-\cos^2 x(1-2\sin^2 x)}{x^2(1+\cos x.\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x+2\sin^2 x\cos^2 x}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{\sin^2 x+2\sin^2 x\cdot\cos^2 x}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right).\frac{(1+2\cos^2 x)}{(1+\cos x\sqrt{\cos 2x})} = 1\times\frac{3}{2} = \frac{3}{2}$$ ■

EXAMPLE 9 Find the value of $\lim_{x\to 0}\dfrac{1-\cos(1-\cos(1-\cos x)))}{x^8}$

Solution In this problem, we can take help of $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$, which is very easy to establish.

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2}\times\frac{(1+\cos x)}{(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \frac{1}{2}\left(\text{As } \lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1 \text{ and } \lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{2}\right)$$

Now, this limit can be generalized as

$$\lim_{f(x)\to 0}\frac{1-\cos(f(x))}{f^2(x)} = \frac{1}{2}$$

Coming back to our problem,

$\lim_{x\to 0}\frac{1-\cos(1-\cos(1-\cos x))}{x^8}$; In this expression, we can see multiple compositions.

Hence we can assume following let $1 - \cos(1 - \cos x) = f(x)$ and $1 - \cos x = g(x)$

$\therefore$ expression becomes

$$\lim_{x\to 0}\frac{1-\cos(f(x))}{x^8}=\lim_{x\to 0}\frac{1-\cos(f(x))}{f^2(x)\times x^8}\times\frac{(1-\cos(g(x))^2}{(g(x))^4}(1-\cos x)^4$$

$$=\lim_{x\to 0}\frac{1-\cos(f(x))}{f^2(x)}\times\left(\frac{1-\cos(g(x))}{(g(x))^2}\right)^2\left(\frac{1-\cos x}{x^2}\right)^4$$

$$=\frac{1}{2}\times\frac{1}{2^2}\times\frac{1}{2^4}=\frac{1}{128}$$

■

Limits involving $\lim_{x\to 0}\frac{a^x-1}{x}$ and $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

> **THEOREM 8—Limit of Ratio $\frac{a^x-1}{x}$ as $x\to 0$**
>
> $$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

Proof Here we can use the expansion of $f(x) = a^x$ around $x = 0$

$$a^x = e^{x\ln a} = 1+\frac{x\ln a}{1!}+\frac{x^2\ln^2 a}{2!}+\frac{x^3\ln^3 a}{3!}+\ldots\infty\ (a>0)$$

Now,

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{x\to 0}\frac{\left(1+\frac{x\ln a}{1!}+\frac{x^2\ln^2 a}{2!}+\ldots\ldots\right)-1}{x}$$

$$=\lim_{x\to 0}\frac{\frac{x\ln a}{1!}+\frac{x^2\ln^2 a}{2!}+\ldots\ldots}{x}$$

$$=\operatorname*{lt}_{x\to 0}\left(\ln a+\frac{x\ln^2 a}{2!}+\ldots.\right)$$

$$=\ln a$$

$\therefore$ $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a(a>0)$ and for $a = e$;

$$\lim_{x\to 0}\frac{e^x-1}{x}=\ln e=1$$

■

> **THEOREM 9—Limit of Ratio $\frac{\ln(1+x)}{x}$ as $x\to 0$**
>
> $$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

Proof $\lim_{x\to 0}\frac{\ln(1+x)}{x}$ in this we can use the expansion for

$$f(x) = \ln(1+x)\ (x \in (-1, 1]),\ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$$

Now,
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots..\infty}{x}$$

$$= \lim_{x\to 0} 1 - \frac{x}{2} + \frac{x^2}{3} + \ldots..\infty = 1$$

$\therefore$
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

■

EXAMPLE 10 Find $\lim_{x\to 0}(1+x)^{1/x}$

Solution $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)}$ (As we have already seen $\lim_{x\to 0}\frac{\ln(1+x)}{x}$)

$\therefore$
$$\lim_{x\to 0} e^{\frac{\ln(1+x)}{x}} = e$$

$\therefore$
$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Generalized form for 1^∞ (Base is approaching 1)

Suppose the problem is to find $\lim_{x\to a}(f(x)^{g(x)}$ and when $x \to a$, $f(x) \to 1$ and $g(x) \to \infty$. Clearly the given problem acquires indeterminate form 1^∞. Now, in such cases we can proceeds as follows.

$$\lim_{x\to a}(f(x))^{g(x)}$$

$$= \lim_{x\to a}\left((1 + f(x) - 1)^{\frac{1}{f(x)-1}}\right)^{g(x)(f(x)-1)}$$

Clearly, as $x \to a$ $f(x) - 1 \to 0$

$\therefore$
$$\lim_{x\to a}(1 + f(x) - 1)^{\frac{1}{f(x)-1}} = e$$

$\therefore$
$$\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} e^{g(x)\cdot(f(x)-1)}$$

Clearly it is only for 1^∞ (Base approaches 1)

■

EXAMPLE 11 Find the value of $\lim_{x\to 0}\left(\frac{6}{2+\sqrt{16+x}}\right)^{\cot x}$

Solution Here, $\lim_{x\to 0}\left(\frac{6}{2+\sqrt{16+x}}\right)^{\cot x}$ is of the form 1^∞

$\therefore$
$$\lim_{x\to 0}\left(\frac{6}{2+\sqrt{16+x}}\right)^{\cot x}$$

$$= \lim_{x\to 0} e^{\cot x\left(\frac{6}{2+\sqrt{16+x}}-1\right)} \left(\because \lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} e^{g(x)(f(x)-1)}\right)$$

$$= \lim_{x\to 0} e^{\cot x\left(\frac{6-2-\sqrt{16+x}}{2+\sqrt{16+x}}\right)}$$

$$= \lim_{x\to 0} e^{\cot x\left(\frac{4-\sqrt{16+x}}{2+\sqrt{16+x}}\right)\left(\frac{4+\sqrt{16+x}}{4+\sqrt{16+x}}\right)}$$

$$= \lim_{x\to 0} e^{\frac{\cos x}{\sin x}\left(\frac{-x}{(2+\sqrt{16+x})(4+\sqrt{16+x})}\right)}$$

$$= e^{-\frac{1}{48}}$$ ■

EXAMPLE 12 Find the value of $\lim_{x\to\infty}\left(\frac{x+7}{x+2}\right)^{x+3}$.

Solution Clearly as $x \to \infty$; $\frac{x+7}{x+2} = \frac{1+\frac{7}{x}}{1+\frac{2}{x}} \to 1$ and $x+3 \to \infty$

$$\therefore \quad \lim_{x\to\infty}\left(\frac{x+7}{x+2}\right)^{x+3} = \lim_{x\to\infty} e^{(x+3)\left(\frac{x+7}{x+2}-1\right)} = \lim_{x\to\infty} e^{(x+3)\frac{5}{x+2}}$$

$$= \lim_{x\to\infty} e^{5\left(\frac{1+\frac{3}{x}}{1+\frac{2}{x}}\right)} = e^5$$ ■

EXAMPLE 13 Find the value of $\lim_{x\to 0}\left(\sin^2\frac{\pi}{2-\alpha x}\right)^{\sec^2\left(\frac{\pi}{2-\beta\pi}\right)}$.

Solution Clearly as $x \to 0$, $\sin^2\frac{\pi}{2-\alpha x} \to 1$

and $$\sec^2\left(\frac{\pi}{2-\beta x}\right) \to \infty$$

$$\therefore \quad \lim_{x\to 0}\left(\sin^2\left(\frac{\pi}{2-\alpha x}\right)\right)^{\sec^2\left(\frac{\pi}{2-\alpha x}\right)} = \lim_{x\to 0} e^{\sec^2\left(\frac{\pi}{2-\beta x}\right)\cdot\left(\sin^2\frac{\pi}{2-\alpha x}-1\right)}$$

$$= \lim_{x\to 0} e^{-\left(\frac{\cos\left(\frac{\pi}{2-\alpha x}\right)}{\cos\left(\frac{\pi}{2-\beta x}\right)}\right)^2} = e^{-\ell^2}$$

where $$\ell = \lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2-\alpha x}\right)}{\cos\left(\frac{\pi}{2-\beta x}\right)} = \lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-\alpha x}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-\beta x}\right)}$$

$$= \lim_{x\to 0} \frac{\sin\left(\dfrac{\pi.(-\alpha x)}{2(2-\alpha x)}\right)}{\sin\left(\dfrac{\pi-(\beta x)}{2(2-\beta x)}\right)} \times \frac{\pi\left(\dfrac{-\beta x}{2(1-\beta x)}\right)}{\pi\left(\dfrac{-\alpha x}{2(2-\alpha x)}\right)} \times \frac{\pi\left(\dfrac{-\alpha x}{2(2-\alpha x)}\right)}{\pi\left(\dfrac{-\beta x}{2(2-\beta x)}\right)}$$

$$\therefore \qquad \ell = \frac{\alpha}{\beta}$$

$$\therefore \qquad \lim_{x\to 0}\left(\sin^2\left(\frac{\pi}{2-\alpha x}\right)\right)^{\sec^2\left(\frac{\pi}{2-\beta x}\right)} = e^{-\frac{\alpha^2}{\beta^2}}$$

■

Limits of functions having built-in limit with them

EXAMPLE 14 If $f(x) = \lim\limits_{n\to\infty} \dfrac{\sin(\pi x^2) + (x+2)^n \cdot \tan x}{x^2 + (x+2)^n}$, then find $\lim\limits_{x\to -1} f(x)$ (where $n \in N$)

Solution This can be termed as limit of a function $f(x)$ having built-in limit.

Now, $$f(x) = \lim_{n\to\infty} \frac{\sin(\pi x^2) + (x+2)^n \cdot \tan x}{x^2 + (x+2)^n}$$

Clearly the part affected is $(x+2)^n$

Now, when $$\lim_{n\to\infty} (x+2)^n = \begin{cases} \infty & x+2>1 \Rightarrow & x>-1 \\ 1 & x+2=1 \Rightarrow & x=-1 \\ 0 & 0<x+2<1 \Rightarrow & -2<x<-1 \end{cases}$$

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{\dfrac{\sin(\pi x^2)}{(x+2)^n} + \tan x}{\dfrac{x^2}{(x+2)^n} + 1} & \text{for } x>-1 \\ \lim\limits_{n\to\infty} \dfrac{\sin \pi x^2 + 0 \tan x}{x^2 + 0} & \text{for } -2<x<-1 \end{cases}$$

$$\therefore \qquad f(x) = \begin{cases} \tan x & x>-1 \\ \dfrac{\sin \pi x^2}{x^2} & -2<x<-1 \end{cases}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (\tan x) = -\tan 1$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{\sin(\pi x^2)}{x^2} = 0$$

$$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$$

Hence, limit does not exists. ■

EXAMPLE 15 If $\lim_{n\to\infty} \dfrac{n \cdot 5^n}{n(x-3)^n + n \cdot 5^{n+1} - 5^n} = \dfrac{1}{5}$ (where $n \in N$), then find the range of x.

Solution $\lim_{n\to\infty} \dfrac{n \cdot 5^n}{n(x-3)^n + n \cdot 5^{n+1} - 5^n} = \dfrac{1}{5} \Rightarrow \lim_{n\to\infty} \dfrac{1}{\left(\dfrac{x-3}{5}\right)^n + 5 - \dfrac{1}{n}} = \dfrac{1}{5}$

Clearly, the limit will be $\dfrac{1}{5}$ only when $\lim_{n\to\infty} \left(\dfrac{x-3}{5}\right)^n = 0$

and for that $-1 < \dfrac{x-3}{5} < 1$

$\Rightarrow \quad -5 < x - 3 < 5$

$\Rightarrow \quad -2 < x < 8$

$\Rightarrow \quad x \in (-2, 8)$ ■

EXAMPLE 16 Evaluate $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{\sin^3 x}$

Solution Clearly the best option for solving this problem would be expansion of e^x and e^{-x} about $x = 0$.

Now, $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\infty$$

and $$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\infty$$

$\therefore$ $$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots\ldots\infty$$

$\therefore$ $$e^x - e^{-x} - 2x = \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots\ldots\infty$$

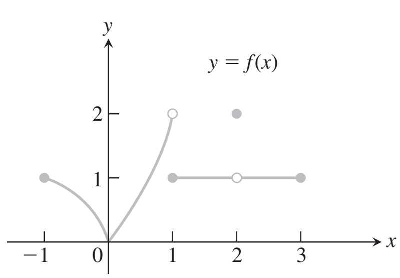
$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin^3 x} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\dfrac{\sin^3 x}{x^3} \times x^3}$$

$$= \lim_{x\to 0} \frac{\dfrac{2x^3}{3!} + \dfrac{2x^5}{5!} + \ldots\infty}{\left(\dfrac{\sin^3 x}{x^3}\right) \times x^3} = \frac{2}{6} = \frac{1}{3}$$ ■

Exercises 2.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 2} f(x)$ does not exist.

c. $\lim_{x\to 2} f(x) = 2$ **d.** $\lim_{x\to 1^-} f(x) = 2$

e. $\lim_{x\to 1^+} f(x) = 1$ **f.** $\lim_{x\to 1} f(x)$ does not exist.

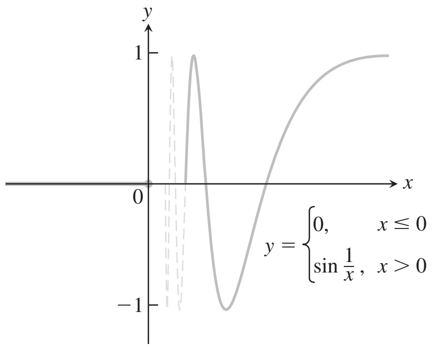
g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

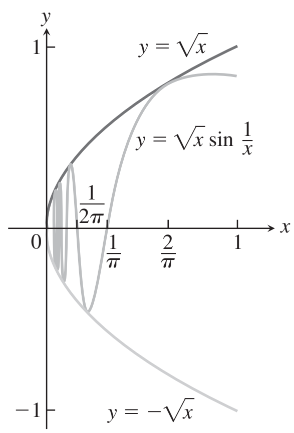
j. $\lim_{x\to -1^-} f(x) = 0$ **k.** $\lim_{x\to 3^+} f(x)$ does not exist.

2. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0. \end{cases}$



a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?
b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?
c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

3. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?
b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?
c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

4. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$
b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.
c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Finding One-Sided Limits Algebraically

Use the graph of the greatest integer function $y = \lfloor x \rfloor$,

5. a. $\lim\limits_{\theta\to 3^+} \frac{\lfloor \theta \rfloor}{\theta}$ b. $\lim\limits_{\theta\to 3^-} \frac{\lfloor \theta \rfloor}{\theta}$

6. a. $\lim\limits_{t\to 4^+} (t - \lfloor t \rfloor)$ b. $\lim\limits_{t\to 4^-} (t - \lfloor t \rfloor)$

Using $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Find the limits in Exercises 7–8.

7. $\lim\limits_{\theta\to 0} \frac{\tan\theta}{\theta^2 \cot 3\theta}$ 8. $\lim\limits_{\theta\to 0} \frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Theory and Examples

9. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

10. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

11. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

12. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

2.5 Continuity

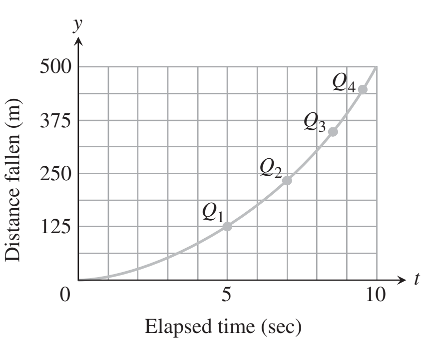


FIGURE 2.34 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

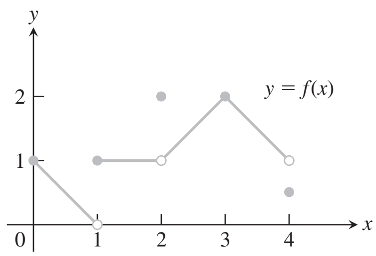


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

EXAMPLE 1 At which numbers does the function f in Figure 2.35 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The breaks appear as jumps, which we identify later as "jump discontinuities." These are numbers for which the function is not continuous, and we discuss each in turn.

Numbers at which the graph of f has breaks:

At $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x\to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x\to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x\to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x\to 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. So no break occurs in the graph of the function at this endpoint, and the function is continuous from the right at $x = 0$.

At $x = 3$, the function has a limit, $\lim_{x\to 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. No break occurs in the graph and the function is continuous at $x = 3$.

At all other numbers $x = c$ in the domain, which we have not considered, the function has a limit equal to the value of the function at the point, so $\lim_{x\to c} f(x) = f(c)$. For example, $\lim_{x\to 5/2} f(x) = f\left(\frac{5}{2}\right) = \frac{3}{2}$. No breaks appear in the graph of the function at any of these remaining numbers and the function is continuous at each of them. ■

The following definitions capture the continuity ideas we observed in Example 1.

DEFINITIONS Let c be a real number on the x-axis.

The function f is **continuous at c** if

$$\lim_{x\to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x\to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x\to c^-} f(x) = f(c).$$

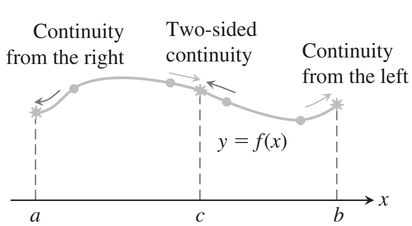


FIGURE 2.36 Continuity at points a, b, and c.

From Theorem 6, it follows immediately that a function f is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval.

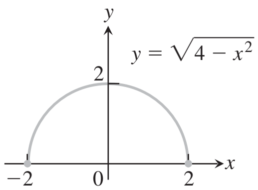


FIGURE 2.37 A function that is continuous over its domain (Example 2).

This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at an interior point c of its domain, we say that f is **discontinuous at c**, and that c is a point of discontinuity of f. Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.37). It is right-continuous at $x = -2$, and left-continuous at $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

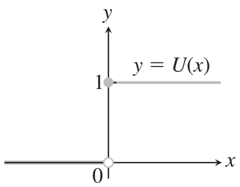


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

We summarize continuity at an interior point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity and continuity at an endpoint of an interval, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

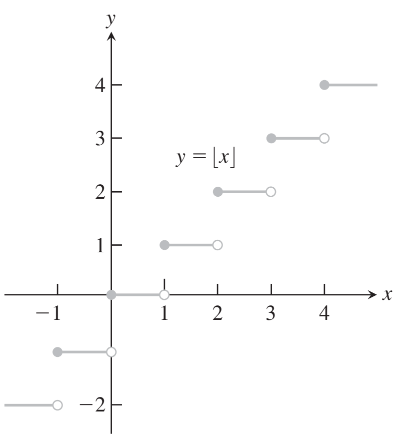


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \qquad \text{and} \qquad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor.$$ ■

Figure 2.40 displays several common types of discontinuities. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b would be continuous if it had $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.40c is **removable**. The function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x \to 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates too much to have a limit as $x \to 0$.

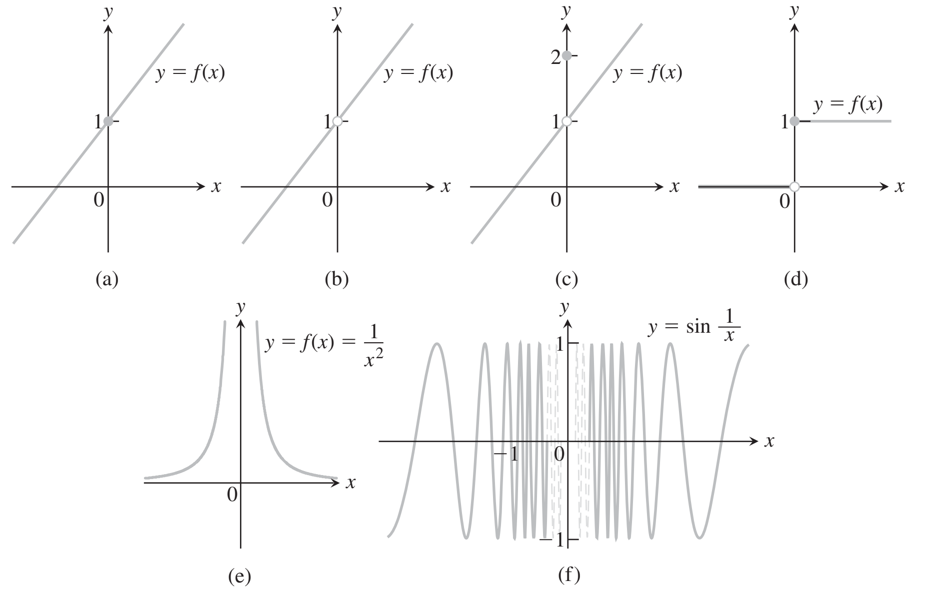


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

Generally, we want to describe the continuity behavior of a function throughout its entire domain, not only at a single point. We know how to do that if the domain is a closed interval. In the same way, we define a **continuous function** as one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain, we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.

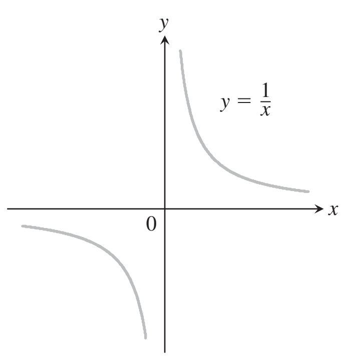


FIGURE 2.41 The function $y = 1/x$ is continuous over its natural domain. It has a point of discontinuity at the origin, so it is discontinuous on any interval containing $x = 0$ (Example 5).

EXAMPLE 5

(a) The function $y = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 10—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Most of the results in Theorem 10 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}(f(x)+g(x)) \\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) && \text{Sum Rule, Theorem 1} \\ &= f(c)+g(c) && \text{Continuity of } f, g \text{ at } c \\ &= (f+g)(c).\end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x\to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 11 of Section 2.2. Both functions are, in fact, continuous everywhere (see Exercise 70). It follows from Theorem 10 that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

EXAMPLE 8 Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & \text{if } x<0 \\ a & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}+1 & \text{if } x>0 \end{cases}$

then the value of "a" for which $f(x)$ is continuous at $x = 0$ is

(a) 8 **(b)** 9 **(c)** $\dfrac{17}{2}$ **(d)** not possible

Solution For the function $f(x)$ to be continuous at $x = 0$

L.H.L. = R.H.L. = value of function

i.e., $$\text{L.H.L.} = \lim_{x\to 0^-}\frac{1-\cos 4x}{x^2} = \lim_{x\to 0^-}\frac{1-\cos 4x}{16x^2}\times 16$$

$$= \frac{1}{2}\times 16 = 8$$

Now, $$\text{R.H.L.} = \lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}+1$$

$$= \lim_{x\to 0^+}\frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}}+4\right)}{(\sqrt{16+\sqrt{x}}-4)(\sqrt{16+\sqrt{x}}+4)}+1$$

$$= \lim_{x\to 0^+}(\sqrt{16+\sqrt{x}}+4)+1=9 \quad \blacksquare$$

Hence L.H.L. ≠ R.H.L., Hence $f(x)$ cannot be made continuous at $x = 0$, so the value of "a" is not possible.

EXAMPLE 9 Let $f(x) = \sum_{r=1}^{n} \text{cosec}(2^r x)$ and $h(x) = f(x) + \cot(2^n x)$

$$\text{if } H(x) = \begin{cases} (\sec x)^{h(x)} + \left(\dfrac{1+\tan x}{1+\sin x}\right)^{\text{cosec}\, x} & x > 0 \\ \ell & x = 0 \\ \dfrac{e^x + e^{-x} - 2\cos x}{x \tan x} & x < 0 \end{cases}$$

Then the value of ℓ for which $H(x)$ is continuous at $x = 0$ is

(a) 1 **(b)** 2 **(c)** 3 **(d)** does not exist

Solution We need to first simplify $f(x)$

given $$f(x) = \sum_{r=1}^{n} \frac{1}{\sin(2^r x)} = \sum_{r=1}^{n} \frac{\sin(2^{r-1}x)}{\sin(2^r x)\sin(2^{r-1}x)}$$

$$= \sum_{r=1}^{n} \frac{\sin((2^r - 2^{r-1})x)}{\sin(2^r x).\sin(2^{r-1}.x)}$$

$$= \sum_{r=1}^{n} \frac{\sin(2^r x)\cos(2^{r-1}x) - \cos(2^r x).\cos(2^{r-1}x)}{\sin(2^r.x)\sin(2^{r-1}x)}$$

$$= \sum_{r=1}^{n} \left(\frac{\sin(2^r x)\cos(2^{r-1}x)}{\sin(2^r x)\sin(2^{r-1}x)} - \frac{\cos(2^r x)\cos(2^{r-1}x)}{\sin(2^r x)\sin(2^{r-1}x)}\right)$$

$$= \sum_{r=1}^{n} (\cot(2^{r-1}x) - \cot(2^r.x))$$

$\cot x - \cot 2x$

$\cot 2x - \cot(2^2 x)$

$\vdots$

$\cot(2^{n-1}x) - \cot(2^n x)$

$\therefore$ $f(x) = \cot x - \cot(2^n x)$

Now, $h(x) = f(x) + \cot(2^n x)$

$= \cot x - \cot(2^n x) + \cot(2^n x) = \cot x$

Also for the continuity of the function $H(x)$

L.H.L. = R.H.L. = value of function

Now, $$\text{R.H.L.} = \lim_{x\to 0^+}(\sec x)^{\cot x} + \left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$$

$$= \lim_{x\to 0^+}\left(e^{\cot x(\sec x-1)} + e^{\operatorname{cosec} x\left(\frac{1+\tan x}{1+\sin x}-1\right)}\right)$$

$$= \lim_{x\to 0^+}\left(e^{\frac{\cos x}{\sin x}\frac{(1-\cos x)}{\cos x}} + e^{\frac{1}{\sin x}\times\left(\frac{1+\tan x-1-\sin x}{1+\sin x}\right)}\right)$$

$$= \lim_{x\to 0^+}\left(e^{\left(\frac{1-\cos x}{x^2}\right)\left(\frac{x}{\sin x}\right).x} + e^{\frac{1}{\sin x}\times\sin x\frac{(1-\cos x)}{(1+\sin x).\cos x}}\right)$$

$$= e^{1/2\times 1\times 0} + e^{0/1} = 2$$

Now, $$\text{L.H.L.} = \lim_{x\to 0^-}\left(\frac{e^x+e^{-x}-2\cos x}{x\tan x}\right)$$

For evaluating such kind of limits, it is advisable remember the expansions of function e^x, e^{-x}, $\cos x$, $\tan x$, etc.

$$\therefore \quad \lim_{x\to 0^-}\left(\frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....\right)+\left(1-\frac{x}{1!}+\frac{x^2}{2!}-....\right)-2\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-....\right)}{x\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+......\right)}\right)$$

$$= \lim_{x\to 0^-}\frac{2x^2+(\text{higher powers of } x)}{x^2\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+......\right)} = 2$$

$\therefore$ L.H.L. = R.H.L. = 2

Hence, the functional value at $x = 0$ is 2

$\therefore$ $l = 2$

Hence correct answer is (B). ■

EXAMPLE 10 If $f(x) = \begin{cases} \dfrac{\sin x - x^2 - \{x\}\{-x\}}{x\cos x - x^2 - \{x\}\{-x\}} & x \neq 0 \\ k & x = 0 \end{cases}$

(where $\{\cdot\}$ denotes fractional part function then the value of k for which $f(x)$ is continuous at $x = 0$ is

(a) 1/2 **(b)** 1/3 **(c)** 1 **(d)** Not possible

Solution For the function $f(x)$ to be continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0) = k$$

Now, we will have to separately find the left- and right-hand limits as $\{x\}$ is involved.

$$\text{L.H.L.} = \lim_{x\to 0^-}\frac{\sin x - x^2 - \{x\}\{-x\}}{x\cos - x^2 - \{x\}\{-x\}}$$

$$= \lim_{x\to 0^-}\frac{\sin x - x^2 - (x-[x])(-x-[-x])}{x\cos x - x^2 - (x-[x])(-x-[-x])}$$

As $\quad x \to 0^-$

$\Rightarrow \quad [x] = -1$ and $[-x] = 0$

$$= \lim_{x\to 0^-} \frac{\sin x - x^2 - (x+1)(-x)}{s\cos x - x^2 - (x+1)(-x)}$$

$$= \lim_{x\to 0^-} \frac{\sin x - x^2 + x^2 + x}{s\cos x - x^2 + x^2 + x}$$

$$= \lim_{x\to 0^-} \frac{\frac{\sin x}{x} + 1}{\cos x + 1} = \frac{2}{2} = 1$$

Also $\quad$ R.H.L. $= \lim_{x\to 0^+} \dfrac{\sin x - x^2 - \{x\}\{-x\}}{x\cos x - x^2 - \{x\}\{-x\}}$

$$= \lim_{x\to 0^+} \frac{\sin x - x^2 - (x-[x])(-x-[-x])}{x\cos x - x^2 - (x-[x])(-x-[x])}$$

Again if $\quad x \to 0^+$

$\Rightarrow \quad [x] = 0$ and $[-x] = -1$

$$= \lim_{x\to 0^+} \frac{\sin x - x^2 - x(-x+1)}{x\cos x - x^2 - x(-x+1)}$$

$$= \lim_{x\to 0^+} \frac{\sin x - x^2 + x^2 - x}{x\cos x - x^2 + x^2 - x}$$

Applying the expansion of $\sin x$ and $\cos x$, we get

$$= \lim_{x\to 0^+} \frac{\left(x - \frac{x^3}{3!} + \ldots\right) - x}{x\left(\left(1 - \frac{x^2}{2!} + \ldots.\right) - 1\right)}$$

$$= \lim_{x\to 0^+} \frac{1}{6} \times \frac{2}{1} = \frac{1}{3}$$

Hence there does not exist any k for which $f(x)$ is continuous at $x = 0$ L.H.L. $\neq$ R.H.L.

$\therefore$ (d) ■

Composites

All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.42). In this case, the limit as $x \to c$ is $g(f(c))$.

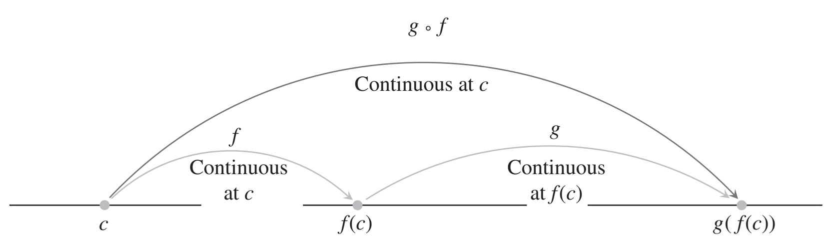


FIGURE 2.42 Composites of continuous functions are continuous.

> **THEOREM 11—Composite of Continuous Functions** If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Intuitively, Theorem 11 is reasonable because if x is close to c, then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied.

EXAMPLE 11 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$. The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 10).

(d) Because the sine function is everywhere-continuous, the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.43). ■

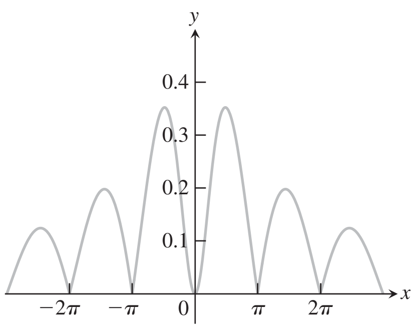


FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 11).

Theorem 11 is actually a consequence of a more general result, which we now state and prove.

> **THEOREM 12—Limits of Continuous Functions** If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then
>
> $$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

Proof Let $\epsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1.$$

Since $\lim_{x\to c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \epsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, this proves that $\lim_{x\to c} g(f(x)) = g(b)$. ■

EXAMPLE 12 As an application of Theorem 12, we have the following calculation:

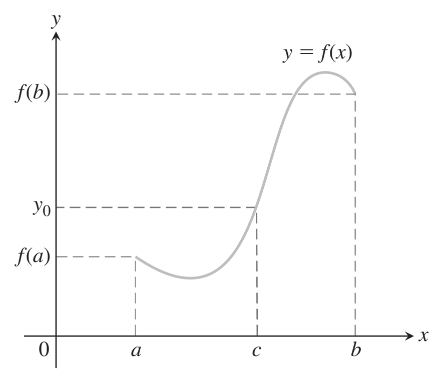
$$\begin{aligned}\lim_{x\to\pi/2}\cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) &= \cos\left(\lim_{x\to\pi/2}2x+\lim_{x\to\pi/2}\sin\left(\frac{3\pi}{2}+x\right)\right)\\ &= \cos\left(\pi+\sin 2\pi\right)=\cos\pi=-1.\end{aligned}$$

■

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *Intermediate Value Property*. A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

THEOREM 13—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Theorem 13 says that continuous functions over *finite closed* intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts.

The continuity of f on the interval is essential to Theorem 13. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.44 (choose y_0 as any number between 2 and 3).

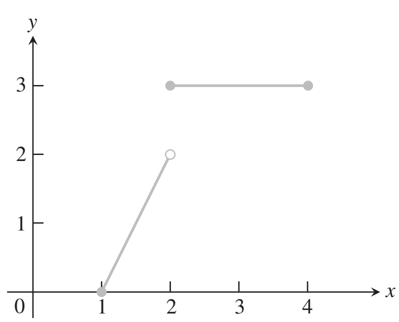


FIGURE 2.44 The function

$$f(x)=\begin{cases}2x-2, & 1\le x<2\\ 3, & 2\le x\le 4\end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

A Consequence for Graphing: Connectedness Theorem 13 implies that the graph of a function continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps like the graph of the greatest integer function (Figure 2.39), or separate branches like the graph of $1/x$ (Figure 2.41).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 13 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$. ■

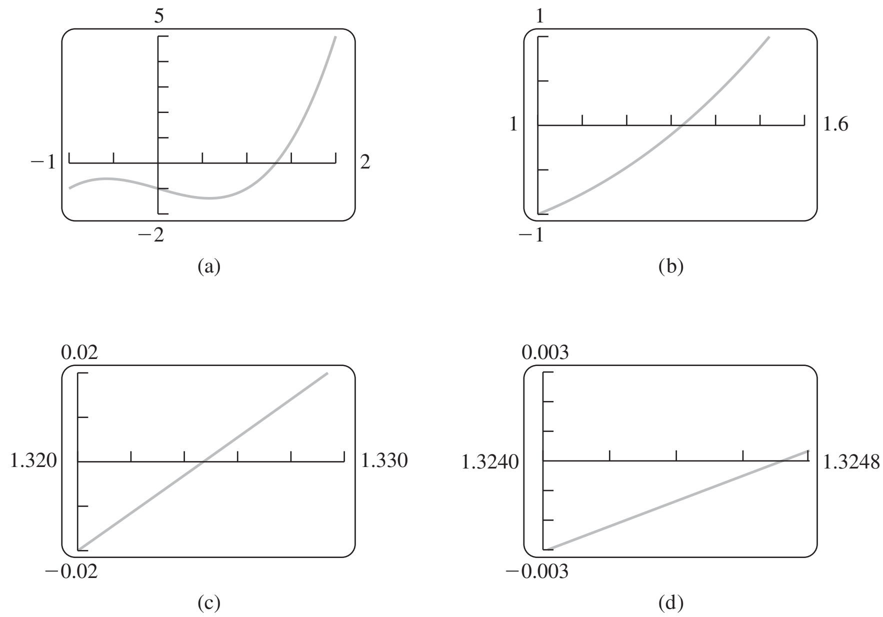


FIGURE 2.45 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

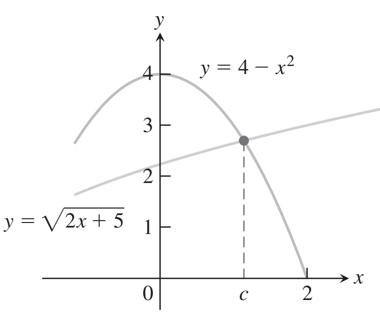


FIGURE 2.46 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x + 5} = 4 - x^2$ (Example 14).

EXAMPLE 14 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution (Figure 2.46).

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 = 4,$$

and set $f(x) = \sqrt{2x + 5} + x^2$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is the composite of the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2$, and the quadratic function is continuous for all values of x. It follows that $f(x) = \sqrt{2x + 5} + x^2$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} \approx 2.24$ and $f(2) = \sqrt{9} + 4 = 7$, and note that f is also continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 4$ is between the numbers 2.24 and 7, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 4$. That is, the number c solves the original equation. ■

EXAMPLE 15 Prove that the function $f(x) = a\sqrt{x-1} + b\sqrt{2x-1} - \sqrt{2x^2 - 3x + 1}$, where $a + 2b = 2$ and $a, b \in R$ always has a root in (1,5) $\forall\, b \in R$

Solution Given function is a continuous function.

Clearly $f(1) = b$ and $f(5) = 2a + 3b - 6 = 2(a + 2b) - b - 6 = 4 - b - 6 = -(b + 2)$

Now $f(1).f(5) = -b(b + 2)$, which is negative $\forall\ b \in R - (-2, 0)$. Hence according to intermediate value theorem, $f(x)$ has always a root. Also if $b \in (-2, 0)$ which implies the function has a negative value at $x = 1$. Now if we are able to prove that function attains a positive value for any $x \in (1, 5)$ then again using IVT we can claim a root in that interval which will be some sub-interval of (1, 5).

Now $f(2) = a + b\sqrt{3} - \sqrt{3} = 2 - 2b + b\sqrt{3} - \sqrt{3}$ (As $a + 2b = 2$) $= 2 - \sqrt{3} - b(2 - \sqrt{3})$ $= (1 - b)(2 - \sqrt{3})$

Clearly $f(2)$ is positive for $b \in (-2, 0)$. Hence $f(x)$ always has a root in (1, 5) $\forall\ b \in R$. ■

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f, to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since the origin is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \to 0$ (Theorem 7), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

The function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \to 0} \frac{\sin x}{x} = F(0),$$

so it meets the requirements for continuity (Figure 2.47).

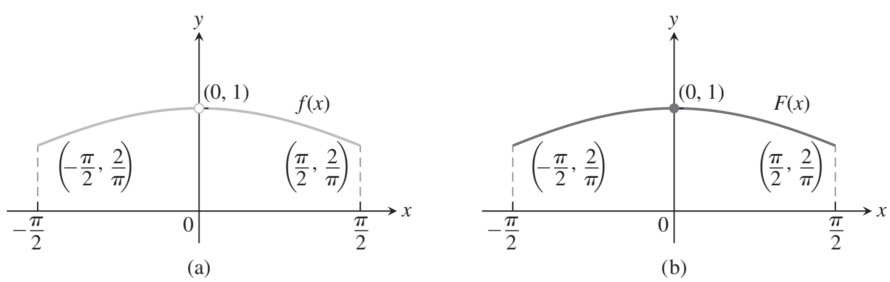


FIGURE 2.47 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \le x \le \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$.

More generally, a function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f, continuous extensions are often found by canceling common factors in the numerator and denominator.

EXAMPLE 16 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

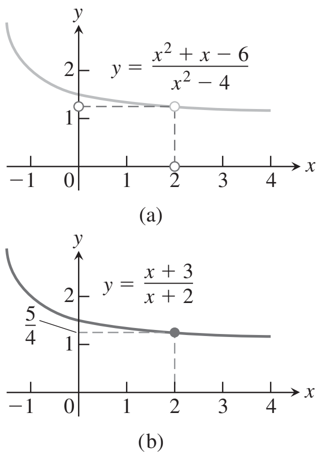


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 16).

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

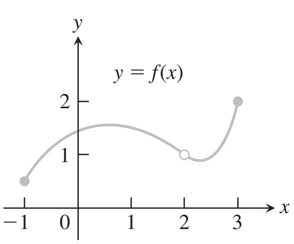
The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed. ■

Exercises 2.5

Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

1.



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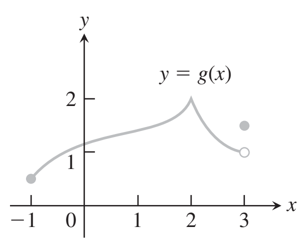

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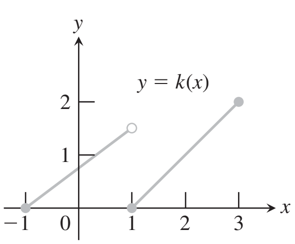

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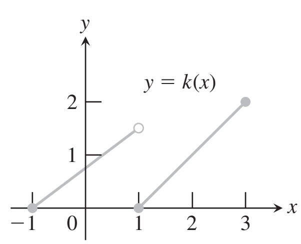
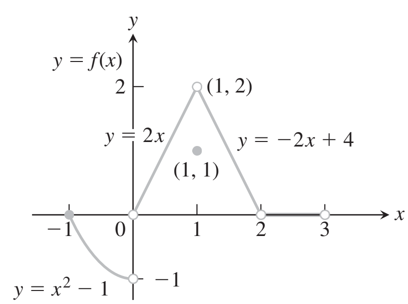


Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \leq x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. a. Does $f(-1)$ exist?

b. Does $\lim_{x \to -1^+} f(x)$ exist?

c. Does $\lim_{x \to -1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

6. a. Does $f(1)$ exist?

b. Does $\lim_{x \to 1} f(x)$ exist?

c. Does $\lim_{x \to 1} f(x) = f(1)$?

d. Is f continuous at $x = 1$?

7. a. Is f defined at $x = 2$? (Look at the definition of f.)

b. Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4 **12.** Exercise 2, Section 2.4

At what points are the functions in Exercises 13–30 continuous?

13. $y = \dfrac{1}{x-2} - 3x$

14. $y = \dfrac{1}{(x+2)^2} + 4$

15. $y = \dfrac{x+1}{x^2 - 4x + 3}$

16. $y = \dfrac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

29. $g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$

30. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 31–38. Are the functions continuous at the point being approached?

31. $\lim\limits_{x \to \pi} \sin(x - \sin x)$

32. $\lim\limits_{t \to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

33. $\lim\limits_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

34. $\lim\limits_{x \to 0} \tan\left(\dfrac{\pi}{4} \cos(\sin x^{1/3})\right)$

35. $\lim\limits_{t \to 0} \cos\left(\dfrac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$

36. $\lim\limits_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

37. $\lim\limits_{x \to 0} \sin \sqrt{\dfrac{\cos^2 x - \cos x}{x}}$

38. $\lim\limits_{x \to 0} \sec\left(\dfrac{\pi(\sin 2x - \sin x)}{3x}\right)$

Continuous Extensions

39. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

40. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

41. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

42. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

43. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

44. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

45. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

46. For what value of b is

$$g(x) = \begin{cases} \dfrac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x?

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x?

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

Theory and Examples

49. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

50. Explain why the equation $\cos x = x$ has at least one solution.

51. Roots of a cubic Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

52. A function value Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

53. Solving an equation If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals **(a)** π; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

54. Explain why the following five statements ask for the same information.

a. Find the roots of $f(x) = x^3 - 3x - 1$.

b. Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c. Find all the values of x for which $x^3 - 3x = 1$.

d. Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e. Solve the equation $x^3 - 3x - 1 = 0$.

55. Removable discontinuity Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

56. Nonremovable discontinuity Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

57. A function discontinuous at every point

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

58. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

59. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

60. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 11? Give reasons for your answer.

61. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

62. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

63. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

64. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \ne 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

65. Prove that f is continuous at c if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

66. Use Exercise 65 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 67–72 to prove that each equation has a solution.

67. $x^3 - 3x - 1 = 0$

68. $2x^3 - 2x^2 - 2x + 1 = 0$

69. $x(x - 1)^2 = 1$ (one root)

70. $\sqrt{x} + \sqrt{1 + x} = 4$

71. $\cos x = x$ (one root).

72. $2 \sin x = x$ (three roots).

2.6 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*, which are not limits as before, but rather a new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these limit ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

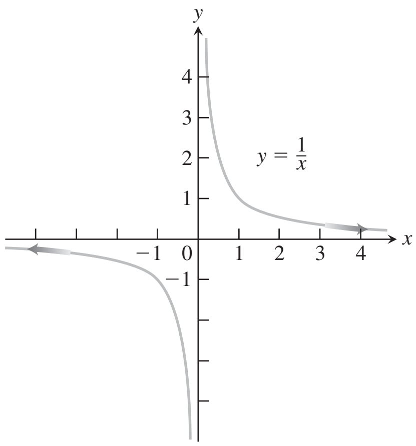


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and negative infinity*. Here are precise definitions.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

Intuitively, $\lim_{x \to \infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, $\lim_{x \to -\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

The strategy for calculating limits of functions as $x \to \pm\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x \to \pm\infty} k = k \qquad \text{and} \qquad \lim_{x \to \pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the second result in Example 1.

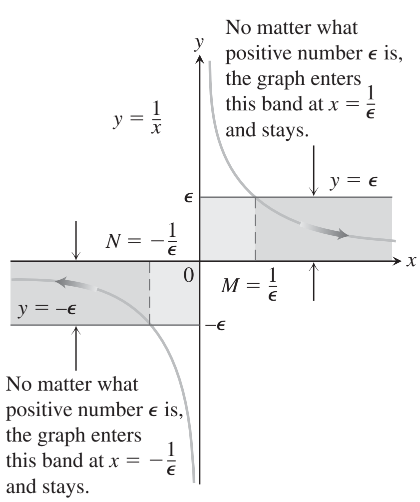


FIGURE 2.50 The geometry behind the argument in Example 1.

EXAMPLE 1 Show that

(a) $\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$ **(b)** $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.50). This proves $\lim_{x \to \infty}(1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.50). This proves $\lim_{x\to-\infty}(1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 14 All the Limit Laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

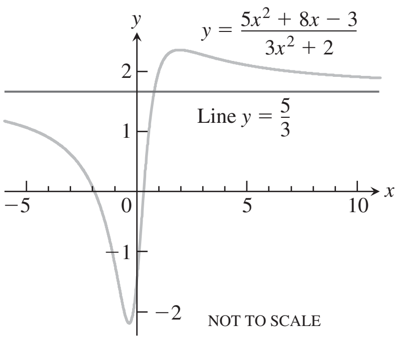


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

EXAMPLE 2 The properties in Theorem 14 are used to calculate limits in the same way as when x approaches a finite number c.

(a)
$$\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$
$$= 5 + 0 = 5 \qquad \text{Known limits}$$

(b)
$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$
$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product Rule}$$
$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$
■

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

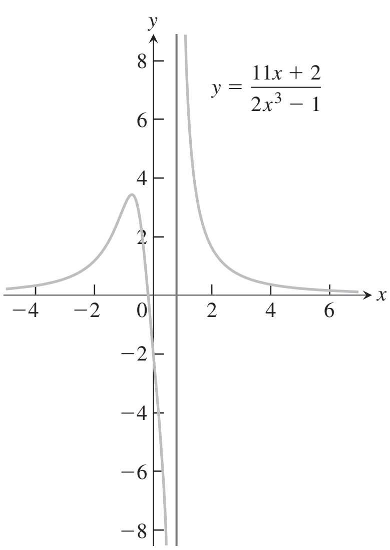


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a)
$$\lim_{x\to\infty}\frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty}\frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$
$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} \qquad \text{See Fig. 2.51.}$$

(b)
$$\lim_{x\to-\infty}\frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty}\frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} \qquad \text{Divide numerator and denominator by } x^3.$$
$$= \frac{0 + 0}{2 - 0} = 0 \qquad \text{See Fig. 2.52.}$$
■

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 9 and 13.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.49), we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

We say that the x-axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

> **DEFINITION** A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.51 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x\to\infty} f(x) = \frac{5}{3} \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = \frac{5}{3}.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $\displaystyle \lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$

For $x < 0$: $\displaystyle \lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x. ■

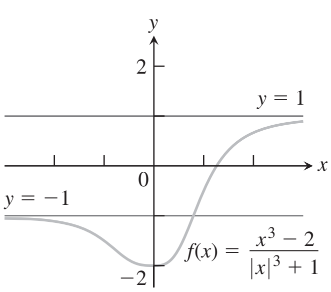


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 5 Find **(a)** $\lim_{x\to\infty} \sin(1/x)$ and **(b)** $\lim_{x\to\pm\infty} x\sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.49). Therefore,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0^+}\frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{x\to -\infty} x\sin\frac{1}{x} = \lim_{t\to 0^-}\frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.54, and we see that the line $y = 1$ is a horizontal asymptote. ■

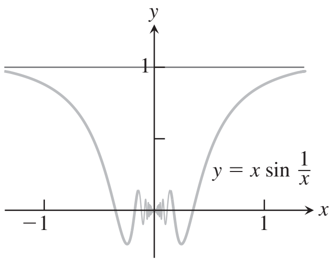


FIGURE 2.54 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 5b).

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

EXAMPLE 6 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.55).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

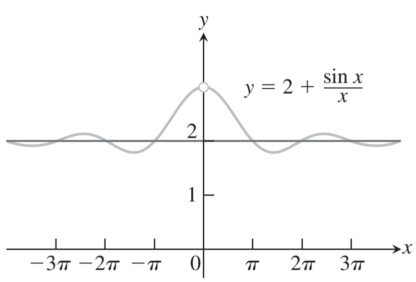


FIGURE 2.55 A curve may cross one of its asymptotes infinitely often (Example 6).

We can investigate the behavior of $y = f(1/x)$ as $x \to 0$ by investigating $y = f(t)$ as $t \to \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\displaystyle\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor$.

Solution We let $t = 1/x$ so that

$$\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor = \lim_{t\to\infty}\frac{1}{t}\lfloor t\rfloor$$

From the graph in Figure 2.56, we see that $t - 1 \le \lfloor t\rfloor \le t$, which gives

$$1 - \frac{1}{t} \le \frac{1}{t}\lfloor t\rfloor \le 1 \qquad \text{Multiply inequalities by } \frac{1}{t} > 0.$$

It follows from the Sandwich Theorem that

$$\lim_{t\to\infty}\frac{1}{t}\lfloor t\rfloor = 1,$$

so 1 is the value of the limit we seek. ■

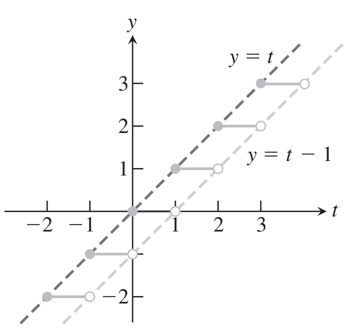


FIGURE 2.56 The graph of the greatest integer function $y = \lfloor t\rfloor$ is sandwiched between $y = t - 1$ and $y = t$.

EXAMPLE 8 Find $\displaystyle\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic result:

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}.$$

As $x\to\infty$, the denominator in this last expression becomes arbitrarily large, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} = \lim_{x\to\infty}\frac{-\dfrac{16}{x}}{1+\sqrt{\dfrac{x^2}{x^2}+\dfrac{16}{x^2}}} = \frac{0}{1+\sqrt{1+0}} = 0.$$

■

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x\to\pm\infty$.

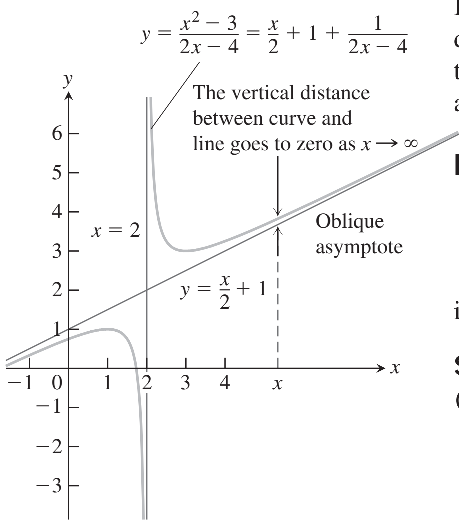


FIGURE 2.57 The graph of the function in Example 9 has an oblique asymptote.

EXAMPLE 9 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2-3}{2x-4}$$

in Figure 2.57.

Solution We are interested in the behavior as $x\to\pm\infty$. We divide $(2x-4)$ into (x^2-3):

$$\begin{array}{r@{}l} & \frac{x}{2}+1 \\ 2x-4\,\big) & x^2-3 \\ & \underline{x^2-2x} \\ & \quad 2x-3 \\ & \quad \underline{2x-4} \\ & \qquad\quad 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\left(\frac{x}{2}+1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x-4}\right)}_{\text{remainder}}.$$

As $x\to\pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2}+1$$

an asymptote of the graph of f (Figure 2.57). The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x\to 2$ (where the denominator is zero), as shown in the graph. ■

Notice in Example 9 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

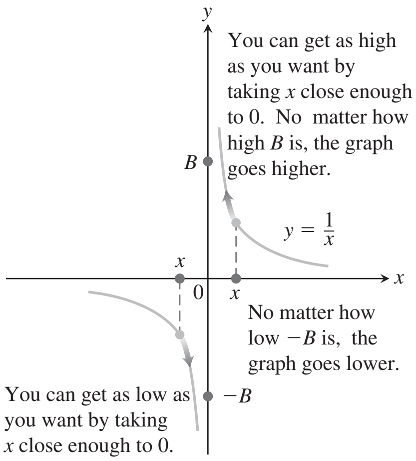


FIGURE 2.58 One-sided infinite limits: $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.58). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.58.) We write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative*.

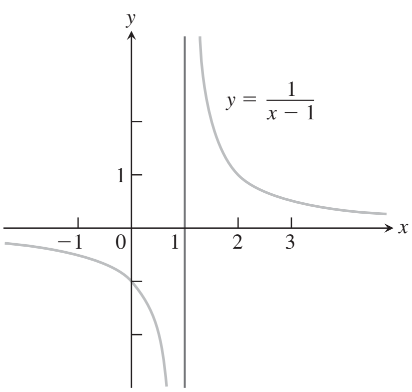


FIGURE 2.59 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 10).

EXAMPLE 10 Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.59). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ■

EXAMPLE 11 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \qquad \text{as} \qquad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.60). This means that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0} (1/x^2) = \infty$. ■

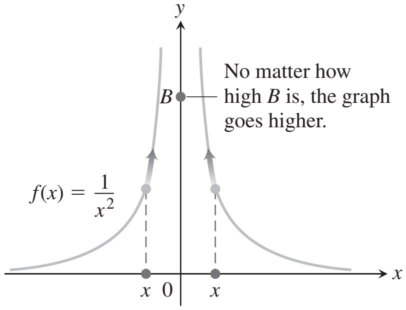


FIGURE 2.60 The graph of $f(x)$ in Example 11 approaches infinity as $x \to 0$.

EXAMPLE 12 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ — The values are negative for $x > 2$, x near 2.

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ — The values are positive for $x < 2$, x near 2.

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. — See parts (c) and (d).

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

EXAMPLE 13 Find $\lim_{x\to-\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x\to-\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x\to-\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty, \qquad x^{-n} \to 0,\ x - 3 \to -\infty\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

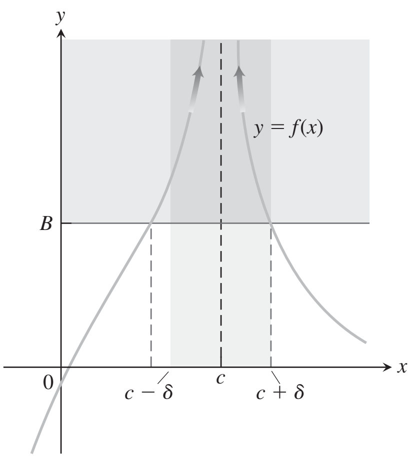


FIGURE 2.61 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

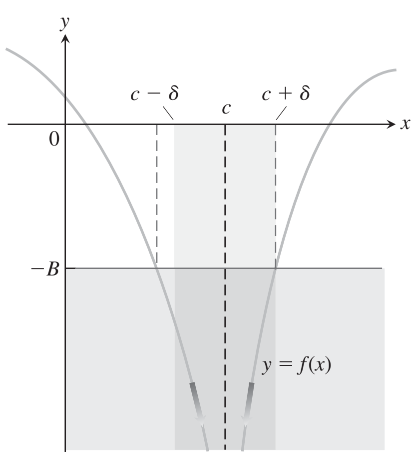


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.61 and 2.62 accompany these definitions.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -B.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 14 Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if} \qquad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \ge B.$$

Therefore, by definition,

$$\lim_{x\to 0} \frac{1}{x^2} = \infty. \qquad \blacksquare$$

Vertical Asymptotes

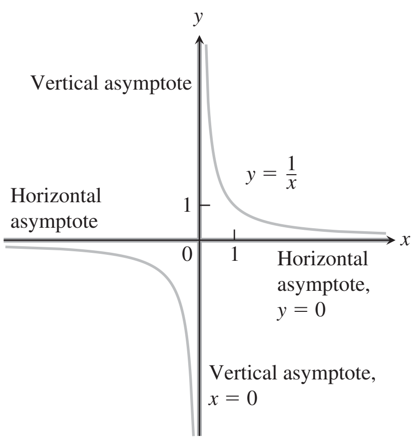


FIGURE 2.63 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.63). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

EXAMPLE 15 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{r} 1 \\ x + 2\,\overline{)\,x + 3} \\ \underline{x + 2} \\ 1 \end{array}$$

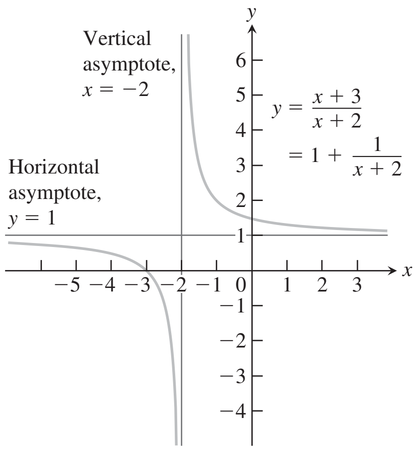


FIGURE 2.64 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 15.

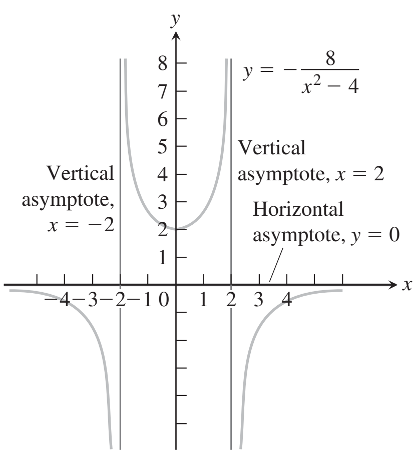


FIGURE 2.65 Graph of the function in Example 16. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x + 2}.$$

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.64). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ■

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.65). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

EXAMPLE 17 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.66).

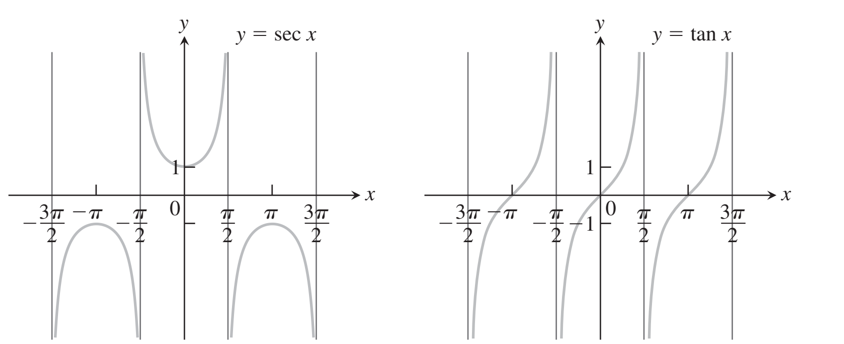


FIGURE 2.66 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 17). ■

Dominant Terms

In Example 9 we saw that by long division we could rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For x near 2, this term is very large in absolute value.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x - 4)$ dominates when x is near 2. **Dominant terms** like these help us predict a function's behavior.

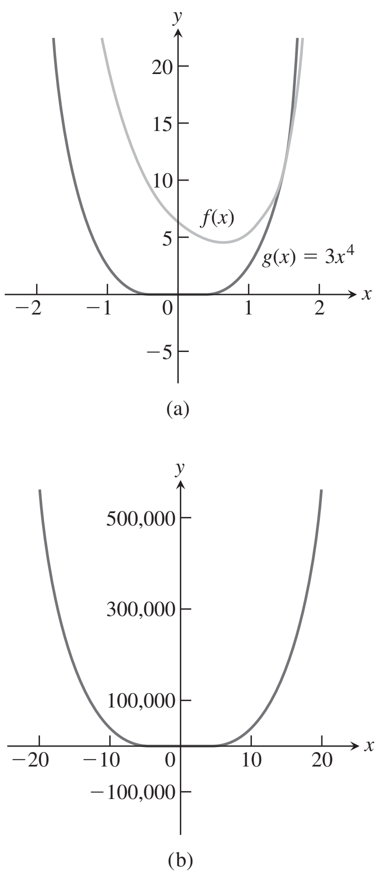


FIGURE 2.67 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 18).

EXAMPLE 18 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.67a), but appear as virtually identical on a larger scale (Figure 2.67b).

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty}\left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1,\end{aligned}$$

which means that f and g appear nearly identical when $|x|$ is large. ■

Summary

In this chapter we presented several important calculus ideas that are made meaningful and precise by the concept of the limit. These include the three ideas of the exact rate of change of a function, the slope of the graph of a function at a point, and the continuity of a function. The primary methods used for calculating limits of many functions are captured in the algebraic Limit Laws of Theorem 1 and in the Sandwich Theorem, all of which are proved from the precise definition of the limit. We saw that these computational rules also apply to one-sided limits and to limits at infinity. However, to calculate complicated limits such as

$$\lim_{x\to 0}\left(1 + \frac{1}{x}\right)^x, \qquad \lim_{x\to 0}\frac{x - \sin x}{x^3}, \qquad \text{and} \qquad \lim_{x\to 0} x^{1/x},$$

techniques other than simple algebra are required. The *derivative* is one of the tools we need to calculate limits such as these, and this notion is the central subject of our next chapter.

Exercises 2.6

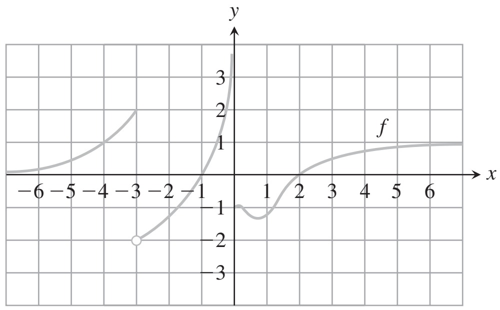
Finding Limits

1. For the function f whose graph is given, determine the following limits.

   a. $\lim_{x\to 2} f(x)$ b. $\lim_{x\to -3^+} f(x)$ c. $\lim_{x\to -3^-} f(x)$

   d. $\lim_{x\to -3} f(x)$ e. $\lim_{x\to 0^+} f(x)$ f. $\lim_{x\to 0^-} f(x)$

   g. $\lim_{x\to 0} f(x)$ h. $\lim_{x\to \infty} f(x)$ i. $\lim_{x\to -\infty} f(x)$



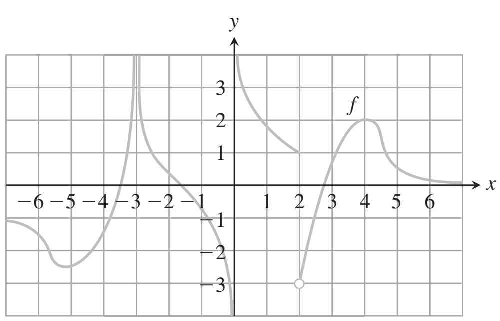
2. For the function f whose graph is given, determine the following limits.

   a. $\lim_{x\to 4} f(x)$ b. $\lim_{x\to 2^+} f(x)$ c. $\lim_{x\to 2^-} f(x)$

   d. $\lim_{x\to 2} f(x)$ e. $\lim_{x\to -3^+} f(x)$ f. $\lim_{x\to -3^-} f(x)$

   g. $\lim_{x\to -3} f(x)$ h. $\lim_{x\to 0^+} f(x)$ i. $\lim_{x\to 0^-} f(x)$

   j. $\lim_{x\to 0} f(x)$ k. $\lim_{x\to \infty} f(x)$ l. $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \dfrac{2}{x} - 3$

4. $f(x) = \pi - \dfrac{2}{x^2}$

5. $g(x) = \dfrac{1}{2 + (1/x)}$

6. $g(x) = \dfrac{1}{8 - (5/x^2)}$

7. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

8. $h(x) = \dfrac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty} \dfrac{\sin 2x}{x}$

10. $\lim_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$

11. $\lim_{t\to-\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

12. $\lim_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

13. $f(x) = \dfrac{2x + 3}{5x + 7}$

14. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \dfrac{x + 1}{x^2 + 3}$

16. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

20. $g(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

21. $f(x) = \dfrac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$

22. $h(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

24. $\lim_{x\to-\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\lim_{x\to-\infty} \left(\dfrac{1 - x^3}{x^2 + 7x}\right)^5$

26. $\lim_{x\to\infty} \sqrt{\dfrac{x^2 - 5x}{x^3 + x - 2}}$

27. $\lim_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

28. $\lim_{x\to\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

30. $\lim_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\lim_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

32. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

33. $\lim_{x\to\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

34. $\lim_{x\to-\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

35. $\lim_{x\to\infty} \dfrac{x - 3}{\sqrt{4x^2 + 25}}$

36. $\lim_{x\to-\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to 0^+} \dfrac{1}{3x}$

38. $\lim_{x\to 0^-} \dfrac{5}{2x}$

39. $\lim_{x\to 2^-} \dfrac{3}{x - 2}$

40. $\lim_{x\to 3^+} \dfrac{1}{x - 3}$

41. $\lim_{x\to -8^+} \dfrac{2x}{x + 8}$

42. $\lim_{x\to -5^-} \dfrac{3x}{2x + 10}$

43. $\lim_{x\to 7} \dfrac{4}{(x - 7)^2}$

44. $\lim_{x\to 0} \dfrac{-1}{x^2(x + 1)}$

45. **a.** $\lim_{x\to 0^+} \dfrac{2}{3x^{1/3}}$ **b.** $\lim_{x\to 0^-} \dfrac{2}{3x^{1/3}}$

46. **a.** $\lim_{x\to 0^+} \dfrac{2}{x^{1/5}}$ **b.** $\lim_{x\to 0^-} \dfrac{2}{x^{1/5}}$

47. $\lim_{x\to 0} \dfrac{4}{x^{2/5}}$ **48.** $\lim_{x\to 0} \dfrac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\lim_{x\to(\pi/2)^-} \tan x$ **50.** $\lim_{x\to(-\pi/2)^+} \sec x$

51. $\lim_{\theta\to 0^-} (1 + \csc\theta)$ **52.** $\lim_{\theta\to 0} (2 - \cot\theta)$

Find the limits in Exercises 53–58.

53. $\lim \dfrac{1}{x^2 - 4}$ as

a. $x \to 2^+$ **b.** $x \to 2^-$
c. $x \to -2^+$ **d.** $x \to -2^-$

54. $\lim \dfrac{x}{x^2 - 1}$ as

a. $x \to 1^+$ **b.** $x \to 1^-$
c. $x \to -1^+$ **d.** $x \to -1^-$

55. $\lim \left(\dfrac{x^2}{2} - \dfrac{1}{x}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$
c. $x \to \sqrt[3]{2}$ **d.** $x \to -1$

56. $\lim \dfrac{x^2 - 1}{2x + 4}$ as

a. $x \to -2^+$ **b.** $x \to -2^-$
c. $x \to 1^+$ **d.** $x \to 0^-$

57. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \to 0^+$ **b.** $x \to 2^+$
c. $x \to 2^-$ **d.** $x \to 2$
e. What, if anything, can be said about the limit as $x \to 0$?

58. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \to 2^+$ **b.** $x \to -2^+$
c. $x \to 0^-$ **d.** $x \to 1^+$
e. What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 59–62.

59. $\lim \left(2 - \dfrac{3}{t^{1/3}}\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

60. $\lim \left(\dfrac{1}{t^{3/5}} + 7\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

61. $\lim \left(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x - 1)^{2/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$
c. $x \to 1^+$ **d.** $x \to 1^-$

62. $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x - 1)^{4/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$
c. $x \to 1^+$ **d.** $x \to 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \dfrac{1}{x - 1}$ **64.** $y = \dfrac{1}{x + 1}$

65. $y = \dfrac{1}{2x + 4}$ **66.** $y = \dfrac{-3}{x - 3}$

67. $y = \dfrac{x + 3}{x + 2}$ **68.** $y = \dfrac{2x}{x + 1}$

Inventing Graphs and Functions

In Exercises 69–72, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

69. $f(0) = 0, f(1) = 2, f(-1) = -2, \lim_{x\to-\infty} f(x) = -1$, and $\lim_{x\to\infty} f(x) = 1$

70. $f(0) = 0, \lim_{x\to\pm\infty} f(x) = 0, \lim_{x\to 0^+} f(x) = 2$, and $\lim_{x\to 0^-} f(x) = -2$

71. $f(0) = 0, \lim_{x\to\pm\infty} f(x) = 0, \lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to 1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = -\infty$

72. $f(2) = 1, f(-1) = 0, \lim_{x\to\infty} f(x) = 0, \lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to-\infty} f(x) = 1$

In Exercises 73–76, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

73. $\lim_{x\to\pm\infty} f(x) = 0, \lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

74. $\lim_{x\to\pm\infty} g(x) = 0, \lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

75. $\lim_{x\to-\infty} h(x) = -1, \lim_{x\to\infty} h(x) = 1, \lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

76. $\lim_{x\to\pm\infty} k(x) = 1, \lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

77. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to-\infty} (f(x)/g(x))$? Give reasons for your answer.

78. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

79. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 80–86.

80. $\lim_{x\to\infty} \left(\sqrt{x + 9} - \sqrt{x + 4}\right)$

81. $\lim_{x\to\infty} \left(\sqrt{x^2 + 25} - \sqrt{x^2 - 1}\right)$

82. $\lim_{x\to-\infty} \left(\sqrt{x^2 + 3} + x\right)$

83. $\lim_{x\to-\infty} \left(2x + \sqrt{4x^2 + 3x - 2}\right)$

84. $\lim_{x\to\infty} \left(\sqrt{9x^2 - x} - 3x\right)$

85. $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

86. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 87 and 88.

87. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

88. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 89–92.

89. $\lim_{x\to 0}\frac{-1}{x^2} = -\infty$

90. $\lim_{x\to 0}\frac{1}{|x|} = \infty$

91. $\lim_{x\to 3}\frac{-2}{(x-3)^2} = -\infty$

92. $\lim_{x\to -5}\frac{1}{(x+5)^2} = \infty$

93. Here is the definition of **infinite right-hand limit**.

> We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$c < x < c + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 93 to prove the limit statements in Exercises 94–98.

94. $\lim_{x\to 0^+}\frac{1}{x} = \infty$

95. $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

96. $\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$

97. $\lim_{x\to 2^+}\frac{1}{x-2} = \infty$

98. $\lim_{x\to 1^-}\frac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 99–104. Include the graphs and equations of the asymptotes.

99. $y = \frac{x^2}{x-1}$

100. $y = \frac{x^2+1}{x-1}$

101. $y = \frac{x^2-4}{x-1}$

102. $y = \frac{x^2-1}{2x+4}$

103. $y = \frac{x^2-1}{x}$

104. $y = \frac{x^3+1}{x^2}$

Chapter 2 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
3. Give an informal or intuitive definition of the limit
$$\lim_{x\to c} f(x) = L.$$
Why is the definition "informal"? Give examples.
4. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
5. What function behaviors might occur for which the limit may fail to exist? Give examples.
6. What theorems are available for calculating limits? Give examples of how the theorems are used.
7. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
8. What is the value of $\lim_{\theta\to 0}((\sin\theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.
9. What exactly does $\lim_{x\to c} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given f, L, c, and $\epsilon > 0$ in the precise definition of limit.
10. Give precise definitions of the following statements.
    a. $\lim_{x\to 2^-} f(x) = 5$ **b.** $\lim_{x\to 2^+} f(x) = 5$
    c. $\lim_{x\to 2} f(x) = \infty$ **d.** $\lim_{x\to 2} f(x) = -\infty$
11. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?
12. How can looking at the graph of a function help you tell where the function is continuous?
13. What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?
14. What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.
15. What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.
16. What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?
17. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.

18. What exactly do $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$ mean? Give examples.

19. What are $\lim_{x\to\pm\infty} k$ (k a constant) and $\lim_{x\to\pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.

20. How do you find the limit of a rational function as $x \to \pm\infty$? Give examples.

21. What are horizontal and vertical asymptotes? Give examples.

Chapter 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0,$ and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $f(t)$ are defined for all t and that $\lim_{t\to t_0} f(t) = -7$ and $\lim_{t\to t_0} g(t) = 0$. Find the limit as $t \to t_0$ of the following functions.

a. $3f(t)$ **b.** $(f(t))^2$

c. $f(t)\cdot g(t)$ **d.** $\dfrac{f(t)}{g(t) - 7}$

e. $\cos(g(t))$ **f.** $|f(t)|$

g. $f(t) + g(t)$ **h.** $1/f(t)$

4. Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to0} f(x) = 1/2$ and $\lim_{x\to0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a. $-g(x)$ **b.** $g(x)\cdot f(x)$

c. $f(x) + g(x)$ **d.** $1/f(x)$

e. $x + f(x)$ **f.** $\dfrac{f(x)\cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x\to0} g(x)$ must have if the given limit statements hold.

5. $\displaystyle\lim_{x\to0}\left(\frac{4 - g(x)}{x}\right) = 1$ **6.** $\displaystyle\lim_{x\to-4}\left(x \lim_{x\to0} g(x)\right) = 2$

7. On what intervals are the following functions continuous?

a. $f(x) = x^{1/3}$ **b.** $g(x) = x^{3/4}$

c. $h(x) = x^{-2/3}$ **d.** $k(x) = x^{-1/6}$

8. On what intervals are the following functions continuous?

a. $f(x) = \tan x$ **b.** $g(x) = \csc x$

c. $h(x) = \dfrac{\cos x}{x - \pi}$ **d.** $k(x) = \dfrac{\sin x}{x}$

Finding Limits

In Exercises 9–13, find the limit or explain why it does not exist.

9. $\displaystyle\lim \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

a. as $x \to 0$ **b.** as $x \to 2$

In Exercises 10–13, find the limit of $g(x)$ as x approaches the indicated value.

10. $\displaystyle\lim_{x\to0^+} (4g(x))^{1/3} = 2$ **11.** $\displaystyle\lim_{x\to\sqrt{5}} \frac{1}{x + g(x)} = 2$

12. $\displaystyle\lim_{x\to1} \frac{3x^2 + 1}{g(x)} = \infty$ **13.** $\displaystyle\lim_{x\to-2} \frac{5 - x^2}{\sqrt{g(x)}} = 0$

Limits at Infinity

Find the limits in Exercises 14–23.

14. $\displaystyle\lim_{x\to\infty} \frac{2x + 3}{5x + 7}$ **15.** $\displaystyle\lim_{x\to-\infty} \frac{2x^2 + 3}{5x^2 + 7}$

16. $\displaystyle\lim_{x\to-\infty} \frac{x^2 - 4x + 8}{3x^3}$ **17.** $\displaystyle\lim_{x\to\infty} \frac{1}{x^2 - 7x + 1}$

18. $\displaystyle\lim_{x\to-\infty} \frac{x^2 - 7x}{x + 1}$ **19.** $\displaystyle\lim_{x\to\infty} \frac{x^4 + x^3}{12x^3 + 128}$

20. $\displaystyle\lim_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

21. $\displaystyle\lim_{\theta\to\infty} \frac{\cos\theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

22. $\displaystyle\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

23. $\displaystyle\lim_{x\to\infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

Horizontal and Vertical Asymptotes

24. Use limits to determine the equations for all vertical asymptotes.

a. $y = \dfrac{x^2 + 4}{x - 3}$

b. $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1}$

c. $y = \dfrac{x^2 + x - 6}{x^2 + 2x - 8}$

25. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \dfrac{1 - x^2}{x^2 + 1}$ **b.** $f(x) = \dfrac{\sqrt{x} + 4}{\sqrt{x + 4}}$

c. $g(x) = \dfrac{\sqrt{x^2 + 4}}{x}$ **d.** $y = \sqrt{\dfrac{x^2 + 9}{9x^2 + 1}}$

Chapter 2 Single Choice Questions

1. The value of $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$ equals

a. –ln 3 **b.** 0 **c.** –ln 2 **d.** –ln 5

2. Let $P(x) = a_1x + a_2x^2 + a_3x^3 + ... + a_{100}x^{100}$, where $a_1 = 1$ and $a_i \in R\ \forall\ i = 2, 3, 4,\ldots 100$. Then $\lim_{x\to 0} \frac{\sqrt[100]{1+P(x)}-1}{x}$ has the value equal to

a. 100 **b.** $\frac{1}{100}$ **c.** 1 **d.** 5050

3. Given that $\prod_{n=1}^{n} \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$

Let $f(x) = \begin{cases} \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right), & x \in (0,\pi) - \left\{\frac{\pi}{2}\right\} \\ \frac{2}{\pi}, & x = \frac{\pi}{2} \end{cases}$

Then which one of the following alternative is true?

a. $f(x)$ has non-removable discontinuity of infinite type at $x = \frac{\pi}{2}$

b. $f(x)$ has missing point discontinuity at $x = \frac{\pi}{2}$

c. $f(x)$ is continuous at $x = \frac{\pi}{2}$

d. $f(x)$ has non-removable discontinuity of infinite type at $x = \frac{\pi}{2}$

4. Let $f(x) = \lim_{h\to 0} \frac{(\sin(x+h))^{\ln(x+h)} - (\sin x)^{\ln x}}{h}$ then $f\left(\frac{\pi}{2}\right)$ is

a. equal to 0 **b.** equal to 1

c. $\ln\frac{\pi}{2}$ **d.** non-existent

5. The $\lim_{x\to 0}\left(\left(1+\frac{1}{x}\right)^x + \left(\frac{1}{x}\right)^x + (\tan x)^{\frac{1}{x}}\right)$ is equal to

a. 0 **b.** 1 **c.** 2 **d.** 3

6. The value of $\lim_{n\to\infty}\left(\frac{n^{n+1}.\mathrm{e}}{(n+1)^n} - n\right)$

a. 1 **b.** $\frac{1}{2}$ **c.** $\frac{3}{4}$ **d.** 2

7. The value of $\lim_{x\to 0}\left(\frac{\mathrm{e}}{4x} - \frac{\mathrm{e}}{2x(e^{ex}+1)}\right)$ equals

a. $\frac{\mathrm{e}^2}{2}$ **b.** $\frac{\mathrm{e}^2}{4}$ **c.** $\frac{\mathrm{e}^2}{8}$ **d.** $\frac{\mathrm{e}}{2}$

8. If $f(x)$ is a quadratic expression such that $f(-2) = f(2) = 0$ and $f(1) = 6$, then $\lim_{x\to 0} \frac{\sqrt{f(x)} - 2\sqrt{2}}{\ln(\cos x)}$ is equal to

a. – 4 **b.** 4

c. $\frac{1}{\sqrt{2}}$ **d.** $-\frac{1}{\sqrt{2}}$

9. The value of $\lim_{x\to 0}\left(5x^2 + [x^2+1]\right)^{\left(\frac{1}{x^2+\sin^2 x}\right)}$ is

[Note: [y] denotes greatest integer function less than or equal to y.]

a. $\frac{5}{2}$ **b.** $e^{\frac{5}{2}}$ **c.** $\frac{25}{2}$ **d.** $e^{\frac{25}{2}}$

10. If the value of $\lim_{x\to 0^+}\left(\frac{(3/x)+1}{(3/x)-1}\right)^{1/x}$ can be expressed in the form of $e^{\frac{p}{q}}$, where p and q are relatively prime then the value of $(p+q)$, is

a. 2 **b.** 3 **c.** 4 **d.** 5

11. The value of $\lim_{x\to -\infty} \frac{3^{\sin x} + 2x + 1}{\sin x - \sqrt{x^2+1}}$ is equal to

a. – 2 **b.** 0 **c.** 2 **d.** does not exist

12. The value of $\lim_{x\to a}(\log_a x)^{\frac{1}{x-a}}$, where $0 < a < 1$, is equal to

a. $e^{\frac{\ell n\, a}{a}}$ **b.** $e^{a\ln a}$

c. $e^{\frac{1}{a\ell n a}}$ **d.** $e^{\frac{a}{\ell n a}}$

13. Let $f(x) = \sqrt{\frac{\mathrm{sgn}(\alpha x^2 + \alpha x + 1)}{\cot^{-1}(x^2 - \alpha)}}$. If $f(x)$ is continuous for all $x \in R$, then number of integers in the range of α, is

a. 0 **b.** 4 **c.** 3 **d.** 6

[Note: sgn k denotes signum function of k.]

14. Let $f(x) = \begin{cases} [\cos x], & -\pi < x \le 2 \\ x-2, & x > 2 \end{cases}$

Number of points where $f(x)$ is discontinuous in $(-\pi, \infty)$ is

[Note: [k] denotes greatest integer less than or equal to k.]

a. 3 **b.** 4 **c.** 5 **d.** 6

15. Let $f(x) = \begin{cases} \frac{e^x \cdot e^{2x} \cdot e^{2^2 x} .....e^{2^{10}x} - 1}{x}, & x \ne 0 \\ \lambda, & x = 0 \end{cases}$ be a continuous function in $(-\infty, \infty)$ then λ is equal to

a. 1024 **b.** 2048 **c.** $2^{10} - 1$ **d.** $2^{11} - 1$

16. Let $f(x) = \begin{cases} \left(\dfrac{x\sin x + 2\cos 2x}{2}\right)^{\frac{1}{x^2}}, & x \neq 0 \\ e^{-\frac{k}{2}}, & x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then the value of k is

a. 1 **b.** 2 **c.** 3 **d.** 4

17. The value of $\lim\limits_{x\to 2} \dfrac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}}$ is

a. 16 **b.** 8 **c.** 4 **d.** 2

18. If $f(x) = \begin{cases} 2x-1, & -2 \le x < 0 \\ x+2, & 0 \le x \le 2 \end{cases}$ and $g(x) = \begin{cases} [x], & -4 \le x < -2 \\ x+2, & -2 \le x \le 4 \end{cases}$, then

a. $\lim\limits_{x\to 2^+} f(g((x)) = 2$

b. $f(g(x))$ is discontinuous at $x = -2$

c. $f(g(x))$ is not defined at $x = 2$

d. None of these

19. A function $f(x)$ is defined as below

$f(x) = \dfrac{\cos(\sin x) - \cos x}{x^2}$, $x \neq 0$ and $f(0) = a$

$f(x)$ is continuous at $x = 0$ if a equals

a. 0 **b.** 4 **c.** 5 **d.** 6

20. If the function $f(x) = \begin{cases} \left(\cot\left(x + \dfrac{\pi}{4}\right)\right)^{\frac{1}{x}}, & x \neq 0 \\ e^{\alpha}, & x = 0 \end{cases}$ is continuous at $x = 0$, then a equals

a. 2 **b.** – 1 **c.** – 2 **d.** 1

21. Let $f(x) = \begin{cases} \left(\dfrac{1+2x}{e^x}\right)^{\frac{1}{x}}, & x \neq 0 \\ k, & x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then the value of k is

a. 1 **b.** e **c.** e^2 **d.** e^{-2}

22. Let $L = \lim\limits_{x\to 0} \dfrac{3\lambda x + (\lambda - 2)\sin x}{(\sin^{-1} x)^3}$, where $\lambda \in R$. If L is finite then $(L + \lambda)$ is equal to

a. $\dfrac{1}{4}$ **b.** $\dfrac{1}{2}$ **c.** $\dfrac{3}{4}$ **d.** $\dfrac{5}{4}$

23. Let $f(x) = \begin{cases} \dfrac{1}{|x-1|}, & -\infty < x < 0 \\ |x|^2 + 1, & 0 \le x < 1 \\ 2x, & 1 \le x < 2 \\ \dfrac{x^2-1}{x-1}, & \text{otherwise} \end{cases}$

then $f(x)$ is discontinuous in $(-\infty, \infty)$ at

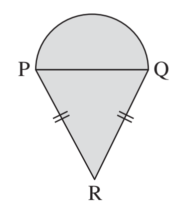
a. two points **b.** three points

c. one point **d.** no point

24. If $f(x) = \begin{cases} (\cot x)^{\cos x} \cdot (\cos x)^{\cot x}, & x \neq \dfrac{\pi}{2} \\ k, & x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$, then k is equal to

a. 0 **b.** $\dfrac{1}{2}$ **c.** 1 **d.** 2

25. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like an ice-cream cone, as shown in the figure, where ∠PRQ = θ. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, then $\lim\limits_{\theta\to 0^+} \dfrac{A(\theta)}{B(\theta)}$ is



a. $\dfrac{\pi}{2}$ **b.** $\dfrac{\pi}{4}$ **c.** 0 **d.** $\dfrac{1}{2}$

26. Let for $k > 0$, $f(x) = \begin{cases} \dfrac{k^x + k^{-x} - 2}{x^2} & \text{if } x > 0 \\ 3\ell n(k - x) - 2 & \text{if } x \le 0 \end{cases}$. If $f(x)$ is continuous at $x = 0$, then k is equal to

a. e or 1 **b.** 1 or 2 **c.** e or e^2 **d.** only e^2

27. The value of $\lim\limits_{x\to 0} (x^{16} + 4x^{12} - 3x^4)\left[\dfrac{1}{x^4}\right]$, (where [·] denotes greatest integer function), is

a. 0 **b.** – 3 **c.** 4 **d.** Non-existent

28. $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{\ell n(\cos(2x^2 - x))}$ is equal to

a. 2 **b.** – 2 **c.** 1 **d.** – 1

29. If $f(x) = \begin{cases} \dfrac{1 - \sin x}{(\pi - 2x)^2} \cdot \dfrac{\log \sin x}{\log(1 + \pi^2 - 4\pi x + 4x^2)}, & x \neq \dfrac{\pi}{2} \\ k, & x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \pi/2$, then k is

a. $-\dfrac{1}{16}$ **b.** $-\dfrac{1}{32}$ **c.** $-\dfrac{1}{64}$ **d.** $-\dfrac{1}{128}$

30. $\lim\limits_{x\to 0} \left[\dfrac{2}{x^3}(\tan x - \sin x)\right]^{2/x^2}$ is

a. e **b.** $\sqrt{e}$ **c.** $\dfrac{1}{e}$ **d.** $\dfrac{1}{\sqrt{e}}$

31. If $a = \left[\lim_{x\to\infty} \cos\sin^{-1}\frac{x}{\sqrt{x^3+x}}\right]$ and $b = \lim_{x\to\infty}\left[\sin\cos^{-1}\frac{x}{\sqrt{x^3+x}}\right]$ (where [·] denotes greatest integer function), then

a. $a+b=1$ **b.** $a+b=2$
c. $a+b=0$ **d.** $a=b$

32. If $\alpha_n \in \left(0, \frac{\pi}{4}\right)$ is a root of the equation $\tan\alpha + \cot\alpha = n$, $n \geq 2$, then $\lim_{x\to\infty}(\sin\alpha_n + \cos\alpha_n)^n$

a. 0 **b.** 1 **c.** 2 **d.** e

33. If $= \lim_{x\to\lambda}\left(2-\frac{\lambda}{x}\right)^{\lambda\tan\left(\frac{\pi x}{2\lambda}\right)}$, $\frac{1}{e}$ then λ is equal to

a. $-\pi$ **b.** π **c.** $\frac{\pi}{2}$ **d.** $-\frac{2}{\pi}$

34. If $L = \lim_{x\to 0}\frac{\ell n(1+3\sin^2 x)(e^{2x}-2e^x+1)\sin(x+1)}{(1-\cos 5x)(\sin^{-1}x)^2(x+1)}$, then $\frac{25L}{6}$ is

a. less than sin 2 **b.** greater than sin 2
c. equal to sin 2 **d.** is equal to 1

35. The value of $\lim_{x\to 0}\frac{e^{x\cos x}-1-x}{\sin x^2}$ is equal to

a. 0 **b.** 1/2 **c.** 1 **d.** 2

36. If the function $f(x) = 2a[2x+5] + b[3x-7]$ is continuous at $x = 1$, then (where [·] denotes greatest integer function and a and b are non-zero)

a. $2a+b=0$ **b.** $a+2b=0$
c. $a+b=0$ **d.** $10a-7b=0$

37. $\lim_{x\to 0}\frac{(\cos x)\ell n(1+x) - x\cos x - \ell n(1+x) + x}{x^4}$ is equal to

a. 0 **b.** $\frac{1}{2}$ **c.** $\frac{1}{4}$ **d.** $\frac{1}{8}$

38. Let $f(x)$ be a quadratic function such that $f(0) = f(1) = 0$ and $f(2) = 1$, then $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}\cos^2 x\right)}{f^2(x)}$ is equal to

a. $\frac{\pi}{2}$ **b.** π **c.** 2π **d.** 4π

39. If $\lim_{n\to\infty}(\sqrt{2n^2+n} - \lambda\sqrt{2n^2-n}) = \frac{1}{\sqrt{2}}$ (where λ is a real number), then

a. $\lambda = 1$ **b.** $\lambda = -1$
c. $\lambda = \pm 1$ **d.** $\lambda \in (-\infty, 1)$

40. If a_n $(n \geq 1)$ is a sequence of real number such that $a_1 = 12$ and $a_{n+1} = a_n\left(1+\frac{3}{n+1}\right)$, then $\left(\lim_{r\to\infty}\sum_{k=1}^{r}\frac{1}{a_k}\right)$ is

a. $\frac{1}{12}$ **b.** $\frac{1}{6}$
c. $\frac{1}{3}$ **d.** 1

41. Let $L = \lim_{x\to\infty}\left(x\ell nx + 2x\ell n\left(\sin\frac{1}{\sqrt{x}}\right)\right)$, then the value of $\left(-\frac{2}{L}\right)$ is equal to

a. $-\frac{1}{3}$ **b.** 6 **c.** -3 **d.** $-\frac{1}{6}$

42. $\lim_{x\to 0}\frac{\sin[\pi^2]x - \sin[-\pi^2]x}{\tan^2\sqrt{x}}$ is equal to (where [·] denotes greatest integer function)

a. 0 **b.** 18 **c.** 19 **d.** does not exist

43. $\lim_{x\to 0}\frac{\sin x^{2012} - x^{2012}\cos x^{2012} + (x^3)^{2012}}{x^{2012}(e^{5.x^{2012}} - 1 - 5.x^{2012})}$ is equal to

a. $\frac{8}{75}$ **b.** $\frac{1}{3}$ **c.** $\frac{4}{3}$ **d.** $-\frac{1}{3}$

44. If the function $f(x) = \begin{cases} 2x^2-3 & \text{if } 0 < x \leq 1 \\ x^2+bx-1 & \text{if } 1 < x < 2 \end{cases}$ is continuous at every point of its domain, then the value of b is

a. -1 **b.** 0 **c.** 0 **d.** 2

45. Let α, β be the roots of the equation $6x^2 - x - 1 = 0$, then the number of points of discontinuity of the function $g(t) = \frac{1}{(\sin t-\alpha)(\sin t-\beta)}$ on $\left(0, \frac{5\pi}{2}\right]$ are

a. 2 **b.** 10 **c.** 5 **d.** 12

46. If $[x]$ denotes the integral part of x and $f(x) = [x]\left(\frac{\sin\frac{\pi}{[x+3]} + \sin\pi[x+3]}{3+[x]}\right)$ then

a. $f(x)$ is continuous on R
b. $f(x)$ is a periodic function
c. $f'(x)$ exists $\forall\, x \in R$
d. $f(x)$ is discontinuous for all integral points in R

47. Let $f: R \to Q$ be a continuous function such that $f(2) = 7$, then

a. $f(x)$ is always an even function
b. $f(x)$, is always an odd function
c. Nothing can said about $f(x)$ being even or odd
d. $f(x)$ is an increasing function

48. If $f(x) = \begin{cases} -1 & , & x < 0 \\ 0 & , & x = 0 \\ 1 & , & x > 0 \end{cases}$ and $g(x) = \sin x + \cos x$ then number of points of discontinuity of $f(g(x))$ in $(0, 2\pi)$ are

a. 2 **b.** 3 **c.** 4 **d.** 5

49. Let $f(x) = \begin{cases} \sqrt{x^2-1} & ; \quad x \le \sqrt{10} \\ (\sqrt{10})x - 7 & ; \quad \sqrt{10} < x < 5 \\ \sin \pi x & ; \quad 5 \le x < 6 \\ \{x\} & ; \quad 6 \le x \le 7 \end{cases}$,

where $\{\cdot\}$ denotes fractional part function, then number of points where $f(x)$ is discontinuous in $[1, 7]$

a. 0 **b.** 1 **c.** 2 **d.** 3

50. Let $f(x)$ is a continuous function in $[a, b]$ such that $f(a) \ne f(b)$ and $f(a), f(b) > 0$, then which of the following is incorrect ?

a. if $f(c) = \dfrac{f(a)+f(b)}{2}$, then there always exist at least one "c" in (a, b)

b. if $f(c) = \sqrt{f(a)\times f(b)}$ then there always exist at least one "c" in (a, b)

c. If $5f(c) = 3f(a) + 2f(b)$, then there always exist at least one "c" in (a, b)

d. If $2f(c) = 3f(a) - f(b)$, then there always exists at least one "c" in (a, b)

Chapter 2 Multiple Choice Questions

1. If $\lim\limits_{x\to\infty}\left(\sqrt{x^2-x+1}-ax-b\right) = 0$, then for $k \ge 2$, $k \in N$ which of the following is/are correct?

a. $2a + b = 0$

b. $a + 2b = 0$

c. $\lim\limits_{n\to\infty} \sec^{2n}(k!\pi b) = 1$

d. $\lim\limits_{n\to\infty} \sec^{2n}(k!\pi a) = 1$

2. Which of the following statements is/are correct?

a. Let f and g be defined on R and c be any real number. If $\lim\limits_{x\to c} f(x) = b$ and $g(x)$ is continuous at $x = b$ then $\lim\limits_{x\to c} g(f(x)) = g(b)$

b. There exists a function $f: [0, 1] \to R$ which is discontinuous at every point in $[0, 1]$ and $|f(x)|$ is continuous at every point in $[0, 1]$

c. If $f(x)$ and $g(x)$ are two continuous functions defined from $R \to R$ such that $f(r) = g(r)$ for all rational numbers "r" then $f(x) = g(x)\ \forall\ x \in R$

d. If $f(a)$ and $f(b)$ possesses opposite signs then there must exist at least one solution of the equation $f(x) = 0$ in (a, b) provided f is continuous in $[a, b]$.

3. Let $L = \lim\limits_{x\to 1}\dfrac{\sin(6\cos^{-1}x)}{\sqrt{1-x^2}}$ and $M = \lim\limits_{x\to 1}\dfrac{1-\cos(6\cos^{-1}x)}{1-x^2}$. Which of the following is/are correct?

a. $L + M = 24$ **b.** $\dfrac{M}{L} = 3$

c. $L - M = 1$ **d.** $LM = 54$

4. The value of $\lim\limits_{n\to\infty}\left(\dfrac{2x}{\pi}\cot^{-1}(nx) - x\right)$ is/are

a. $x, x > 0$ **b.** $x, x < 0$

c. $-x, x > 0$ **d.** $-x, x < 0$

5. If $\lim\limits_{x\to 1}\dfrac{9^{3^{2\ell nx}} - 9^{2^{\ell nx}}}{\ln x} = \ln(3^p)\cdot\ln\left(\dfrac{9}{q}\right)$ where p and $q \in N$, then the value of $(p + q)$, is

a. 18 **b.** 19 **c.** 20 **d.** 21

6. If $\lim\limits_{x\to\infty}\left(a\sqrt{2x^2+x+1}-bx\right) = 2$ $(a, b \in R)$ then the value of $(a^2 + b^2)$ is greater than

a. 32 **b.** 48 **c.** 64 **d.** 96

7. Which of the following statements is (are) correct?

a. If $\dfrac{\sin x + x}{x} < f(x) < \dfrac{x^2+x+1}{x^2+1}$ for all $x > 5$, then $\lim\limits_{x\to\infty} f(x) = 1$.

b. If f is continuous on $[-1, 1]$ such that $f(-1) = 2$ and $f(1) = 10$ then there exists a number c such that $|c| < 1$ and $f(c) = e^2$, where "e" is Napier constant.

c. If $f(x)$ and $g(x)$ both are discontinuous at $x = c$, then the product function $f(x) \times g(x)$ must be discontinuous at $x = c$.

d. If $f(x)$ and $g(x)$ both are continuous at $x = c$, then $g \circ f(x)$ must be continuous at $x = c$.

8. $\lim\limits_{x\to\infty} 2\left(\sqrt{25x^2+x+5}-5x\right)$ is equal to

a. $\lim\limits_{x\to 0}\dfrac{2x-\log_e(1+x)^2}{5x^2}$

b. $\lim\limits_{x\to 0}\dfrac{e^{-x}-1+x}{x^2}$

c. $\lim\limits_{x\to 0}\dfrac{2(1-\cos x^2)}{5x^2}$

d. $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{5}}{x}$

9. Given

$$f(x) = \begin{cases} \dfrac{a(1-x\sin x)+b\cos x+7}{x^2} & \text{if } x < 0 \\ = 5 & \text{if } x = 0 \\ = \left\{1+\left(\dfrac{cx+dx^3}{x^2}\right)\right\}^{1/x} & \text{if } x > 0 \end{cases}$$

such that $f(x)$ is continuous at $x = 0$ then,

a. $a = -3$ **b.** $b = -4$

c. $c = 1$ **d.** $d = \log_e 5$

10. $\lim_{x\to-\infty} \dfrac{x^5 \sin\left(\dfrac{1}{\pi x^2}\right) + 3|x|^2 + 70}{|x|^3 + 7|x| + 18}$ is equal to

a. $-\dfrac{1}{\pi}$ **b.** 0

c. ∞ **d.** None of these

11. Function whose jump (non-negative difference of LHL and RHL) of discontinuity is greater than or equal to one is/are

a. $f(x) = \begin{cases} \dfrac{(e^{1/x}+1)}{(e^{1/x}-1)} & ; \quad x<0 \\ \dfrac{(1-\cos x)}{x} & ; \quad x>0 \end{cases}$

b. $g(x) = \begin{cases} \dfrac{(x^{1/3}-1)}{x^{1/2}-1} & ; \quad x<0 \\ \dfrac{\ln x}{(x-1)} & ; \quad \dfrac{1}{2}<x<1 \end{cases}$

c. $u(x) = \begin{cases} \dfrac{\sin^{-1} 2x}{\tan^{-1} 3x} & ; \quad x \in \left(0, \dfrac{1}{2}\right] \\ \dfrac{|\sin x|}{x} & ; \quad x<0 \end{cases}$

d. $v(x) = \begin{cases} \log_3(x+2) & ; \quad x>2 \\ \log_{1/2}(x^2+5) & ; \quad x<2 \end{cases}$

12. Let $f(x) = g(x) \times \dfrac{2e^{1/x}+3e^{2/x}+4e^{3/x}}{4e^{1/x}+3e^{2/x}+2e^{3/x}}$, $x \neq 0$ be such that $\lim_{x\to0} f(x)$ exists, then $g(x)$ can be

a. $g(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & ; \quad x>0 \\ \dfrac{1-\cos 2x}{x^2} & ; \quad x<0 \end{cases}$

b. $g(x) = \begin{cases} [x] & ; \quad x>0 \\ 1+[x] & ; \quad x<0 \end{cases}$ (where [·] denotes greatest integer function)

c. $g(x) = \begin{cases} \exp.\left(\dfrac{1-e^x}{x^2}\right) & ; \quad x>0 \\ (1+\tan^2 x)^{1/x^3} & ; \quad x<0 \end{cases}$

d. $g(x) = \begin{cases} \dfrac{|x|}{x} & ; \quad x>0 \\ \dfrac{\sin 4x}{|\sin x|} & ; \quad x<0 \end{cases}$

13. If $A = \lim_{x\to0} \dfrac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$ and $B = \lim_{x\to0} \dfrac{[|x|]}{x}$, then (where [·] denotes greatest integer function)

a. $A = 1$ **b.** A does not exist

c. $B = 0$ **d.** $B = 1$

14. If $\lim_{x\to0} \dfrac{k+\cos \ell x}{x^2}$ exists and has the value equal to –4 then the ordered pair (k, ℓ) is

a. $(-1, 2\sqrt{2})$ **b.** $(-1, 2)$

c. $(-1, +-2\sqrt{2})$ **d.** $(-1, -2)$

15. $\lim_{x\to\infty} \dfrac{x^p + x^{p-1} + 1}{x^q + x^{q-2} + 2}$, where $p > 0$, $q > 0$ is

a. 0 if $p > q$ **b.** 1 if $p = q$

c. infinite if $p > q$ **d.** 1 if $p > q$

16. If x_i, $i = 1, 2, \ldots, n$ are the point of discontinuity of $f(x) = \{x[x]^2\}$ in (0, 2) then (where [·] and {·} denotes greatest integer and fractional part function, respectively.)

a. $\left[\sum_{i=1}^{n} x_i^2\right] = 7$

b. $\left[\sum_{i=1}^{n} x_i^2\right] = 8$

c. $\left[\sum_{i=1}^{n} x_i^2\right] = 4$

d. Number of $x_i \in N$ is equal to zero

17. If function $f(x) = \begin{cases} \text{sgn}(ax) & ; \quad x<0 \\ x+1 & ; \quad 0 \le x \le 1 \\ \dfrac{2x}{|x-b|} & ; \quad x>1 \end{cases}$ is discontinuous exactly at two points in R, then which options(s) can be true

a. $a = 0, b = 2$

b. $a < 0, b > 3$

c. $a > 0, b = 0$

d. $a > 0, b < -1$

18. If functions g and h are defined as $g(x) = \begin{cases} x^2+1 & x \in Q \\ px^2 & x \notin Q \end{cases}$ and $h(x) = \begin{cases} px & x \in Q \\ 2x+q & x \notin Q \end{cases}$. If $(g + h)(x)$ is continuous at $x = 1$ and $x = 3$, then

a. $3p + q = 2$ **b.** $3p + q = 1$

c. $3p - q = 2$ **d.** $3p - q = 1$

Chapter 2 Passage Type Questions

Passage 1

Let $f(x) = \lim\limits_{x\to\infty}(1 - \sin x + \sqrt[n]{e}\sin x)^n\ n \in N$

$$a = \frac{2}{11}\lim_{x\to 0}\left(\left[\frac{\sin x}{x}\right] + \left[\frac{2\sin x}{x}\right] + \left[\frac{3\sin x}{x}\right] + \dots + \left[\frac{11\sin x}{x}\right]\right)$$

$$b = \lim_{x\to 0}\left(\frac{x^2}{\left[\frac{\tan x}{x}\right] - \cos x}\right)$$

[Note: $[k]$ denotes greatest integer function less than or equal to k.]

1. The value of $(a + b)$ is equal to

 a. 2 **b.** 6 **c.** 10 **d.** 12

2. Number of integral values of λ so that the equation $bx^2 - b^2x + \lambda = 0$ has roots α, β such that $1 < \alpha < 2$ and $2 < \beta < 3$ is

 a. 0 **b.** 1 **c.** 2 **d.** 3

3. If $\lim\limits_{u\to 0}\ [1 + u\ln(1 + k^2)]^{1/u} = 2k\ln^2 f(x)$, $k > 0$ and $x \in (0, \pi)$, then the value of $(x + k)$ is equal to

 a. $1 + \frac{\pi}{4}$ **b.** $1 + \frac{\pi}{2}$ **c.** 1 **d.** $\frac{\pi}{2}$

Passage 2

Let $f(x) = \begin{cases} x+3 & ; & -2 < x < 0 \\ 4 & ; & x = 0 \\ 2x+5 & ; & 0 < x < 1 \end{cases}$

then

1. $\lim\limits_{x\to 0^-} f([x - \tan x])$ is ($[\cdot]$ denotes greatest integer function)

 a. 2 **b.** 4 **c.** 5 **d.** None of these

2. $\lim\limits_{x\to 0^+} f\left(\left\{\frac{x}{\tan x}\right\}\right)$ is ($\{\cdot\}$ denotes fractional part of function)

 a. 4 **b.** 5 **c.** 7 **d.** None of these

Passage 3

A certain function $f(x)$ has the property that $f(3x) = a\,f(x)$ for all positive real values of x and $f(x) = 1 - |x - 2|$ for $1 \le x \le 3$

1. $\lim\limits_{x\to 2}(f(x))^{\operatorname{cosec}\left(\frac{\pi x}{2}\right)}$ is

 a. $\frac{2}{\pi}$ **b.** $-\frac{2}{\pi}$ **c.** $e^{2/x}$ **d.** Non-existent

2. If the total area bounded by $y = f(x)$ and x-axis in $[1, \infty)$ converges to a finite quantity, then the range of a is

 a. $(-1, 1)$ **b.** $\left(-\frac{1}{2}, \frac{1}{2}\right)$

 c. $\left(-\frac{1}{3}, \frac{1}{3}\right)$ **d.** $\left(-\frac{1}{4}, \frac{1}{4}\right)$

Passage 4

If $f(x) = \cot^{-1}\left(\frac{1}{x} + 2x\right) + \cot^{-1}\left(\frac{1}{x} + 6x\right) + \cot^{-1}\left(\frac{1}{x} + 12x\right) + \cot^{-1}\left(\frac{1}{x} + 20x\right) + \dots \infty, \forall x \in (0, \infty)$

$$a = \lim_{x\to 0}\frac{\ln(1 + 2x + 4x^2 + 6x^3)}{\ln(1 + x + x^2 + x^3)}$$

$$b = \lim_{x\to 0}\frac{\sqrt[4]{1+x} - \sqrt[5]{1+x}}{\tan^{-1}x}$$

$$c = \lim_{x\to 0}\frac{\prod_{r=1}^{10}(1 + rx) - 1}{11x}$$

1. $f(x)$ is

 a. $\cot^{-1}x$ **b.** $\cot^{-1}\frac{x}{2}$

 c. $\frac{1}{2}\cot^{-1}x$ **d.** $\tan^{-1}\frac{x}{2}$

2. Which is correct?

 a. $f(a) > f(b)$ **b.** $f(b) > f(a)$

 c. $f(a) + f(10b) < \frac{\pi}{2}$ **d.** $ab = 10$

3. If $\sin^{-1}\frac{c}{x} + \sin^{-1}\frac{6a}{x} = \frac{\pi}{2}$, then the value of x is

 a. 3 **b.** 10 **c.** 13 **d.** 15

Passage 5

Consider a sequence $\{a_n\}$ such that $a_1 = 0$ and $\frac{1}{1 - a_{n+1}} - \frac{1}{1 - a_n} = 2n + 1$ ($n = 1, 2, 3, \dots$). Also $b_k = \sqrt{\frac{k+1}{k}}\left(1 - \sqrt{a_{k+1}}\right)$ ($k = 1, 2, 3, \dots$)

On the basis of above information, answer the following questions.

1. $\lim\limits_{n\to\infty} a_n$ is equal to

 a. 0 **b.** 1 **c.** 2 **d.** 3

2. $\prod\limits_{n=2}^{\infty} a_n$ is equal to

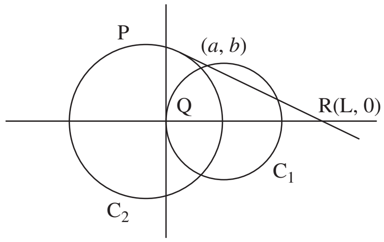
 a. $\frac{1}{2}$ **b.** 1 **c.** $\frac{3}{4}$ **d.** Infinite

3. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} b_k$ is equal to

 a. $\sqrt{2}$ **b.** 2 **c.** $\sqrt{2} - 1$ **d.** $\sqrt{2} + 1$

Passage 6

The figure shows a fixed circle C_1 and shrinking circle C_2.The equation of the fixed circle is $(x-1)^2 + y^2 = 1$. The center of shrinking circle C_2 is at origin and it cuts positive y-axis at P. The points of intersection of C_1 and C_2 is $Q(a, b)$, where $b > 0$, PQ cuts x-axis at $R(L, 0)$. Let r be the radius of circle C_2 and let $L = f(r)$. On the basic of above information, answer the following questions:



1. $\lim\limits_{r\to 0} f(r)$ is equal to

   a. 0 **b.** 2 **c.** 4 **d.** 8

2. If $f(r)\cdot g(r) = r^2$, then $\lim\limits_{\theta\to 0}\dfrac{g(2\sin\theta)}{\theta^2}$ is

   a. 0 **b.** 1 **c.** 1/2 **d.** 2

3. Range of f is

   a. [2, 4] **b.** (2, 4] **c.** [2, 4) **d.** (2, 4)

Passage 7

Given $g(x) = \sin\left(\dfrac{1}{x}\right)$, $h(x) = x^2 + 2x + (\lambda + 1)$ and $u(x) = \dfrac{1}{x} + \cos\dfrac{1}{x^2}$

Let $f(x) = \lim\limits_{n\to\infty}\dfrac{x^{2n+1}g(x) + h(x)}{x^{2n} + 3xu(x)}$

On the basis of above information, answer the following questions:

1. If $\lim\limits_{x\to 0} f(x) = 2$, then the value of λ is

   a. 10 **b.** 5 **c.** 2 **d.** does not exist

2. $\lim\limits_{x\to\infty} f(x)$ is equal to

   a. 0 **b.** 2 **c.** 5 **d.** 1

3. If $\lim\limits_{x\to -2} f(x) = L$, then $[L]$ (where [·] denotes greatest integer function) is equal to

   a. 0 **b.** 1 **c.** –1 **d.** 2

Chapter 2 Matrix Match Type Questions

1. Match the function in column-I with its behavior at $x = 0$ in column-II, where [·] denotes greatest integer function and sgn(x) denotes signum function.

Column-I	Column-II
(a) $f(x) = [x]\,[1 + x]$	(p) LHL exist at $x = 0$
(b) $f(x) = [-x]\,[1 + x]$	(q) RHL exist at $x = 0$
(c) $f(x) = (\text{sgn}(x))\,[2 - x]\,[1 + [x]]$	(r) Continuous at $x = 0$
(d) $f(x) = [\cos x]$	(s) $\lim\limits_{x\to 0} f(x)$ exists but function is discontinuous at $x = 0$
	(t) $\lim\limits_{x\to 0} f(x)$ does not exist

2.

Column-I	Column-II
(a) $\lim\limits_{n\to\infty}\left[\sum\limits_{r=1}^{n}\dfrac{1}{2^r}\right]$ is equal to (where [·] denotes greatest integer function)	(p) 0
(b) $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{2x-\pi}}$	(q) 1

(*Continued*)

Column-I	Column-II
(c) $\lim\limits_{n\to\infty}\left[n^2\sin\left(\pi\cos^2\left(\tan\sin\dfrac{1}{n}\right)\right)\right]$ is equal to (where [·] denotes greatest integer function)	(r) 2
(d) The absolute value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin^2 x}-\cos x}{1-\sqrt{1+\tan^2 x}}$ is	(s) 3

3.

Column-I	Column-II
(a) For $f(x) = \left[\dfrac{x}{[x]}\right]$ at $x = 2$	(p) L.H.L. = 0
(b) For $g(x) = \left[x\left[\dfrac{1}{x}\right]\right]$ at $x = \dfrac{1}{2}$	(q) R.H.L. = 1
(c) For $h(x) = \left[\dfrac{[x]}{x}\right]$ at $x = 2$ (where [·] denotes greater integer function)	(r) Limit exists
	(s) Function is discontinuous

4.

Column-I	Column-II
(a) If $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{1/x} = e^2$, then	(p) $a+b=4$
(b) If $\lim\limits_{x\to 0}(1+ax+bx^2)^{2/x} = e^3$, then	(q) $a+b=12$
(c) If $\lim\limits_{x\to 0}\dfrac{ae^x - b}{x} = 2$, then	(r) $ab=2$
(d) If $\lim\limits_{x\to 0}\dfrac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$ exists and finite then	(s) $a+b=3$

5.

Column-I	Column-II
(a) If $f(x) = \text{sgn}(x^2+ax+1)$ has exactly one point of discontinuity then a can be	(p) 0
[**Note:** sgn x denotes the signum function of x.]	(q) 1
(b) $\lim\limits_{x\to 0}\sin^{-1}\sin\cot^{-1}\left(\dfrac{1}{x}\right)$ equal to	(r) 2
(c) If $\lim\limits_{x\to 0}\dfrac{a\tan 3x + (1-\cos 2x)}{x+\sin x+\tan x} = 1$, then the value of a is equal to	(s) 3
(d) If $\lim\limits_{x\to 1}\sec^{-1}\left(\dfrac{k}{\ell nx} - \dfrac{k}{x-1}\right) = \dfrac{\pi}{3}$, then k is	(t) 4

Chapter 2 Integer Type Questions

1. Find the sum of an infinite geometric series whose first term is $\lim\limits_{x\to 0}\sum\limits_{k=1}^{2011}\dfrac{\left\{\dfrac{x}{\tan x}+2k\right\}}{2011}$ and whose common ratio is the value of $\dfrac{\lim\limits_{x\to 0} e^{\tan^3 x} - e^{x^3}}{2\ln(1+x^3\sin^2 x)}$ [Note: $\{y\}$ denotes fractional part of y.]

2. For a certain value of C, the $\lim\limits_{x\to+\infty}(x^4+5x^3+3)^C - x$ is finite and non-zero quantity L. Find $(L - C)$

3. Let $a, b, c \in R$. If

$$f(x) = \begin{cases} \dfrac{\sin(ax^2+bx+c)}{x^2-1}, \text{where } (a+b+c)=\pi, & \text{if } x<1 \\ -1, & \text{if } x=1 \\ a\,\text{sgn}(x+1)\cos(2x-2)+bx^2, & \text{if } 1<x\le 2 \end{cases}$$

is continuous at $x = 1$, then find the value of $\left(\dfrac{a^2+b^2}{5}\right)$.
[**Note:** sgn k denotes sigum function of k.]

4. Let $f(x) = \begin{cases} \dfrac{\alpha\cot x}{x} + \dfrac{\beta}{x^2}, & 0<|x|\le 1 \\ \dfrac{1}{3}, & x=0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then find the value of $(\alpha^2+\beta^2)$

5. The value of $\lim\limits_{x\to 0}\dfrac{(140)^x-(35)^x-(28)^x-(20)^x+7^x+5^x+4^x-1}{x\sin^2 x}$ $= 2\ln 2\ln k\ln 7$, then $k =$

6. Let $f(x)$ = signum (x) and $g(x) = x(x^2 - 10x + 21)$, then the number of points of points of discontinuity of $f(g(x))$ is

7. If $L = \lim\limits_{n\to\infty}\prod\limits_{n=2}^{n}\dfrac{(n^2+1)^2}{n^4+4}$ then the value of $(2L)$ is:

8. If $L = \lim\limits_{x\to 4}\left(\dfrac{\log_5\left(\dfrac{5x}{4}\right)}{\log_5\left(\dfrac{20}{x}\right)}\right)^{\left(\log_5\left(\frac{x}{4}\right)\right)^{-1}}$, then $[L]$ is (where $[\cdot]$ greatest integer function)

9. $L = \lim\limits_{x\to 0}\dfrac{\text{cosec}\,x\tan^{-1}(3\tan x) - 3\cos^2(x/2)}{\tan^2(x/2)}$, then sum of the digits of $|L|$ is

10. Let $f(x) = g(x)\cdot\text{sgn}(x^3+6x^2+11x+6)$, where sgn denotes signum function and $f(x) = ax^3+x^2+x+1$, then minimum number of points of discontinuity of the function $f(x)$ is

11. If $f(x) = \begin{cases} \lambda + \tan[x] & , \text{ is } x>0 \\ 2 & , \text{ is } x=0 \\ \mu + \left[\dfrac{\tan x}{x}\right] & , \text{ is } x<0 \end{cases}$ (where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer and fractional part function, respectively). If $f(x)$ is continuous at $x = 0$, then $(\lambda+\mu)$ is equal to

12. If ABC is a right triangle at A with AC ≤ AB and h_a, w_a, m_a are the lengths of its altitude, angle bisector and median from vertex A, respectively, then the value of $\lim\limits_{b\to c}\left(\dfrac{m_a-h_a}{w_a-h_a}\right)$ is

13. If $f(x) = \sum_{k=1}^{n} \frac{1}{\sin 2^k x}$ and $g(x) = f(x) + \frac{1}{\tan 2^n x}$, then the value of $\lim_{x\to 0^+}\left((\cos x)^{g(x)} + \left(\frac{1}{\cos x}\right)^{\frac{1}{\sin x}}\right)$ is

14. If $\lim_{x\to 0}\frac{(ax+b)-\sqrt{4+\sin x}}{\tan x} = \frac{27}{4}$, where a and b are constants. Then find the value of $(a - b)$

15. Find the absolute value of $\lim_{x\to 0}\frac{2^{\arctan x} - 2^{\arcsin x}}{2^{\tan x} - 2^{\sin x}}$

16. Consider the sequence $(a_n)_{n\geq 1}$ such that $a_0 = 2$ and $a_{n-1} - a_n = \frac{n}{(n+1)!}$. Evaluate $\lim_{n\to\infty}((n+1)!\ \ln a_n)$.

17. If $\lim_{x\to -2}\frac{3x^2+\lambda x+2}{x^2+3x+2}$ exist and is equal to L, then the value of $(\lambda + L)$ is

18. Let $\ell = \lim_{x\to 0}\cot^2 x(\sqrt{2\cos^2 x + 3\cos x + 4} - \sqrt{\cos^2 x + 7\cos x + 1})$, then $(2010\,\ell)$ is

19. If $\lambda = \lim_{x\to 0}\left[\min(t^2 - 6t + 10)\frac{\tan^{-1} x}{x}\right]$ (where $[\cdot]$ denotes greatest integer function) then (2014λ) is

20. Let $f(x) = \begin{cases} \dfrac{a^x + b^x + c^x + d^x - 4}{4}, & x > 0 \\ 3\log(k - x) - 15\log k, & x < 0 \end{cases}$ (where a, b, c, d, $k > 0$). If $f(x)$ is continuous at $x = 0$, then the value of $(k^3\sqrt[4]{abcd}\,)$ is equal to

21. Let $f(x) = \begin{cases} a^{\frac{\sin 2x}{\sin 3x}} & ; \ x < 0 \\ b & ; \ x = 0 \\ \left(1+\dfrac{2x}{3}\right)^{\frac{1}{x}} & ; \ x > 0 \end{cases}$

If "f" is continuous at $x = 0$, then $\log(ab^3)$ is

22. If $f(x) = \begin{cases} \left(\tan\left(\dfrac{4x^2}{a}\right) + \sec\left(\dfrac{9x}{b}\right)\right)^{\frac{ab}{x^2}} & x \neq 0 \\ e^3 & x = 0 \end{cases}$ continuous at $x = 0\ \forall\ b \in R - \{0\}\ (a \neq 0)$ then the value of $\left(\frac{1}{a_{\max}}\right)$ is

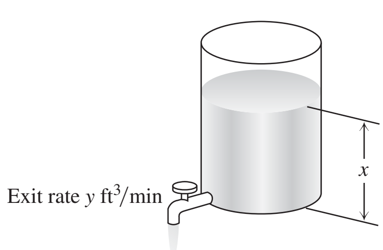
23. If $L = \lim_{x\to 0}\frac{(\sin x - x)^2 + 1 - \cos x^3}{x^5 \sin x}$ then $\frac{72L}{19}$ is equal to

24. Consider the sequence $(a_n)n \geq 1$, such that $\sum_{k=1}^{n} a_k = \frac{3n^2+9n}{2}$ $\forall\ n \geq 1$. If $\lim_{n\to\infty}\frac{1}{na_n}\sum_{k=1}^{n} a_k = \frac{p}{q}$ (where p and q are coprime), then $(p + q)$ is equal to

25. If $f(x) = \begin{cases} \dfrac{\cos(x+\cos x)}{(\pi - 2x)^3} & ; \ x \neq \dfrac{\pi}{2} \\ a & ; \ x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then $144a$ is equal to

Chapter 2 Additional and Advanced Exercises

1. Controlling the flow from a draining tank Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

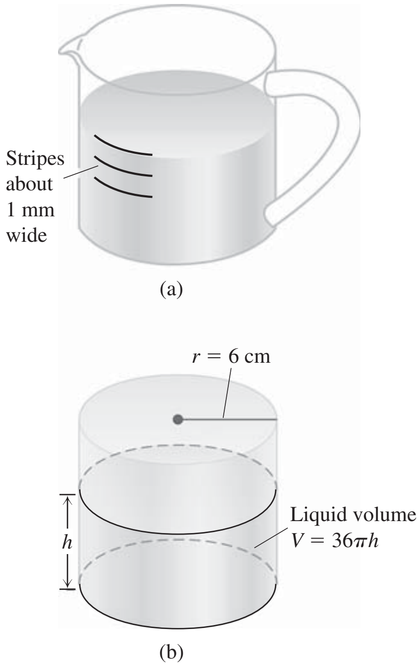
a. within 0.2 ft^3/min of the rate $y_0 = 1$ ft^3/min?

b. within 0.1 ft^3/min of the rate $y_0 = 1$ ft^3/min?

2. Stripes on a measuring cup The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water ($1000\ \text{cm}^3$) with an error of no more than 1% ($10\ \text{cm}^3$)?



A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm

3. **One-sided limits** If $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$, find
   a. $\lim_{x\to 0^+} f(x^3 - x)$
   b. $\lim_{x\to 0^-} f(x^3 - x)$
   c. $\lim_{x\to 0^+} f(x^2 - x^4)$
   d. $\lim_{x\to 0^-} f(x^2 - x^4)$
4. **Limits and continuity** Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).
   a. If $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} g(x)$ does not exist, then $\lim_{x\to c}(f(x) + g(x))$ does not exist.
   b. If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} (f(x) + g(x))$ does not exist.
   c. If f is continuous at x, then so is $|f|$.
   d. If $|f|$ is continuous at c, then so is f.

In Exercises 5 and 6, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

5. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$
6. $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$
7. **A function continuous at only one point** Let
$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$
   a. Show that f is continuous at $x = 0$.
   b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.
8. **The Dirichlet ruler function** If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by
$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$
   For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.
   a. Show that f is discontinuous at every rational number in $[0, 1]$.
   b. Show that f is continuous at every irrational number in $[0, 1]$. (*Hint*: If ϵ is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \ge \epsilon$.)
   c. Sketch the graph of f. Why do you think f is called the "ruler function"?
9. **Antipodal points** Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth's equator where the temperatures are the same? Explain.
10. If $\lim_{x\to c} (f(x) + g(x)) = 3$ and $\lim_{x\to c} (f(x) - g(x)) = -1$, find $\lim_{x\to c} f(x)g(x)$.
11. **Roots of a quadratic equation that is almost linear** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \ne 0$, one positive and one negative:
$$r_+(a) = \frac{-1 + \sqrt{1 + a}}{a}, \qquad r_-(a) = \frac{-1 - \sqrt{1 + a}}{a},$$
   a. What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?
   b. What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?
   c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.
   d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05.
12. **Root of an equation** Show that the equation $x + 2\cos x = 0$ has at least one solution.
13. **Bounded functions** A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \le N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \ge M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.
   a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \le B$ for all x in D.
   b. Suppose that f is bounded from above by N. Show that if $\lim_{x\to c} f(x) = L$, then $L \le N$.
   c. Suppose that f is bounded from below by M. Show that if $\lim_{x\to c} f(x) = L$, then $L \ge M$.

14. Max $\{a, b\}$ and min $\{a, b\}$

a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, max $\{a, b\}$ gives the larger of the two numbers a and b.

b. Find a similar expression for min $\{a, b\}$, the smaller of a and b.

Generalized Limits Involving $\dfrac{\sin\theta}{\theta}$

The formula $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ can be generalized. If $\lim_{x\to c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly c itself, then

$$\lim_{x\to c}\frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x^2} = 1$

b. $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}\frac{\sin x^2}{x^2}\lim_{x\to 0}\frac{x^2}{x} = 1\cdot 0 = 0$

c. $\displaystyle\lim_{x\to -1}\frac{\sin(x^2-x-2)}{x+1} = \lim_{x\to -1}\frac{\sin(x^2-x-2)}{(x^2-x-2)}\cdot$

$$\lim_{x\to -1}\frac{(x^2-x-2)}{x+1} = 1\cdot\lim_{x\to -1}\frac{(x+1)(x-2)}{x+1} = -3$$

d. $\displaystyle\lim_{x\to 1}\frac{\sin\left(1-\sqrt{x}\right)}{x-1} = \lim_{x\to 1}\frac{\sin\left(1-\sqrt{x}\right)}{1-\sqrt{x}}\frac{1-\sqrt{x}}{x-1} =$

$$1\cdot\lim_{x\to 1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{(x-1)\left(1+\sqrt{x}\right)} = \lim_{x\to 1}\frac{1-x}{(x-1)\left(1+\sqrt{x}\right)} = -\frac{1}{2}$$

Find the limits in Exercises 15–20.

15. $\displaystyle\lim_{x\to 0}\frac{\sin(1-\cos x)}{x}$

16. $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{\sin\sqrt{x}}$

17. $\displaystyle\lim_{x\to 0}\frac{\sin(\sin x)}{x}$

18. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2+x)}{x}$

19. $\displaystyle\lim_{x\to 2}\frac{\sin(x^2-4)}{x-2}$

20. $\displaystyle\lim_{x\to 9}\frac{\sin\left(\sqrt{x}-3\right)}{x-9}$

3

Derivatives

OVERVIEW In the beginning of Chapter 2, we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can define these ideas precisely and see that both are interpretations of the *derivative* of a function at a point. We then extend this concept from a single point to the *derivative function*, and we develop rules for finding this derivative function easily, without having to calculate any limits directly. These rules are used to find derivatives of many of the common functions reviewed in Chapter 1, as well as various combinations of them.

The derivative is one of the key ideas in calculus, and is used to study a wide range of problems in mathematics, science, economics, and medicine. These problems include finding points where a continuous function is zero, calculating the velocity and acceleration of a moving object, determining how the rate of flow of a liquid into a container changes the level of the liquid within it, describing the path followed by a light ray going from a point in air to a point in water, finding the number of items a manufacturing company should produce in order to maximize its profits, studying the spread of an infectious disease within a given population, or calculating the amount of blood the heart pumps in a minute based on how well the lungs are functioning.

3.1 Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

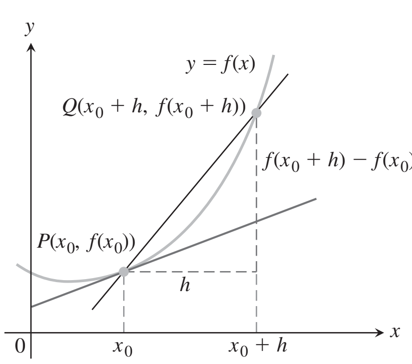


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

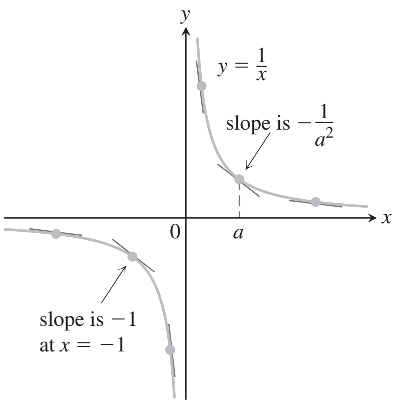


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

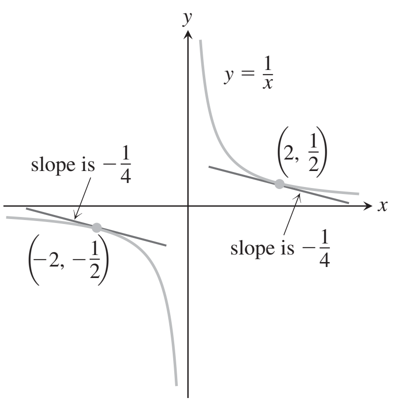


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P. Let's look at another example.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-h}{ha(a+h)} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off becoming more and more horizontal. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

The notation $f'(x_0)$ is read "f prime of x_0."

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. Derivative of the linear function $f(x) = mx + b$ at any point x_0 is simply the slope of the line, so

$$f'(x_0) = m,$$

which is consistent with our definition of slope.

If we interpret the difference quotient as an average rate of change (Section 2.1), the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$f'(1) = \lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

Our original estimate of 32 ft/sec in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

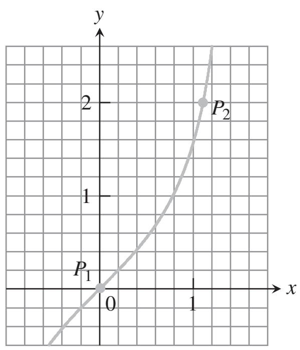
In the next sections, we allow the point x_0 to vary across the domain of the function f.

Exercises 3.1

Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points P_1 and P_2.

1.



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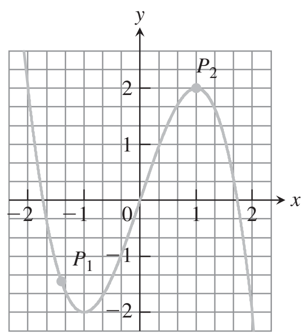

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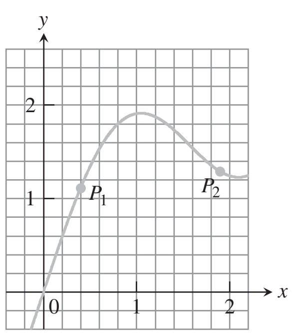

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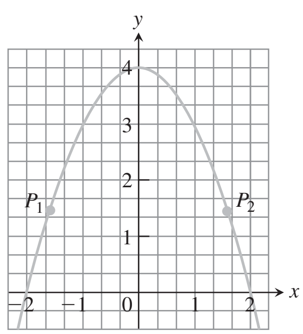
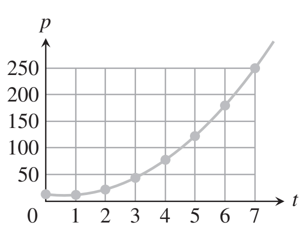


Interpreting Derivative Values

5. Growth of yeast cells In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number P of cells present at hourly intervals. The number after t hours is shown in the accompanying figure.

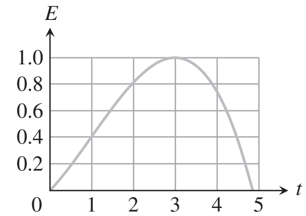


a. Explain what is meant by the derivative $P'(5)$. What are its units?

b. Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.

c. The quadratic curve capturing the trend of the data points (see Section 1.4) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.

6. Effectiveness of a drug On a scale from 0 to 1, the effectiveness E of a pain-killing drug t hours after entering the bloodstream is displayed in the accompanying figure.



a. At what times does the effectiveness appear to be increasing? What is true about the derivative at those times?

b. At what time would you estimate that the drug reaches its maximum effectiveness? What is true about the derivative at that time? What is true about the derivative as time increases in the 1 hour *before* your estimated time?

7. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x - 1)$.

8. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

9. Object dropped from a tower An object is dropped from the top of a 100-m-high tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

10. Speed of a rocket At t sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

11. Circle's changing area What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

12. Ball's changing volume What is the rate of change of the volume of a ball $\left(V = (4/3)\pi r^3\right)$ with respect to the radius when the radius is $r = 2$?

Testing for Tangents

13. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

14. Does the graph of

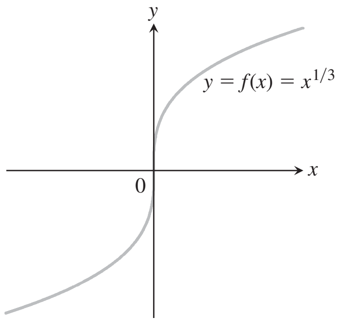
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{1/3}-0}{h}$$
$$=\lim_{h\to 0}\frac{1}{h^{2/3}}=\infty.$$

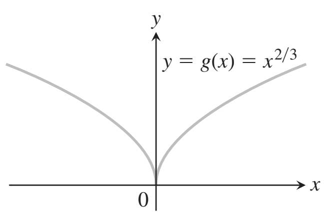


VERTICAL TANGENT AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{h^{2/3}-0}{h}$$
$$=\lim_{h\to 0}\frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT AT ORIGIN

15. Does the graph of

$$f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

16. Does the graph of

$$U(x)=\begin{cases}0, & x<0\\ 1, & x\ge 0\end{cases}$$

have a vertical tangent at the point (0, 1)? Give reasons for your answer.

T Graph the curves in Exercises 17–26.

a. Where do the graphs appear to have vertical tangents?

b. Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 15 and 16.

17. $y = x^{2/5}$

18. $y = x^{4/5}$

19. $y = x^{1/5}$

20. $y = x^{3/5}$

21. $y = 4x^{2/5} - 2x$

22. $y = x^{5/3} - 5x^{2/3}$

23. $y = x^{2/3} - (x - 1)^{1/3}$

24. $y = x^{1/3} + (x - 1)^{1/3}$

25. $y=\begin{cases}-\sqrt{|x|}, & x\le 0\\ \sqrt{x}, & x>0\end{cases}$

26. $y = \sqrt{|4 - x|}$

3.2 The Derivative as a Function

HISTORICAL ESSAY
The Derivative

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

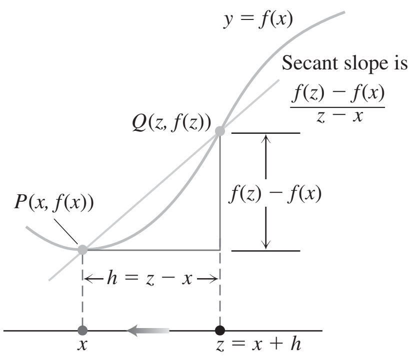
$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists.



Derivative of f at x is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z\to x}\frac{f(z) - f(x)}{z - x}$$

FIGURE 3.4 Two forms for the difference quotient.

We use the notation $f(x)$ in the definition to emphasize the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at x**. If f' exists at every point in the domain of f, we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z\to x}\frac{f(z) - f(x)}{z - x}$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx}f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Derivative of the Reciprocal Function

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad x \neq 0$$

Here are two more examples in which we allow x to be any point in the domain of f.

EXAMPLE 1 Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x + h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \qquad \text{Definition}$$

$$= \lim_{h\to 0}\frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)} \qquad \text{Simplify.}$$

$$= \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \qquad \text{Cancel } h \neq 0.$$

■

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

Solution

(a) We use the alternative formula to calculate f':

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope $1/4$ (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$ ■

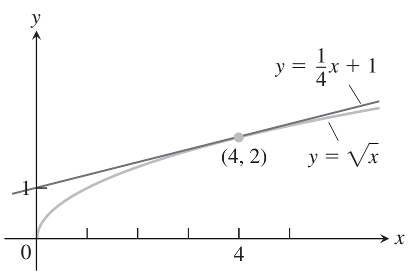


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio (until we introduce the idea of "differentials" in Section 3.9).

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \frac{dy}{dx}\bigg|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{d}{dx}f(x)\bigg|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \frac{d}{dx}\sqrt{x}\bigg|_{x=4} = \frac{1}{2\sqrt{x}}\bigg|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Graphing the Derivative

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f. That is, we plot the points $(x, f'(x))$ in the xy-plane and connect them with a smooth curve, which represents $y = f'(x)$.

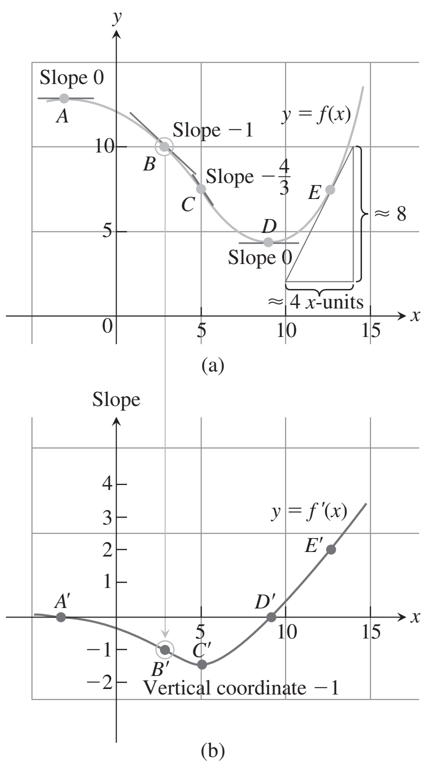


FIGURE 3.6 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on. The slope at E is approximately $8/4 = 2$. In (b) we see that the rate of change of f is negative for x between A' and D'; the rate of change is positive for x to the right of D'.

EXAMPLE 3 Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

Solution We sketch the tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as sketched in Figure 3.6b. ■

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. Because of Theorem 6, Section 2.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

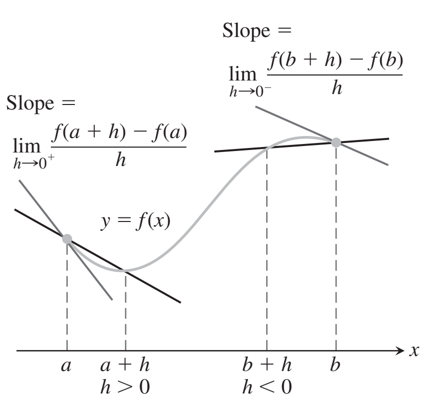


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m,\ |x| = x$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h\to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h\to 0^-} -1 = -1.
\end{aligned}$$

■

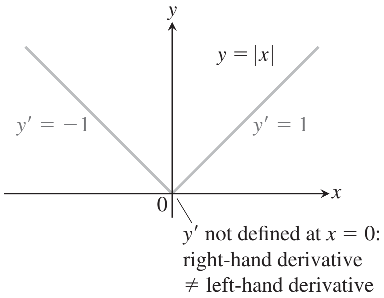


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

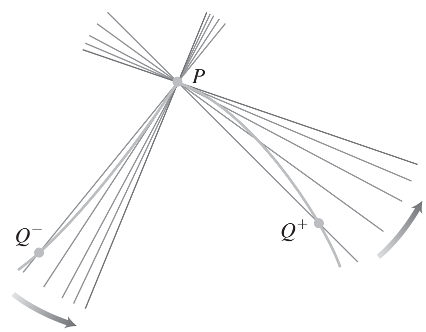
We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

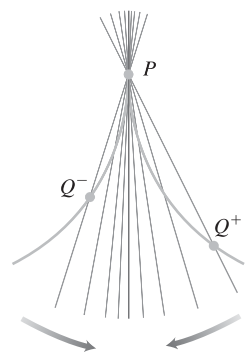
Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $\left(h, \sqrt{h}\right)$ on a graph of $y = \sqrt{x}$ approach ∞, the graph has a *vertical tangent* at the origin. (See Figure 1.11 on page 7.) ■

When Does a Function *Not* Have a Derivative at a Point?

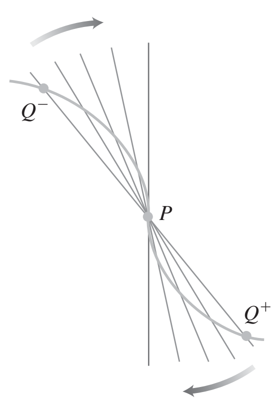
A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. Thus differentiability is a "smoothness" condition on the graph of f. A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



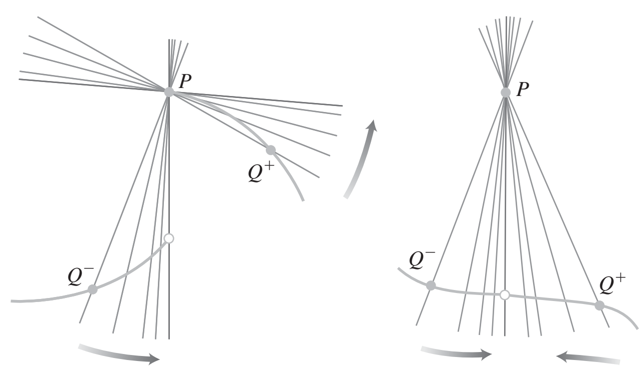
1. a *corner*, where the one-sided derivatives differ.



2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity* (two examples shown).

Another case in which the derivative may fail to exist occurs when the function's slope is oscillating rapidly near P, as with $f(x) = \sin(1/x)$ near the origin, where it is discontinuous (see Figure 2.31).

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1—Differentiability Implies Continuity** If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or equivalently, that $\lim_{h\to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

■

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.5).

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4.

This can be summarized as

If $f'\left(a^+\right) = p$ and $f'(a^-) = q$ then,

1. If $p = q$ where p and q are finite quantities then $f(x)$ is differentiable at $x = a$ and hence it is continuous at $x = a$
2. If $p \neq q$ where p and q are finite quantities then $f(x)$ is not differentiable at $x = a$ but is still continuous at $x = a$

Proof: Given that $f'\left(a^+\right) = p$ is a finite quantity that implies $\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$ exist and is finite quantity. Now in this limit denominator, i.e., $h \to 0$ and overall limit is finite hence numerator, i.e., $f(a + h) - f(a)$ must be zero.

Hence, $\lim_{h\to 0^+} f(a+h) = f(a)$. Similarly we can prove that $\lim_{h\to 0^-} f(a+h) = f(a)$. Hence $f(x)$ is continuous at $x = a$. ■

> **THEOREM 2** If $f(x)$ and $g(x)$ both are derivable at $x = a$, then $f(x) \pm g(x)$, $f(x)\cdot g(x)$ and $\frac{f(x)}{g(x)}(g(a) \neq 0)$ will also be differentiable at $x = a$.

> **THEOREM 3** If $f(x)$ is derivable at $x = a$ but $g(x)$ is not derivable at $x = a$ then $f(x) \pm g(x)$ is not derivable at $x = a$. However, $f(x)\cdot g(x)$ and $\dfrac{f(x)}{g(x)}$ may or may not be differentiable at $x = a$.

For example, $f(x) = x$ is differentiable at $x = 0$ and $g(x) = |x|$ is non-differentiable at $x = 0$ but the product function $x|x|$ is differentiable at $x = 0$ which can be easily checked by defining the product function $x|x|$.

> **THEOREM 4** If $f(x)$ as well $g(x)$ both are non-derivable at $x = a$ then nothing definite can be said about $f(x) \pm g(x)$, $f(x).g(x)$ and $\dfrac{f(x)}{g(x)} (g(a) \neq 0)$.

> **THEOREM 5** If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$, then the product function $F(x) = f(x) \cdot g(x)$ will be derivable at $x = a$.

Proof

$$F'(a^+) = \operatorname*{Lim}_{h\to 0} \frac{f(a+h)g(a+h)-0}{h} = f'(a)\cdot g(a)$$

$$F'(a^-) = \operatorname*{Lim}_{h\to 0} \frac{f(a-h)g(a-h)-0}{-h} = f'(a)\cdot g(a)$$

■

EXAMPLE 6 Check the differentiability of the function $f(x) = e^{|x|}$ at $x = 0$.

Solution To solve such kind of questions we can check the right-hand derivative and left-hand derivative of the function.

R.H.D. of $f(x)$ at $x = 0$ is given by:

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{e^{|h|}-e^0}{h} = \lim_{h\to 0^+} \frac{e^h-1}{h} = 1$$

L.H.D. of $f(x)$ at $x = 0$ is given by:

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{e^{|h|}-e^0}{h} = \lim_{h\to 0^-} \frac{e^{-h}-1}{h} = -\lim_{h\to 0^-} \frac{e^{-h}-1}{-h} = -1$$

Clearly we can see $f'(0^-) \neq f'(0^+)$, Hence $f(x)$ is not derivable at $x = 0$. ■

EXAMPLE 7 Consider

$$f(x) = \begin{cases} \left(\ell n\left(e\left|[x]+[-x]\right|\right)^x\right)\cdot\left(\dfrac{2e^{\left(\frac{\{x\}+\{-x\}}{|x|}\right)}-5}{3+e^{\frac{1}{|x|}}}\right) & \text{for } x<0 \\ 0 & \text{for } x=0 \\ x\cdot\dfrac{1-e^{|x|+\{x\}}}{|x|+\{x\}} & \text{for } x>0 \end{cases}$$

where $[\cdot]$, $\{\cdot\}$ represents integral and fractional part functions, respectively. Compute the right-hand derivative and left-hand derivative at $x = 0$ and comment on the continuity and derivability at $x = 0$.

Solution $f'(0^+) = \underset{h\to 0}{\text{Limit}} \dfrac{h\left(\dfrac{1-e^{h+h}}{h+h}\right)-0}{h} = \dfrac{\left(1-e^{2h}\right)}{(2h)h}h = -1$

$$f'(0^-) = \underset{h\to 0}{\text{lt}} \frac{-h \cdot \ln\left(e\left|-1+0\right|\right) \cdot \left(\dfrac{2e^{\frac{1-h+h}{|-h|}}-5}{3+e^{\frac{1}{h}}}\right)}{-h}$$

$$= \underset{h\to 0}{\text{Limit}} \frac{-h \cdot \left(\dfrac{2e^{\frac{1}{h}}-5}{3+e^{\frac{1}{h}}}\right)}{-h} = \underset{h\to 0}{\text{Lim}} \frac{2-5e^{-\frac{1}{h}}}{3e^{-\frac{1}{h}}+1} = 2$$

Hence f is continuous but not derivable at $x = 0$ ■

Determination of Function Satisfying the Given Functional Rule

1. Write down the expression $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
2. Manipulate $f(x + h) - f(x)$ in such a way that given functional rule is applicable. Now apply the functional rule and simplify right hand side to get $f'(x)$ as *a* function of x along with constants if any.
3. Integrate $f'(x)$ to get $f(x)$ as *a* function of x and *a* constant of integration in some cases *a* differential equation is formed which can be solved to get $f(x)$.
4. To evaluate constant of integration, we can used initial conditions which are given or can be obtained from functional equation.

EXAMPLE 8 Let $f(x)$ be a differentiable function satisfying $f(x/y) = f(x) - f(y)$ for all x and $y > 0$. If $f'(1) = 1$, then find $f(x)$.

Solution We will apply the above steps.

Step 1: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

Step 2: Clearly, if we replace $x \to x + h$ and $y \to x$ in the functional rule, we can apply the function rule to get $f(x + h) - f(x) = f\left(\dfrac{x+h}{x}\right) = f\left(1+\dfrac{h}{x}\right)$

$$\therefore \qquad f'(x) = \lim_{x\to 0} \frac{f\left(1+\dfrac{h}{x}\right)}{h}$$

Clearly the main challenge lies in evaluating this limit.

$$f'(x) = \lim_{x\to 0} \frac{f\left(1+\dfrac{h}{x}\right)-f(1)}{\left(\dfrac{h}{x}\right)\times x}$$

(Clearly, we can obtain $f(1) = 0$ from the functional equation by putting $x = 1$ and $y = 1$)

$$\Rightarrow \qquad f'(x) = \frac{f'(1)}{x} = \frac{1}{x}$$

Step 3: Integrating both sides, w.r.t. x, we get

$$\int f'(x)\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x$$

$$\Rightarrow \quad f(x) = \ln x + C$$

Step 4: Since, $f(1) = 0$

$$f(1) = \ln(1) + \mathrm{C}$$

$$\Rightarrow \quad 0 = 0 + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad f(x) = \ln x$$

Aliter: we can also differentiate the functional equation w.r.t. y treating x as constant; we get

$$f'\left(\frac{x}{y}\right) \times x \times \left(-\frac{1}{y^2}\right) = -f'(y)$$

Put $y = 1$, we get

$$f'(x) \times x = f'(1)$$

$$\Rightarrow \quad f'(x) = 1/x$$

Integrating both sides w.r.t. x, we get

$$\int f'(x)\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x$$

$$\Rightarrow \quad f(x) = \ln x + \mathrm{C}$$

Now, $f(1) = 0$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad \boxed{f(x) = \ln x}$$

■

EXAMPLE 9 A differentiable function $f(x)$ satisfies $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$. Find $f(x)$

Solution $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f\left(\frac{3x+3\times 0}{3}\right)}{h} = \frac{\lim\limits_{h\to 0}\left(\frac{f(3x)+f(3h)}{3}\right) - \left(\frac{f(3x)+f(0)}{3}\right)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{f(3h)-f(0)}{3h} = f'(0) = K \text{ (suppose)}$$

Integrating both sides w.r.t. x, we get

$$f(x) = Kx + C$$

Now, Put $x = 0$, $y = 0$ in the functional equation, we get

$$f(0) = \frac{2f(0)}{3}$$

$$\Rightarrow \quad f(0) = 0$$

$\therefore \quad f(0) = K \times 0 + C$

$\Rightarrow \quad C = 0$

$\therefore \quad \boxed{f(x) = Kx}$

where $\quad \boxed{K = f'(0)}$

Aliter 1: We can differentiate the given functional equation w.r.t y treating x as constant.

$$f'\left(\frac{x+y}{3}\right) \times \frac{1}{3} = \frac{f'(y)}{3}$$

Put $y = 0$; we get

$$f'\left(\frac{x}{3}\right) = f'(0)$$

$\Rightarrow \quad f'(x) = K$

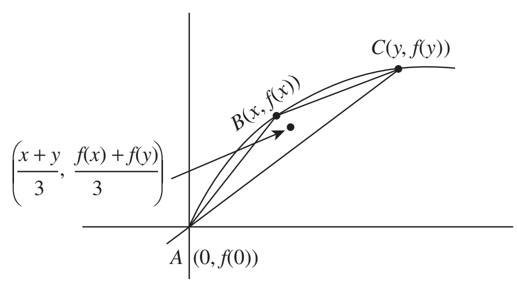
$\Rightarrow$ integrating both side w.r.t. x, we get

$$f(x) = Kx + C$$

Also $\quad f(0) = 0$

$\Rightarrow \quad f(x) = Kx$

Aliter 2: We can take a random graph of $f(x)$ and are selecting three points $(0, f(0))$, $(x, f(x))$ and $(y, f(y))$ on this graph. Also $f(0) = 0$



Calculating the centroid of ΔABC, we get $\left(\frac{x+y}{3}, \frac{f(x)+f(y)}{3}\right)$. It is obvious from graph that $f\left(\frac{x+y}{3}\right) > \frac{f(x)+f(y)}{3}$; but in questions $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$ which suggest $f(x)$ must be straight line passing through origin. ■

Exercises 3.2

Finding Derivative Functions and Values

In Exercises 1–2, find the indicated derivatives.

1. $\frac{dp}{dq}$ if $p = q^{3/2}$

2. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{w^2 - 1}}$

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

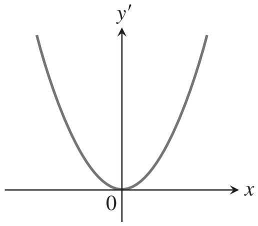
to find the derivative of the functions in Exercises 3–4.

3. $g(x) = \frac{x}{x - 1}$

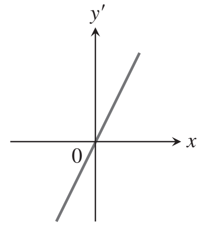
4. $g(x) = 1 + \sqrt{x}$

Graphs

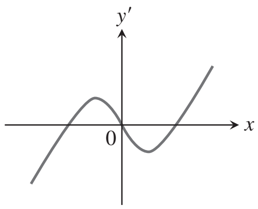
Match the functions graphed in Exercises 5–8 with the derivatives graphed in the accompanying figures (a)–(d).



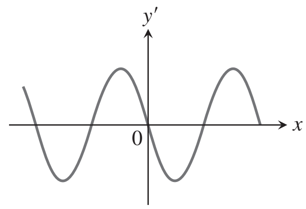
(a)



(b)

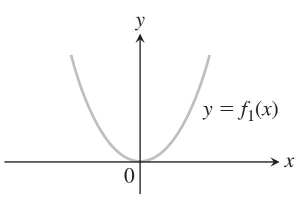


(c)

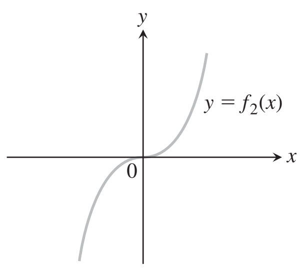


(d)

5.



6.



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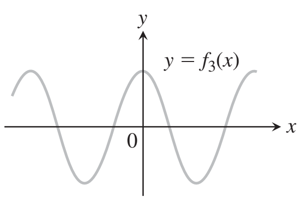
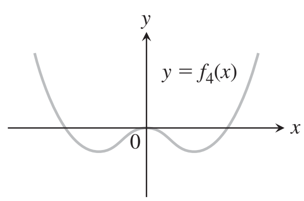
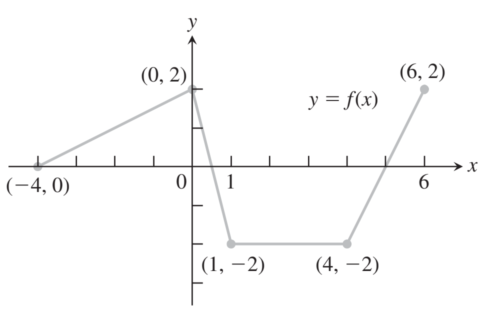


8.



9. **a.** The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



b. Graph the derivative of f.

The graph should show a step function.

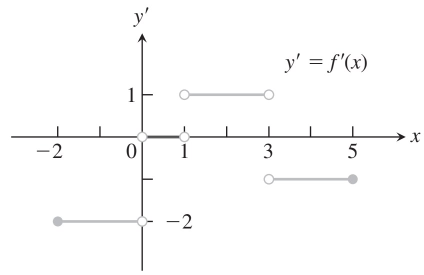
10. Recovering a function from its derivative

a. Use the following information to graph the function f over the closed interval $[-2, 5]$.

i) The graph of f is made of closed line segments joined end to end.

ii) The graph starts at the point $(-2, 3)$.

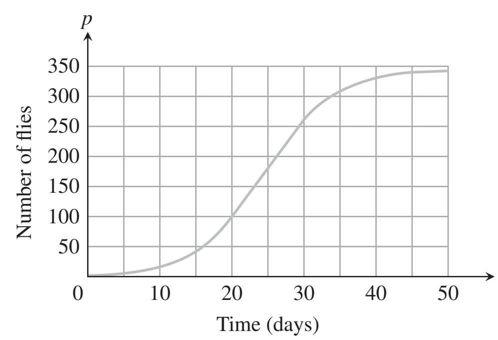
iii) The derivative of f is the step function in the figure shown here.



b. Repeat part (a), assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

11. Fruit flies (*Continuation of Example 4, Section 2.1.*) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a. Use the graphical technique of Example 3 to graph the derivative of the fruit fly population. The graph of the population is reproduced here.

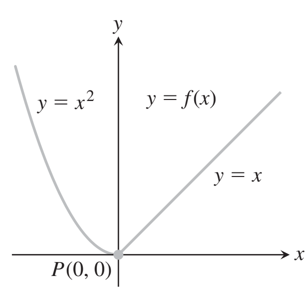


b. During what days does the population seem to be increasing fastest? Slowest?

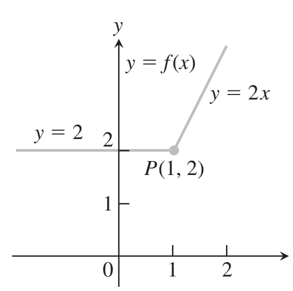
One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 12–15 are not differentiable at the point P.

12.



13.



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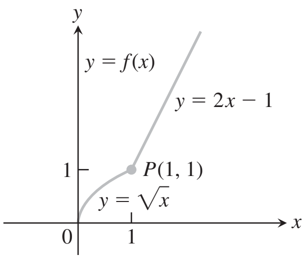
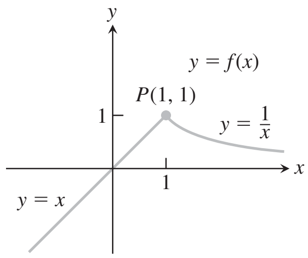


15.



In Exercises 16 and 17, determine if the piecewise-defined function is differentiable at the origin.

16. $f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$

17. $g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$

Differentiability and Continuity on an Interval

Each figure in Exercises 18–23 shows the graph of a function over a closed interval D. At what domain points does the function appear to be

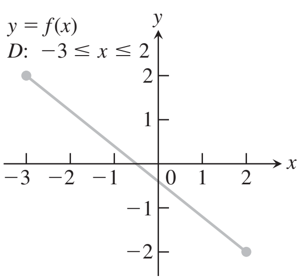
a. differentiable?

b. continuous but not differentiable?

c. neither continuous nor differentiable?

Give reasons for your answers.

18.



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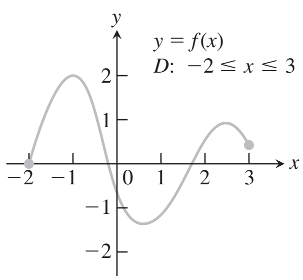
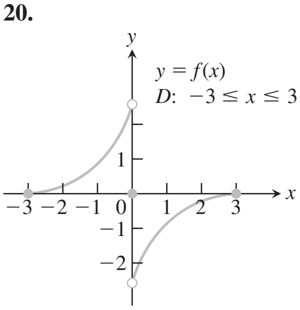
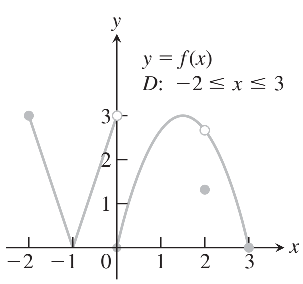


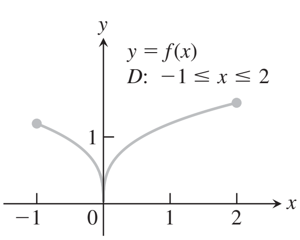
20.



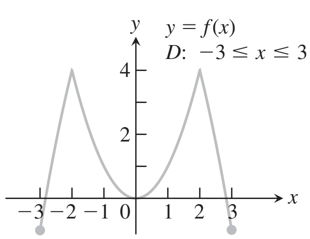
21.



22.



23.



Theory and Examples

24. Derivative of $-f$ Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

25. Derivative of multiples Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

26. Limit of a quotient Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and $g(0) = h(0) = 0$. Can $\lim_{t \to 0} (g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

27. a. Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b. Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

T **28.** Graph $y = 1/\left(2\sqrt{x}\right)$ in a window that has $0 \leq x \leq 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

T **29.** Graph $y = 3x^2$ in a window that has $-2 \leq x \leq 2, 0 \leq y \leq 3$. Then, on the same screen, graph

$$y = \frac{(x + h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

30. Derivative of $y = |x|$ Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

T **31. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x)$$
$$+ (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

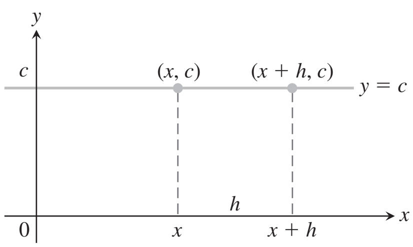


FIGURE 3.9 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

> **Derivative of a Constant Function**
>
> If f has the constant value $f(x) = c$, then
> $$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.9). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0. \qquad \blacksquare$$

From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power x^n. We first prove the rule when n is a positive integer.

> **Derivative of a Positive Integer Power**
>
> If n is a positive integer, then
> $$\frac{d}{dx}x^n = nx^{n-1}.$$

HISTORICAL BIOGRAPHY
Richard Courant
(1888–1972)

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z\to x}\frac{f(z) - f(x)}{z - x} = \lim_{z\to x}\frac{z^n - x^n}{z - x} \\ &= \lim_{z\to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \qquad n \text{ terms} \\ &= nx^{n-1}. \qquad \blacksquare \end{aligned}$$

The Power Rule is actually valid for all real numbers n. We have seen examples for a negative integer and fractional power, but n could be an irrational number as well. To apply the Power Rule, we subtract 1 from the original exponent n and multiply the result by n.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x.

(a) $x^{\sqrt{2}}$ **(b)** $\dfrac{1}{x^4}$ **(c)** $x^{-4/3}$ **(d)** $\sqrt{x^{2+\pi}}$

Applying the Power Rule
Subtract 1 from the exponent and multiply the result by the original exponent.

Solution

(a) $\dfrac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

(c) $\dfrac{d}{dx}\left(x^{-4/3}\right) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$

(d) $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is any real number, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

Proof

$$\begin{aligned}
\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition with } f(x) = cu(x)\\
&= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} && \text{Constant Multiple Limit Property}\\
&= c\frac{du}{dx} && u \text{ is differentiable.}
\end{aligned}$$

■

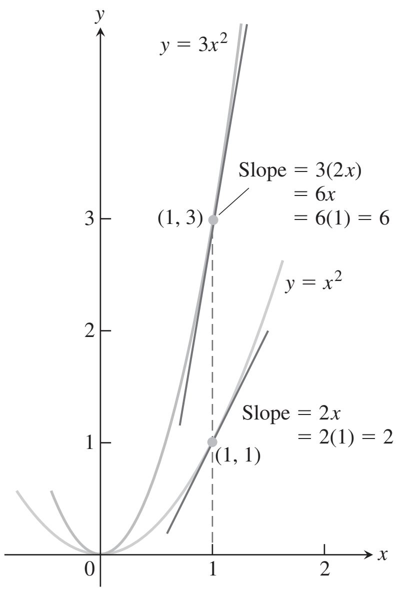


FIGURE 3.10 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

Denoting Functions by u and v
The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. We do not want to use these same letters when stating general differentiation rules, so instead we use letters like u and v that are not likely to be already in use.

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}. \qquad \blacksquare$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}
\frac{d}{dx}[u(x) + v(x)] &= \lim_{h\to 0} \frac{[u(x + h) + v(x + h)] - [u(x) + v(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{u(x + h) - u(x)}{h} + \frac{v(x + h) - v(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{u(x + h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x + h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \qquad \blacksquare
\end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

For instance, to see that the rule holds for three functions we compute

$$\frac{d}{dx}(u_1 + u_2 + u_3) = \frac{d}{dx}((u_1 + u_2) + u_3) = \frac{d}{dx}(u_1 + u_2) + \frac{du_3}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \frac{du_3}{dx}.$$

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

■

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of x.

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0$, 1, and -1. The corresponding points on the curve are (0, 2), (1, 1), and $(-1, 1)$. See Figure 3.11. We will see in Chapter 4 that finding the values of x where the derivative of a function is equal to zero is an important and useful procedure. ■

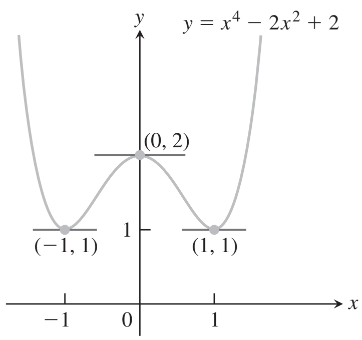


FIGURE 3.11 The curve in Example 4 and its horizontal tangents.

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x)\cdot\frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Equation (3) is equivalent to saying that

$$(fg)' = f'g + fg'.$$

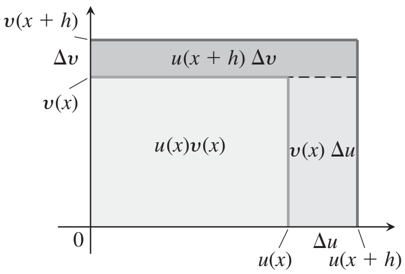
This form of the Product Rule is useful and applies to *dot products* and *cross products* of *vector-valued* functions, studied in Chapter 13.

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x). \tag{3}$$

Picturing the Product Rule

Suppose $u(x)$ and $v(x)$ are positive and increase when x increases, and $h > 0$.



Then the change in the product uv is the difference in areas of the larger and smaller "squares," which is the sum of the upper and right-hand reddish-shaded rectangles. That is,

$$\begin{aligned}\Delta(uv) &= u(x+h)v(x+h) - u(x)v(x)\\ &= u(x+h)\Delta v + v(x)\Delta u.\end{aligned}$$

Division by h gives

$$\frac{\Delta(uv)}{h} = u(x+h)\frac{\Delta v}{h} + v(x)\frac{\Delta u}{h}.$$

The limit as $h \to 0^+$ gives the Product Rule.

EXAMPLE 5 Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution

(a) From the Product Rule with $u = x^2+1$ and $v = x^3+3$, we find

$$\begin{aligned}\frac{d}{dx}\left[\left(x^2+1\right)\left(x^3+3\right)\right] &= \left(x^2+1\right)\left(3x^2\right) + \left(x^3+3\right)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\\ &= 3x^4 + 3x^2 + 2x^4 + 6x\\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = \left(x^2+1\right)\left(x^3+3\right) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ■

EXAMPLE 6 Let $F(x) = f(x)\cdot g(x)\cdot h(x)$. If for some $x = x_0$, $F'(x_0) = 21\,F(x_0)$; $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7g(x_0)$ and $h'(x_0) = k\,h(x_0)$ (where $F(x_0) \neq 0$). then find k.

Solution Clearly $F'(x) = f'gh + fg'h + fgh'$

Now at $x = x_0$; we have

$$\begin{aligned}&F'(x_0) = f'(x_0)g(x_0)h(x_0) + f(x_0)g'(x_0)h(x_0) + f(x_0)g(x_0)h'(x_0)\\ &\Rightarrow 21F(x_0) = 4f(x_0)g(x_0)h(x_0) - 7f(x_0)g(x_0)h(x_0) + kf(x_0)g(x_0)h(x_0)\\ &\Rightarrow 21F(x_0) = (4-7+k)f(x_0)g(x_0)h(x_0)\\ &\Rightarrow 21 = -3+k\\ &\Rightarrow k = 24\end{aligned}$$

■

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]\\ &= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}.\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. Therefore,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

■

The derivative of the quotient of two functions is given by the Quotient Rule.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 7 Find the derivative of $y = \dfrac{t^2 - 1}{t^3 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} \\
&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\
&= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.
\end{aligned}$$

■

Proof of the Derivative Quotient Rule

$$\begin{aligned}
\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x + h)}{v(x + h)} - \dfrac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{v(x)u(x + h) - u(x)v(x + h)}{hv(x + h)v(x)}
\end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x + h) - v(x)u(x) + v(x)u(x) - u(x)v(x + h)}{hv(x + h)v(x)} \\
&= \lim_{h\to 0} \frac{v(x)\dfrac{u(x + h) - u(x)}{h} - u(x)\dfrac{v(x + h) - v(x)}{h}}{v(x + h)v(x)}.
\end{aligned}$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. Exercise 74 outlines another proof. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 8 Find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}\\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$ ■

EXAMPLE 9 Let g be a differentiable function of x. If $f(x) = \dfrac{g(x)}{x^2}$ for $x > 0$, $g(2) = 3$ and $g'(2) = -2$. find $f'(2)$.

Solution Clearly $f'(x) = \dfrac{x^2 g'(x) - g(x)2x}{x^4}$;

Hence $$f'(2) = \frac{4(-2) - 3\cdot 4}{16} = -\frac{5}{4}$$ ■

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x{}^2 f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

How to Read the Symbols for Derivatives

y'	"y prime"
y''	"y double prime"
$\dfrac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\dfrac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"D to the n"

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero. ■

Exercises 3.3

Derivative Calculations

In Exercises 1–2, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

1. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$

2. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 3–4.

3. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$

4. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

5. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

6. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a. $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

Slopes and Tangents

7. a. Normal to a curve Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

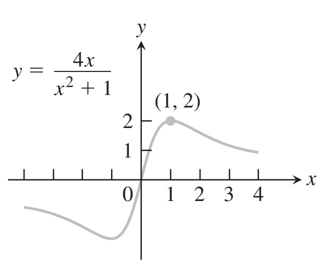
b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. Tangents having specified slope Find equations for the tangents to the curve at the points where the slope of the curve is 8.

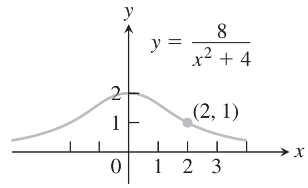
8. a. Horizontal tangents Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

9. Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point (1, 2).



10. Find the tangent to the *Witch of Agnesi* (graphed here) at the point (2, 1).



11. Quadratic tangent to identity function The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find a, b, and c.

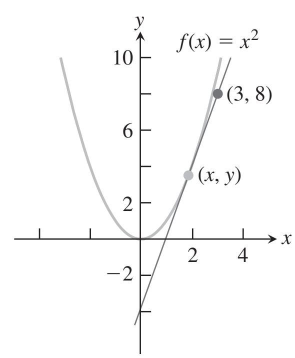
12. Quadratics having a common tangent The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find a, b, and c.

13. Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

14. Find all points (x, y) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.

15. Find all points (x, y) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.

16. Find all points (x, y) on the graph of $f(x) = x^2$ with tangent lines passing through the point (3, 8).



17. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

18. Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

Theory and Examples

For Exercises 19 and 20 evaluate each limit by first converting each to a derivative at a particular x-value.

19. $\displaystyle\lim_{x \to 1} \frac{x^{50} - 1}{x - 1}$

20. $\displaystyle\lim_{x \to -1} \frac{x^{2/9} - 1}{x + 1}$

21. Find the value of a that makes the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \ge 0 \end{cases}$$

22. Find the values of a and b that make the following function differentiable for all x-values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \le -1 \end{cases}$$

23. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Find $P'(x)$.

24. The body's reaction to medicine The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

25. Suppose that the function v in the Derivative Product Rule has a constant value c. What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

26. The Reciprocal Rule

a. The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

b. Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

27. Generalizing the Product Rule The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product uv of two differentiable functions of x.

a. What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b. What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c. What is the formula for the derivative of a product $u_1u_2u_3\cdots u_n$ of a finite number n of differentiable functions of x?

28. Power Rule for negative integers Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

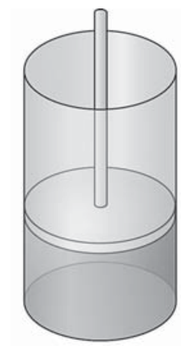
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where m is a positive integer.

29. Cylinder pressure If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV. (See accompanying figure.)



30. The best quantity to order One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2.

3.4 Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x. && \text{Example 5a and Theorem 7, Section 2.4}
\end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\
&= 2x - \cos x
\end{aligned}$$

(b) $y = x^2 \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\
&= x^2 \cos x + 2x \sin x.
\end{aligned}$$

(c) $y = \dfrac{\sin x}{x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\
&= \frac{x \cos x - \sin x}{x^2}
\end{aligned}$$

■

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}
\end{aligned}$$

$$= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 \qquad \text{Example 5a and Theorem 7, Section 2.4}$$

$$= -\sin x.$$

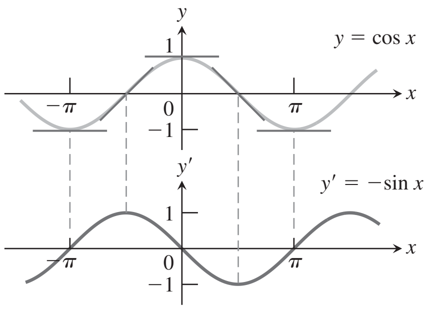


FIGURE 3.12 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Figure 3.12 shows a way to visualize this result in the same way we did for graphing derivatives in Section 3.2, Figure 3.6.

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$

$$= 5 - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

■

Simple Harmonic Motion

The motion of an object or weight bobbing freely up and down with no resistance on the end of a spring is an example of *simple harmonic motion*. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. The next example describes a case in which there are no opposing forces such as friction to slow the motion.

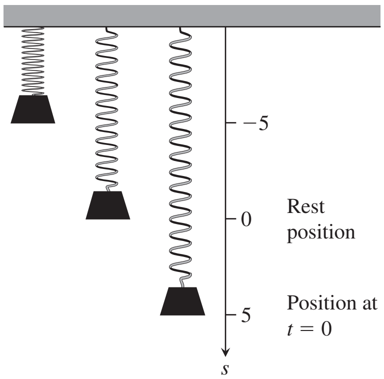


FIGURE 3.13 A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

EXAMPLE 3 A weight hanging from a spring (Figure 3.13) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t?

Solution We have

$$\text{Position:} \qquad s = 5\cos t$$

$$\text{Velocity:} \qquad v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = -5\sin t$$

$$\text{Acceleration:} \quad a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t.$$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of the cosine function.
2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.14. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.
3. The weight is acted on by the spring and by gravity. When the weight is below the rest position, the combined forces pull it up, and when it is above the rest position, they pull it down. The weight's acceleration is always proportional to the negative of its displacement. This property of springs is called *Hooke's Law*, and is studied further in Section 6.5.
4. The acceleration, $a = -5\cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

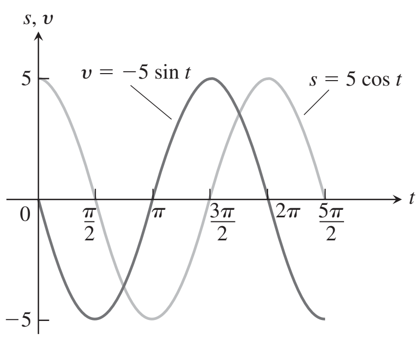


FIGURE 3.14 The graphs of the position and velocity of the weight in Example 3.

EXAMPLE 4 The jerk associated with the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

To show a typical calculation, we find the derivative of the tangent function. The other derivations are left to Exercise 60.

EXAMPLE 5 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned}
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule}\\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}\\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

■

EXAMPLE 6 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$\begin{aligned}
y &= \sec x\\
y' &= \sec x \tan x && \text{Derivative rule for secant function}\\
y'' &= \frac{d}{dx}(\sec x \tan x)\\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Derivative Product Rule}\\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) && \text{Derivative rules}\\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

■

The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.2). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 7 We can use direct substitution in computing limits provided there is no division by zero, which is algebraically undefined.

$$\lim_{x\to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

■

Exercises 3.4

Derivatives

In Exercises 1–3, find dy/dx.

1. $y = x^2 \cos x - 2x \sin x - 2 \cos x$

2. $f(x) = x^3 \sin x \cos x$

3. $g(x) = (2 - x) \tan^2 x$

4. Find y'' if
   a. $y = \csc x$. **b.** $y = \sec x$.

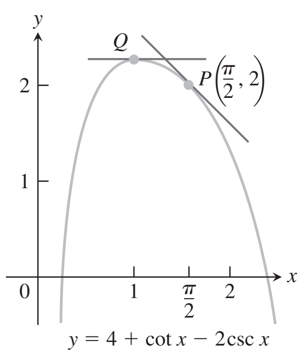
5. Find $y^{(4)} = d^4 y/dx^4$ if
   a. $y = -2 \sin x$. **b.** $y = 9 \cos x$.

6. Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

7. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

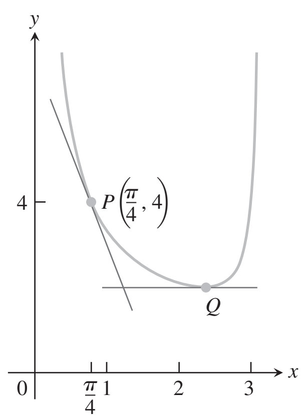
In Exercises 8 and 9, find an equation for **(a)** the tangent to the curve at P and **(b)** the horizontal tangent to the curve at Q.

8.



$y = 4 + \cot x - 2\csc x$

9.



$y = 1 + \sqrt{2} \csc x + \cot x$

Trigonometric Limits

Find the limits in Exercises 10–17.

10. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

11. $\lim_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

12. $\lim_{\theta \to \pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

13. $\lim_{\theta \to \pi/4} \frac{\tan\theta - 1}{\theta - \frac{\pi}{4}}$

14. $\lim_{x \to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$

15. $\lim_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$

16. $\lim_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

17. $\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

Theory and Examples

18. Is there a value of c that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

19. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

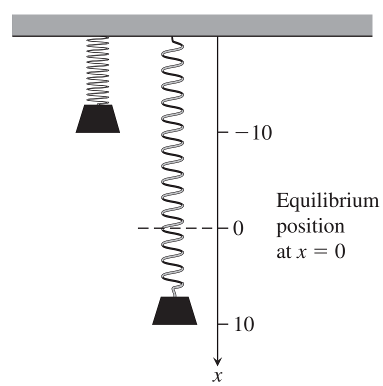
20. By computing the first few derivatives and looking for a pattern, find $d^{999}/dx^{999}(\cos x)$.

21. Derive the formula for the derivative with respect to x of
    a. $\sec x$. **b.** $\csc x$. **c.** $\cot x$.

22. A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10 \cos t,$$

where x is measured in centimeters and t is measured in seconds. See the accompanying figure.



   a. Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

   b. Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

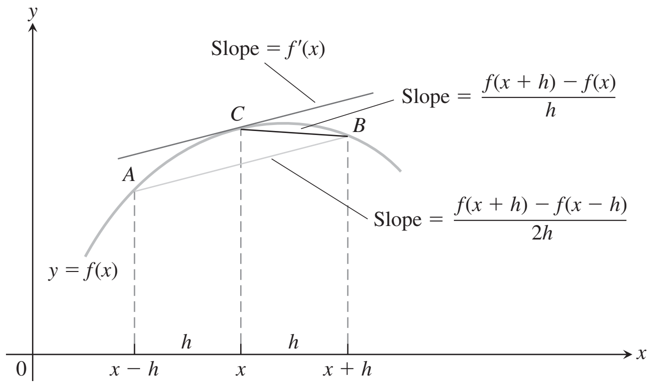
T 23. **Centered difference quotients** The *centered difference quotient*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than the difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

See the accompanying figure.



a. To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 63 for the same values of h.

b. To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 64 for the same values of h.

24. A caution about centered difference quotients (*Continuation of Exercise 23.*) The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

3.5 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composite $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g. We develop the rule in this section.

Derivative of a Composite Function

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

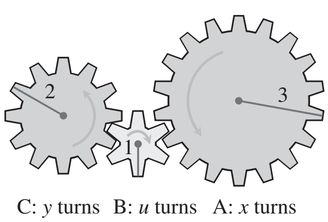


FIGURE 3.15 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. If $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x, then we expect y to change $3/2$ times as fast as x. This effect is much like that of a multiple gear train (Figure 3.15). Let's look at another example.

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \qquad \text{Substitute for } u \\ &= 36x^3 + 12x.\end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) \\ &= 36x^3 + 12x.\end{aligned}$$

■

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 3.16).

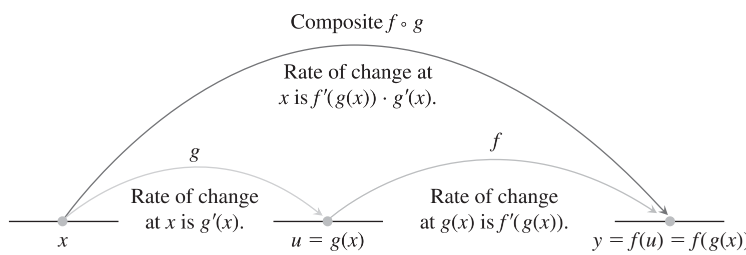


FIGURE 3.16 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

> **THEOREM 2—The Chain Rule** If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and
>
> $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$
>
> In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then
>
> $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$
>
> where dy/du is evaluated at $u = g(x)$.

A Proof of One Case of the Chain Rule:

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \tag{1}$$

and take the limit as $\Delta x \to 0$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} && \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0 \text{ since } g \text{ is continuous.)} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}.
\end{aligned}$$

The problem with this argument is that if the function $g(x)$ oscillates rapidly near x, then Δu can be zero even when $\Delta x \neq 0$, so the cancelation of Δu in Equation (1) would be invalid. A complete proof requires a different approach that avoids this problem, and we give one such proof in Section 3.9. ■

EXAMPLE 2 An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\begin{aligned}
\frac{dx}{du} &= -\sin(u) && x = \cos(u) \\
\frac{du}{dt} &= 2t. && u = t^2 + 1
\end{aligned}$$

By the Chain Rule,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\
&= -\sin(u) \cdot 2t && \frac{dx}{du} \text{ evaluated at } u \\
&= -\sin(t^2 + 1) \cdot 2t \\
&= -2t\sin(t^2 + 1).
\end{aligned}$$

■

Ways to Write the Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$$

"Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 3 Differentiate $\sin(x^2 + x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\substack{\text{inside} \\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside}}}.$$

■

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

HISTORICAL BIOGRAPHY
Johann Bernoulli
(1667–1748)

EXAMPLE 4 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

EXAMPLE 5 Suppose that f is a differentiable function such that $f(2) = 1$ and $f'(2) = 3$ and let $g(x) = f\big(x f(x)\big)$. Find $g'(2)$.

Solution

$$\begin{aligned} g(x) &= f(x \cdot f(x)) \\ g'(x) &= f'\big(x f(x)\big) \cdot \mathrm{D}\big(x f(x)\big) \\ &= f'\big(x f(x)\big)\ [x f'(x) + f(x)] \\ g'(x) &= f'(2)\ [2 \cdot 3 + 1] = 3 \cdot 7 = 21 \end{aligned}$$

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a) $$\begin{aligned} \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\ &= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

(b) $$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$$ Power Chain Rule with $u = 3x - 2, n = -1$

$$= -1(3x-2)^{-2}(3)$$

$$= -\frac{3}{(3x-2)^2}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

(c) $$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$ Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

$$= 5\sin^4 x\cos x$$ ■

Derivative of the Absolute Value Function

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0$$
$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

EXAMPLE 7 In Section 3.2, we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\sqrt{x^2}$$

$$= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2)$$ Power Chain Rule with $u = x^2, n = 1/2, x \neq 0$

$$= \frac{1}{2|x|} \cdot 2x$$ $\sqrt{x^2} = |x|$

$$= \frac{x}{|x|}, \quad x \neq 0.$$ ■

EXAMPLE 8 Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Solution We find the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3}$$

$$= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x)$$ Power Chain Rule with $u = (1-2x), n = -3$

$$= -3(1-2x)^{-4} \cdot (-2)$$

$$= \frac{6}{(1-2x)^4}.$$

At any point (x, y) on the curve, the coordinate x is not $1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1-2x)^4},$$

which is the quotient of two positive numbers. ■

EXAMPLE 9 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians where $x°$ is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x^\circ).$$

See Figure 3.17. Similarly, the derivative of $\cos(x^\circ)$ is $-(\pi/180)\sin(x^\circ)$.

The factor $\pi/180$ would compound with repeated differentiation, showing an advantage for the use of radian measure in computations. ■

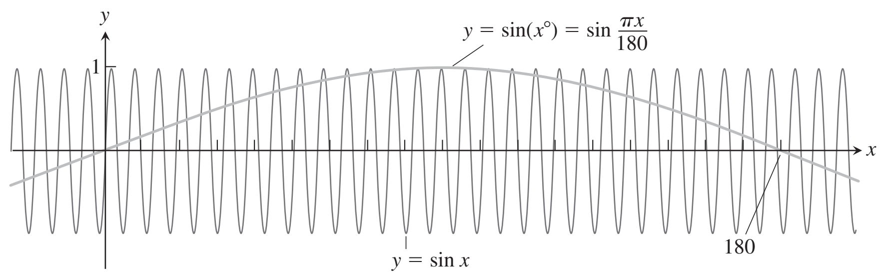


FIGURE 3.17 The function $\sin(x^\circ)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

Exercises 3.5

Derivative Calculations

Find the derivatives of the functions in Exercises 1–10.

1. $y = \frac{1}{x}\sin^{-5}x - \frac{x}{3}\cos^3 x$

2. $y = \frac{1}{18}(3x - 2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

3. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$

4. $y = (4x + 3)^4(x + 1)^{-3}$

5. $y = (2x - 5)^{-1}(x^2 - 5x)^6$

6. $g(x) = \frac{\tan 3x}{(x + 7)^4}$

7. $f(\theta) = \left(\frac{\sin\theta}{1 + \cos\theta}\right)^2$

8. $g(t) = \left(\frac{1 + \sin 3t}{3 - 2t}\right)^{-1}$

9. $r = \sin(\theta^2)\cos(2\theta)$

10. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$

In Exercises 11–13, find dy/dt.

11. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

12. $y = 3t(2t^2 - 5)^4$

13. $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

Second Derivatives

Find y'' in Exercises 14–19.

14. $y = \left(1 + \frac{1}{x}\right)^3$

15. $y = \left(1 - \sqrt{x}\right)^{-1}$

16. $y = \frac{1}{9}\cot(3x - 1)$

17. $y = 9\tan\left(\frac{x}{3}\right)$

18. $y = x(2x + 1)^4$

19. $y = x^2(x^3 - 1)^5$

20. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

21. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

22. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x)$, $x = 2$
b. $f(x) + g(x)$, $x = 3$
c. $f(x)\cdot g(x)$, $x = 3$
d. $f(x)/g(x)$, $x = 2$
e. $f(g(x))$, $x = 2$
f. $\sqrt{f(x)}$, $x = 2$
g. $1/g^2(x)$, $x = 3$
h. $\sqrt{f^2(x) + g^2(x)}$, $x = 2$

23. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $5f(x) - g(x)$, $x = 1$
b. $f(x)g^3(x)$, $x = 0$
c. $\frac{f(x)}{g(x) + 1}$, $x = 1$
d. $f(g(x))$, $x = 0$
e. $g(f(x))$, $x = 0$
f. $(x^{11} + f(x))^{-2}$, $x = 1$
g. $f(x + g(x))$, $x = 0$

3.6 Method of Differentiation

Historically, logarithms played important roles in arithmetic computations, making possible the great seventeenth-century advances in offshore navigation and celestial mechanics. In this section we define the natural logarithm as an integral using the Fundamental Theorem of Calculus. While this indirect approach may at first seem strange, it provides an elegant and rigorous way to obtain the key characteristics of logarithmic and exponential functions.

Definition of the Natural Logarithm Function

The natural logarithm of any positive number x, written as $\ln x$, is defined as an integral.

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

From the Fundamental Theorem of Calculus, $\ln x$ is a continuous function. Geometrically, if $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 3.18). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1, and the function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

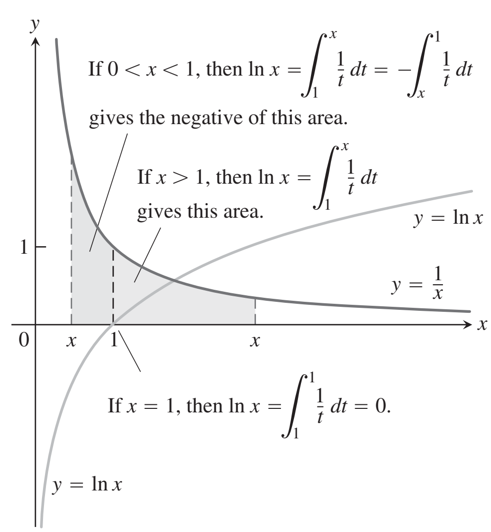


FIGURE 3.18 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 3.1 Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Notice that we show the graph of $y = 1/x$ in Figure 3.18 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function ln x. Several values are given in Table 3.1. There is an important number between $x = 2$ and $x = 3$ whose natural logarithm equals 1. This number, which we now define, exists because ln x is a continuous function and therefore satisfies the Intermediate Value Theorem on $[2, 3]$.

DEFINITION The **number *e*** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

Interpreted geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ equals the area of the unit square. That is, the area of the region shaded blue in Figure 7.1 is 1 sq unit when $x = e$. In the next section, we will see that the number e can be calculated as a limit and has the numerical value $e \approx 2.718281828459045$ to 15 decimal places.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{1}$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive.

If u is a differentiable function of x whose values are positive, so that ln u is defined, then applying the Chain Rule we obtain

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$

EXAMPLE 1 We use Equation (2) to find derivatives.

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3}\cdot\frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3}\cdot 2x = \frac{2x}{x^2 + 3}.$$

(c) Equation (2) with $u = |x|$ gives an important derivative:

$$\begin{aligned}
\frac{d}{dx}\ln|x| &= \frac{d}{du}\ln u \cdot \frac{du}{dx} && u = |x|,\ x \neq 0 \\
&= \frac{1}{u}\cdot\frac{x}{|x|} && \frac{d}{dx}(|x|) = \frac{x}{|x|} \\
&= \frac{1}{|x|}\cdot\frac{x}{|x|} && \text{Substitute for } u. \\
&= \frac{x}{x^2} \\
&= \frac{1}{x}.
\end{aligned}$$

Derivative of ln $|x|$

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \neq 0$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

$$\frac{d}{dx}\ln bx = \frac{1}{x}, \quad bx > 0$$

Notice from Example 1a that the function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant b, provided that $bx > 0$:

$$\frac{d}{dx}\ln bx = \frac{1}{bx}\cdot\frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}.$$

The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down. (See Figure 3.19a.)

We can estimate the value of ln 2 by considering the area under the graph of $y = 1/x$ and above the interval $[1, 2]$. In Figure 3.19(b) a rectangle of height $1/2$ over the interval $[1, 2]$ fits under the graph. Therefore the area under the graph, which is ln 2, is greater than the area, $1/2$, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}.$$

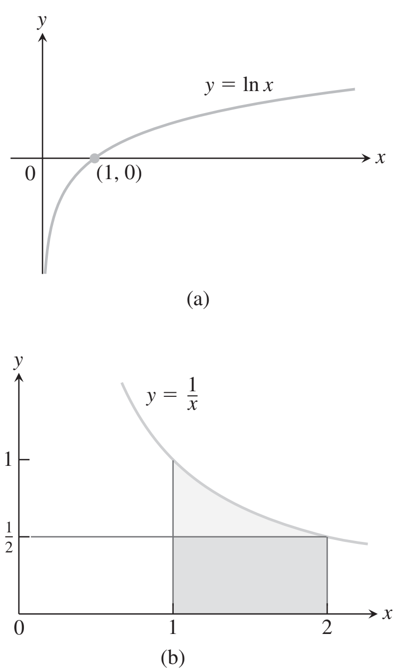


FIGURE 3.19 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \leq x \leq 2$.

This result shows that $\ln(2^n) \to \infty$ as $n \to \infty$. Since $\ln x$ is an increasing function, we get that

$$\lim_{x\to\infty}\ln x = \infty.$$

We also have

$$\lim_{x\to 0^+}\ln x = \lim_{t\to\infty}\ln t^{-1} = \lim_{t\to\infty}(-\ln t) = -\infty. \qquad x = 1/t = t^{-1}$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line, giving the graph of $y = \ln x$ shown in Figure 3.19(a).

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 2 Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln((x^2 + 1)(x + 3)^{1/2}) - \ln(x - 1) && \text{Quotient Rule} \\
&= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Product Rule} \\
&= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). && \text{Power Rule}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (2) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y from the original equation:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$ ■

A direct computation in Example 5, using the Derivative Quotient and Product Rules, would be much longer.

Parametric Differentiation

Figure 3.20 shows the path of a moving particle in the xy-plane. Notice that the path fails the vertical line test, so it cannot be described as the graph of a function of the variable x. However, we can sometimes describe the path by a pair of equations, $x = f(t)$ and $y = g(t)$, where f and g are continuous functions. When studying motion, t usually denotes time. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.

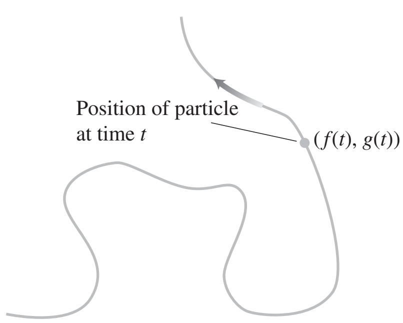


FIGURE 3.20 The curve or path traced by a particle moving in the xy-plane is not always the graph of a function or single equation.

> **DEFINITION** If x and y are given as functions
>
> $$x = f(t), \qquad y = g(t)$$
>
> over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point**. When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval together constitute a **parametrization** of the curve. A given curve can be represented by different sets of parametric equations.

EXAMPLE 3 Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

Solution We make a brief table of values (Table 3.2), plot the points (x, y), and draw a smooth curve through them (Figure 3.21). Each value of t gives a point (x, y) on the curve, such as $t = 1$ giving the point $(1, 2)$ recorded in Table 3.2. If we think of the curve as the path of a moving particle, then the particle moves along the curve in the direction of the arrows shown in Figure 3.21. Although the time intervals in the table are equal, the consecutive points plotted along the curve are not at equal arc length distances. The reason for this is that the particle slows down as it gets nearer to the y-axis along the lower branch of the curve as t increases, and then speeds up after reaching the y-axis at $(0, 1)$ and moving along the upper branch. Since the interval of values for t is all real numbers, there is no initial point and no terminal point for the curve. ■

TABLE 3.2 Values of $x = t^2$ and $y = t + 1$ for selected values of t.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

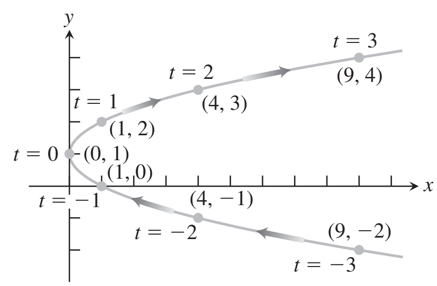


FIGURE 3.21 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 4).

EXAMPLE 4 Identify geometrically the curve in Example 1 (Figure 3.21) by eliminating the parameter t and obtaining an algebraic equation in x and y.

Solution We solve the equation $y = t + 1$ for the parameter t and substitute the result into the parametric equation for x. This procedure gives $t = y - 1$ and

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1.$$

The equation $x = y^2 - 2y + 1$ represents a parabola, as displayed in Figure 3.21. It is sometimes quite difficult, or even impossible, to eliminate the parameter from a pair of parametric equations, as we did here. ■

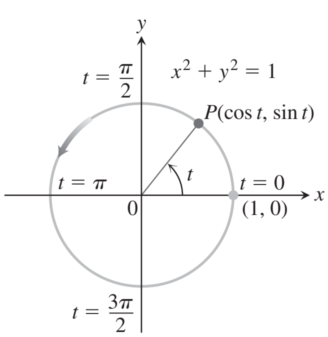


Figure 3.22 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 5).

EXAMPLE 5 Graph the parametric curves

(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$

Solution

(a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the parametric curve lies along the unit circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at $(1, 0)$ and traces the entire circle once counterclockwise (Figure 3.22).

(b) For $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$, we have $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. The parametrization describes a motion that begins at the point $(a, 0)$ and traverses the circle $x^2 + y^2 = a^2$ once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$. The graph is a circle centered at the origin with radius $r = |a|$ and coordinate points $(a\cos t, a\sin t)$. ■

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

If parametric equations define y as a twice-differentiable function of x, we can apply Equation (1) to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}. \qquad \text{Eq. (1) with } y' \text{ in place of } y$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$

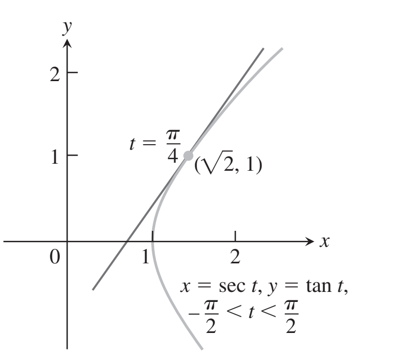


FIGURE 3.23 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

EXAMPLE 6 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $\left(\sqrt{2}, 1\right)$, where $t = \pi/4$ (Figure 3.23).

Solution The slope of the curve at t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}. \qquad \text{Eq. (1)}$$

Setting t equal to $\pi/4$ gives

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{t=\pi/4} &= \frac{\sec(\pi/4)}{\tan(\pi/4)} \\ &= \frac{\sqrt{2}}{1} = \sqrt{2}. \end{aligned}$$

The tangent line is

$$\begin{aligned} y - 1 &= \sqrt{2}\left(x - \sqrt{2}\right) \\ y &= \sqrt{2}\,x - 2 + 1 \\ y &= \sqrt{2}\,x - 1. \end{aligned}$$

■

EXAMPLE 7 Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Finding d^2y/dx^2 in Terms of t

1. Express $y' = dy/dx$ in terms of t.
2. Find dy'/dt.
3. Divide dy'/dt by dx/dt.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Derivative Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3} \qquad \text{Eq. (2)}$$

■

Exercises 3.6

Finding Derivatives

In Exercises 1–30, find the derivative of y with respect to x, t, or θ, as appropriate.

1. $y = \ln 3x$
2. $y = \ln kx$, k constant
3. $y = \ln (t^2)$
4. $y = \ln (t^{3/2})$
5. $y = \ln \dfrac{3}{x}$
6. $y = \ln \dfrac{10}{x}$
7. $y = \ln (\theta + 1)$
8. $y = \ln (2\theta + 2)$
9. $y = \ln x^3$
10. $y = (\ln x)^3$
11. $y = t(\ln t)^2$
12. $y = t\sqrt{\ln t}$
13. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$
14. $y = (x^2 \ln x)^4$
15. $y = \dfrac{\ln t}{t}$
16. $y = \dfrac{1 + \ln t}{t}$
17. $y = \dfrac{\ln x}{1 + \ln x}$
18. $y = \dfrac{x \ln x}{1 + \ln x}$
19. $y = \ln (\ln x)$
20. $y = \ln (\ln (\ln x))$
21. $y = \theta(\sin(\ln \theta) + \cos(\ln \theta))$
22. $y = \ln (\sec \theta + \tan \theta)$
23. $y = \ln \dfrac{1}{x\sqrt{x + 1}}$
24. $y = \dfrac{1}{2}\ln \dfrac{1 + x}{1 - x}$
25. $y = \dfrac{1 + \ln t}{1 - \ln t}$
26. $y = \sqrt{\ln \sqrt{t}}$
27. $y = \ln (\sec (\ln \theta))$
28. $y = \ln \left(\dfrac{\sqrt{\sin \theta \cos \theta}}{1 + 2 \ln \theta}\right)$
29. $y = \ln \left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$
30. $y = \ln \sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

Logarithmic Differentiation

In Exercises 31–44, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

31. $y = \sqrt{x(x + 1)}$
32. $y = \sqrt{(x^2 + 1)(x - 1)^2}$
33. $y = \sqrt{\dfrac{t}{t + 1}}$
34. $y = \sqrt{\dfrac{1}{t(t + 1)}}$
35. $y = \sqrt{\theta + 3}\sin \theta$
36. $y = (\tan \theta)\sqrt{2\theta + 1}$
37. $y = t(t + 1)(t + 2)$
38. $y = \dfrac{1}{t(t + 1)(t + 2)}$
39. $y = \dfrac{\theta + 5}{\theta \cos \theta}$
40. $y = \dfrac{\theta \sin \theta}{\sqrt{\sec \theta}}$
41. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$
42. $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$
43. $y = \sqrt[3]{\dfrac{x(x - 2)}{x^2 + 1}}$
44. $y = \sqrt[3]{\dfrac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}}$

Tangents to Parametrized Curves

In Exercises 45–58, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

45. $x = 2 \cos t, \quad y = 2 \sin t, \quad t = \pi/4$
46. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$
47. $x = 4 \sin t, \quad y = 2 \cos t, \quad t = \pi/4$
48. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
49. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
50. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

51. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$

52. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$

53. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

54. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$

55. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

56. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

57. $x = \dfrac{1}{t+1}, \quad y = \dfrac{t}{t-1}, \quad t = 2$

58. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 59–64 define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

59. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

60. $x = \sqrt{5 - \sqrt{t}}, \quad y(t-1) = \sqrt{t}, \quad t = 4$

61. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

62. $x \sin t + 2x = t, \quad t \sin t - 2t = y, \quad t = \pi$

63. $x = t^3 + t, \quad y + 2t^3 = 2x + t^2, \quad t = 1$

64. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

3.7 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

(See Figures 3.24, 3.25, and 3.26.) These equations define an *implicit* relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This section describes the technique.

Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for y as a function of x to calculate dy/dx in the usual way. Then we differentiate the equations implicitly, and find the derivative to compare the two methods. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines y implicitly as a differentiable function of x so that dy/dx exists.

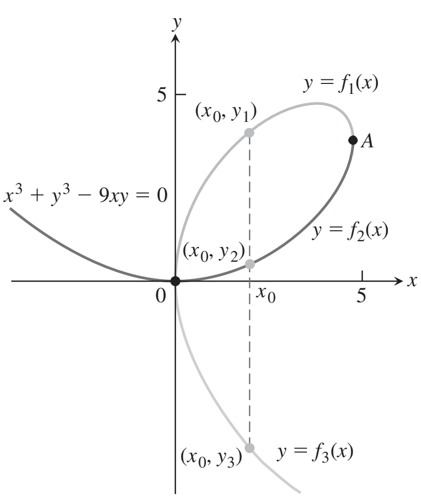


FIGURE 3.24 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.25). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x \qquad \text{The Chain Rule gives } \frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}.$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

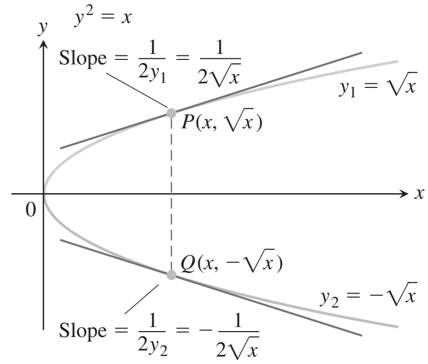


FIGURE 3.25 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

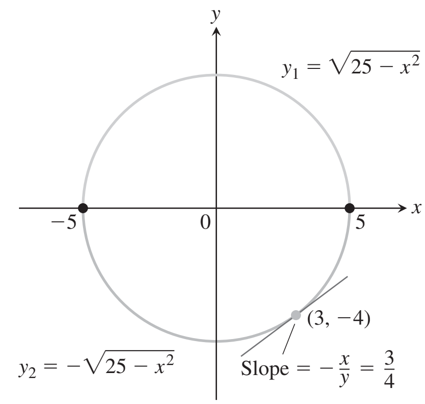


FIGURE 3.26 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2\left(-\sqrt{x}\right)} = -\frac{1}{2\sqrt{x}}. \qquad \blacksquare$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather, it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.26). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

$\frac{d}{dx}\left(-(25 - x^2)^{1/2}\right) =$
$-\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$

We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{See Example 1.}$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Notice that unlike the slope formula for dy_2/dx, which applies only to points below the x-axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope; that is, at all circle points (x, y) where $y \neq 0$. Notice also that the derivative involves *both* variables x and y, not just the independent variable x. ■

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

> **Implicit Differentiation**
>
> **1.** Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
>
> **2.** Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.27).

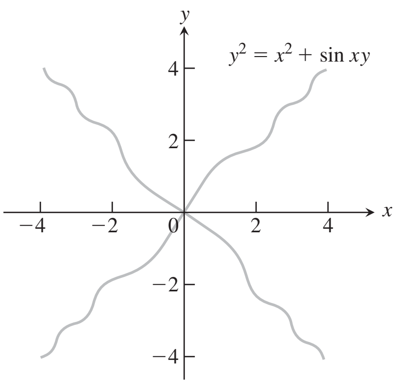


FIGURE 3.27 The graph of the equation in Example 3.

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy) \qquad \ldots \text{ treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right) \qquad \text{Treat } xy \text{ as a product.}$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y \qquad \text{Collect terms with } dy/dx.$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \qquad \text{Solve for } dy/dx.$$

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y, not just the independent variable x. ■

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}\left(2x^3 - 3y^2\right) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

■

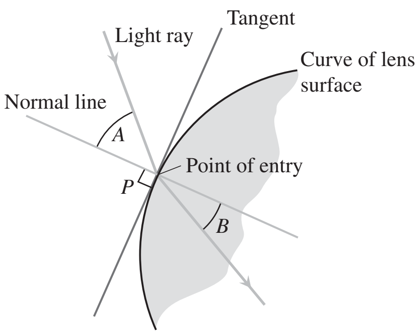


FIGURE 3.29 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.28). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.28, the **normal** is the line perpendicular (also said to be *orthogonal*) to the tangent of the profile curve at the point of entry.

EXAMPLE 5 Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.29).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0 \qquad \text{Differentiate both sides with respect to } x.$$

$$\left(3y^2 - 9x\right)\frac{dy}{dx} + 3x^2 - 9y = 0 \qquad \text{Treat } xy \text{ as a product and } y \text{ as a function of } x.$$

$$3\left(y^2 - 3x\right)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope $4/5$:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

■

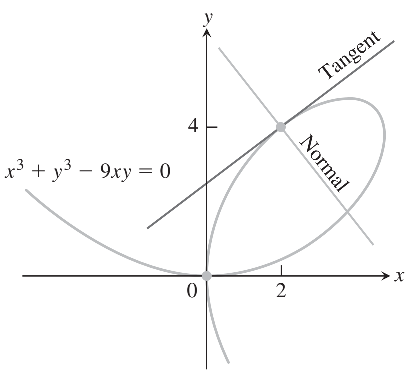


FIGURE 3.29 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

Exercises 3.7

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–5.

1. $x^3 = \dfrac{2x - y}{x + 3y}$

2. $xy = \cot(xy)$

3. $x + \tan(xy) = 0$

4. $x^4 + \sin y = x^3y^2$

5. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$

Derivatives

In Exercises 6–11, use implicit differentiation to find dy/dx and then d^2y/dx^2.

6. $x^2 + y^2 = 1$

7. $x^{2/3} + y^{2/3} = 1$

8. $y^2 = x^2 + 2x$

9. $y^2 - 2x = 1 - 2y$

10. $2\sqrt{y} = x - y$

11. $xy + y^2 = 1$

12. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

13. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

In Exercises 14 and 15, find the slope of the curve at the given points.

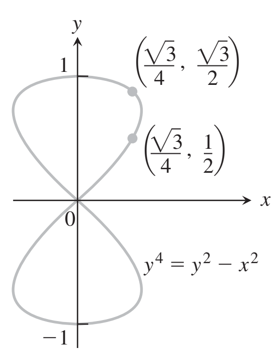
14. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

15. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

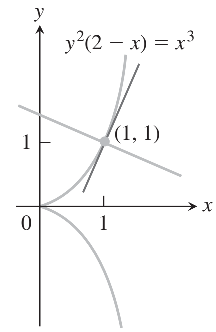
16. **Parallel tangents** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

17. **Normals parallel to a line** Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

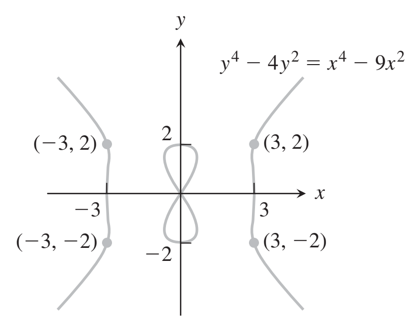
18. **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



19. **The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.



20. **The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



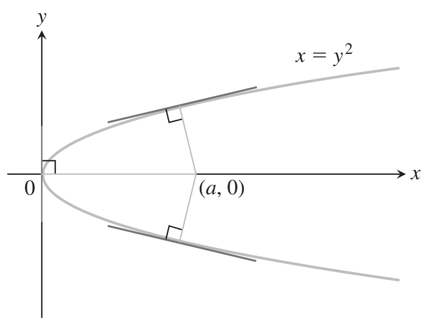
21. **The folium of Descartes** (See Figure 3.24.)
    a. Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.
    b. At what point other than the origin does the folium have a horizontal tangent?
    c. Find the coordinates of the point A in Figure 3.24 where the folium has a vertical tangent.

Theory and Examples

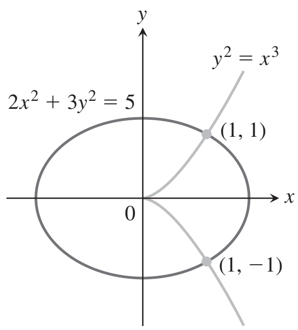
22. **Intersecting normal** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

23. **Power rule for rational exponents** Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,
$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

24. **Normals to a parabola** Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



25. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

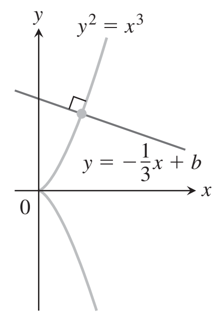


26. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4, \quad x^2 = 3y^2$

b. $x = 1 - y^2, \quad x = \frac{1}{3}y^2$

27. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



3.8 Inverse Trigonometric Functions and their Derivatives

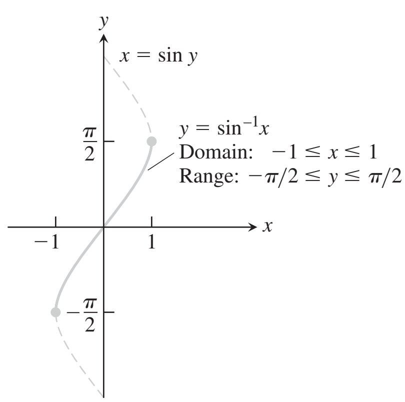


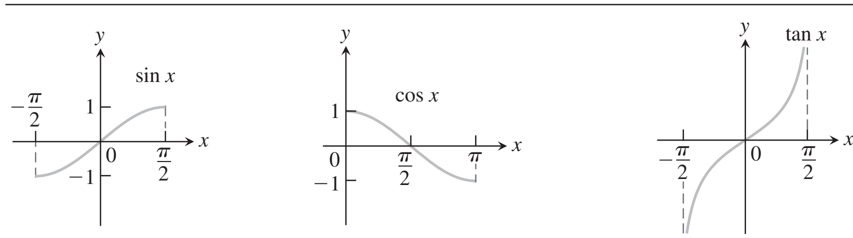
FIGURE 3.30 The graph of $y = \sin^{-1} x$.

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 3.30). Similar domain restrictions can be applied to all six trigonometric functions.

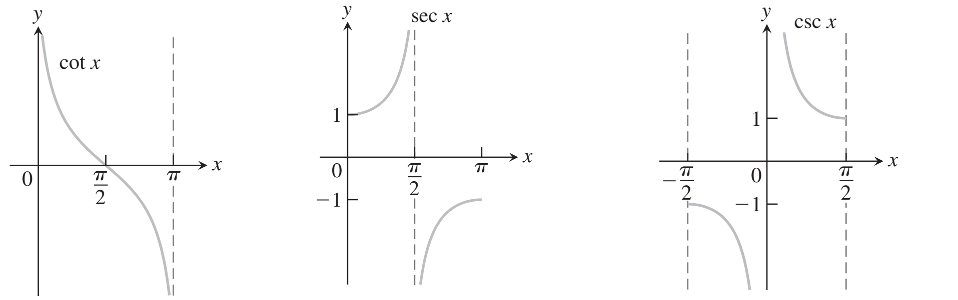
Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{array}{lcl}
y = \sin^{-1} x & \text{or} & y = \arcsin x \\
y = \cos^{-1} x & \text{or} & y = \arccos x \\
y = \tan^{-1} x & \text{or} & y = \arctan x \\
y = \cot^{-1} x & \text{or} & y = \operatorname{arccot} x \\
y = \sec^{-1} x & \text{or} & y = \operatorname{arcsec} x \\
y = \csc^{-1} x & \text{or} & y = \operatorname{arccsc} x
\end{array}
$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. Figure 3.31b shows the graph of $y = \sin^{-1} x$ and Figure 3.32 shows the graphs of all six functions. We now take a closer look at these functions.

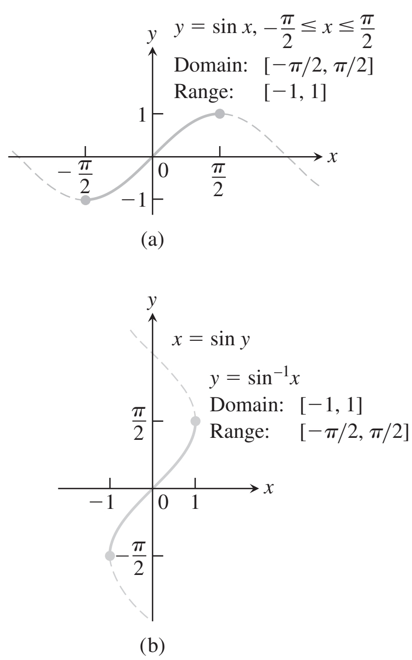


FIGURE 3.31 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

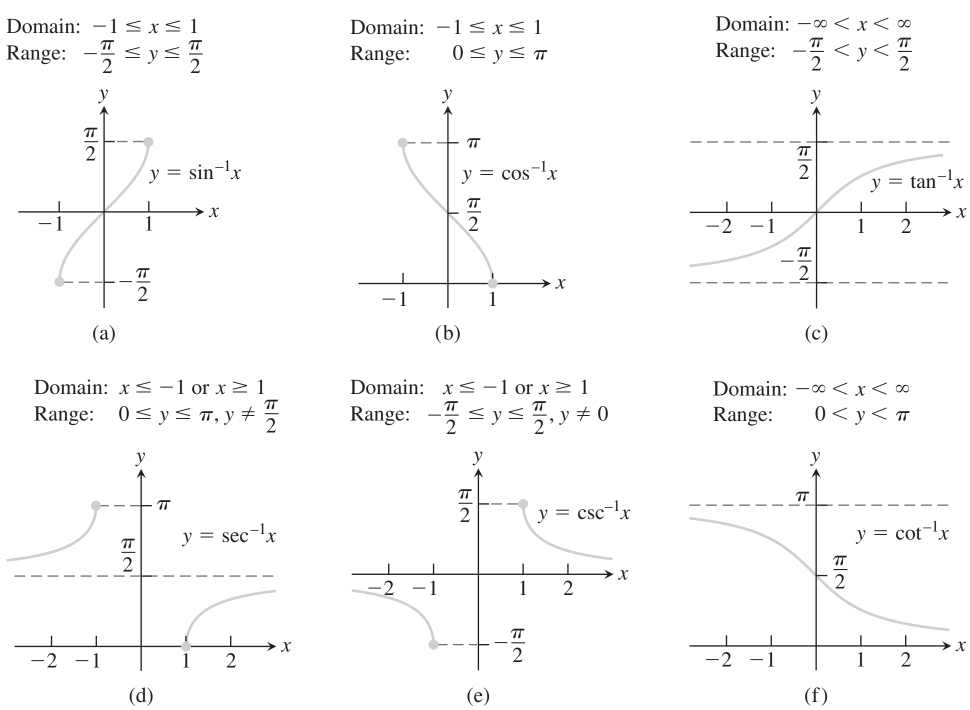
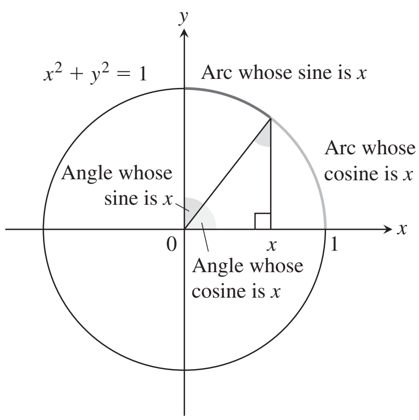


FIGURE 3.32 Graphs of the six basic inverse trigonometric functions.

The "Arc" in Arcsine and Arccosine
For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

DEFINITION

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

The graph of $y = \sin^{-1} x$ (Figure 3.31b) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{1}$$

The graph of $y = \cos^{-1} x$ (Figure 3.33b) has no such symmetry.

EXAMPLE 1 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 3.34a.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 3.34b. ■

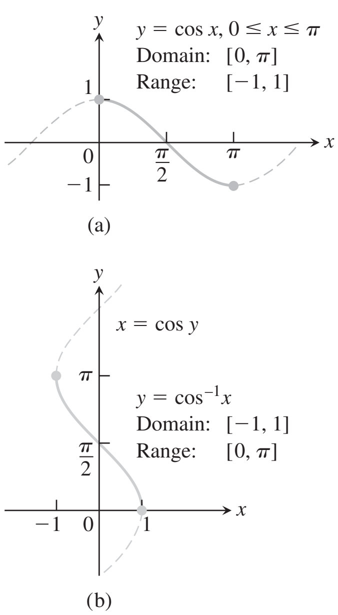


FIGURE 3.33 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

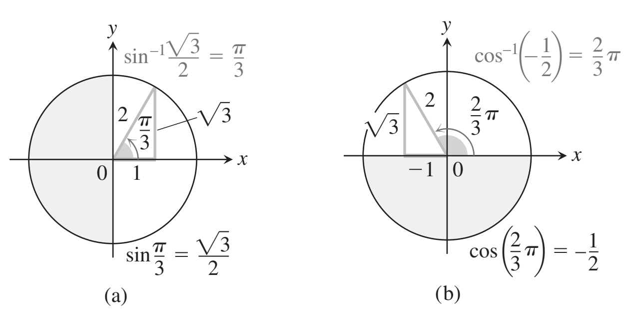


FIGURE 3.34 Values of the arcsine and arccosine functions (Example 1).

Using the same procedure illustrated in Example 1, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

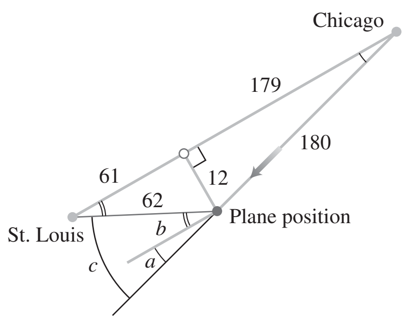


FIGURE 3.35 Diagram for drift correction (Example 2), with distances rounded to the nearest mile (drawing not to scale).

EXAMPLE 2 During a 240 mi airplane flight from Chicago to St. Louis, after flying 180 mi the navigator determines that the plane is 12 mi off course, as shown in Figure 3.35. Find the angle a for a course parallel to the original correct course, the angle b, and the drift correction angle $c = a + b$.

Solution From the Pythagorean theorem and given information, we compute an approximate hypothetical flight distance of 179 mi, had the plane been flying along the original correct course (see Figure 3.35). Knowing the flight distance from Chicago to St. Louis, we next calculate the remaining leg of the original course to be 61 mi. Applying the Pythagorean theorem again then gives an approximate distance of 62 mi from the position of the plane to St. Louis. Finally, from Figure 3.35, we see that $180 \sin a = 12$ and $62 \sin b = 12$, so

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8°$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2°$$

$$c = a + b \approx 15°. \qquad \blacksquare$$

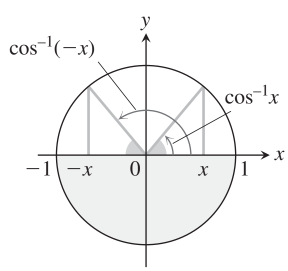


FIGURE 3.36 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

Identities Involving Arcsine and Arccosine

As we can see from Figure 3.36, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{2}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{3}$$

Also, we can see from the triangle in Figure 3.37 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{4}$$

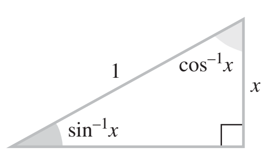


FIGURE 3.37 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

Equation (4) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 3.37. It is, however, a consequence of Equations (1) and (3) (Exercise 113).

Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The arctangent of x is a radian angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x, and so forth. The angles belong to the restricted domains of the tangent, cotangent, secant, and cosecant functions.

DEFINITIONS

$y = \mathbf{tan^{-1}}\, \boldsymbol{x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \mathbf{cot^{-1}}\, \boldsymbol{x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \mathbf{sec^{-1}}\, \boldsymbol{x}$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$y = \mathbf{csc^{-1}}\, \boldsymbol{x}$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$

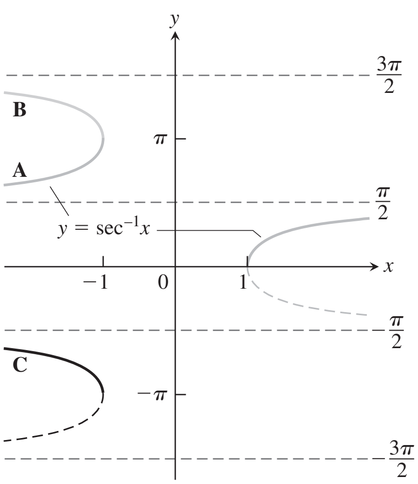


FIGURE 3.38 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

We use open or half-open intervals to avoid values for which the tangent, cotangent, secant, and cosecant functions are undefined. (See Figure 3.32.)

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.32c). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 3.32f). Notice from Figure 3.32c that the graph of the arctangent function has two horizontal asymptotes: one at $y = \pi/2$ and the other at $y = -\pi/2$.

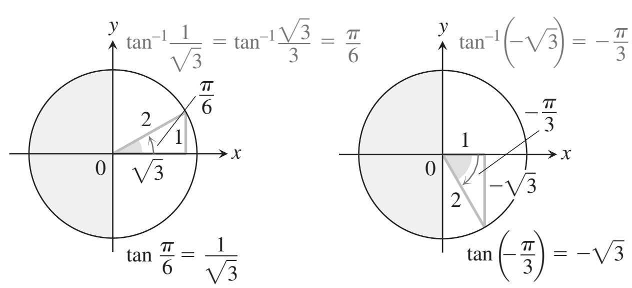
The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 3.32d and 3.32e.

Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some texts choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.38). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{5}$$

by applying Equation (4).

EXAMPLE 3 The accompanying figures show two values of $\tan^{-1} x$.



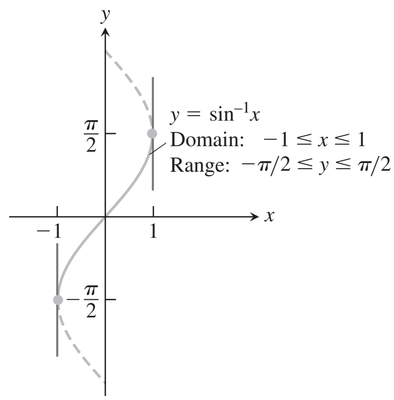


FIGURE 3.39 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

The Derivative of $y = \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 3.39).

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 1 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}
$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$
\boxed{\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.}
$$

EXAMPLE 4 Using the Chain Rule, we calculate the derivative

$$
\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}.
$$
■

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u
\end{aligned}
$$

$$= \frac{1}{1 + \tan^2(\tan^{-1} x)} \qquad \sec^2 u = 1 + \tan^2 u$$

$$= \frac{1}{1 + x^2}. \qquad \tan(\tan^{-1} x) = x$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\boxed{\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\frac{du}{dx}.}$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 1 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}. \qquad \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

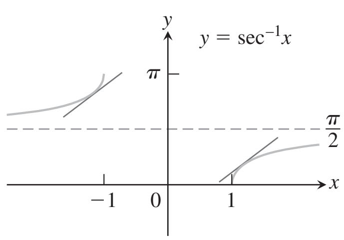


FIGURE 3.40 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

Can we do anything about the $\pm$ sign? A glance at Figure 3.40 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\boxed{\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.}$$

EXAMPLE 5 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned}\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4)\\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) && 5x^4 > 1 > 0\\ &= \frac{4}{x\sqrt{25x^8 - 1}}.\end{aligned}$$

■

Derivatives of the Other Three Inverse Trigonometric Functions

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1}x = \pi/2 - \sin^{-1}x$$
$$\cot^{-1}x = \pi/2 - \tan^{-1}x$$
$$\csc^{-1}x = \pi/2 - \sec^{-1}x$$

We saw the first of these identities in Equation (4). The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1}x$ is calculated as follows:

$$\begin{aligned}\frac{d}{dx}(\cos^{-1}x) &= \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right) && \text{Identity}\\ &= -\frac{d}{dx}(\sin^{-1}x)\\ &= -\frac{1}{\sqrt{1 - x^2}}. && \text{Derivative of arcsine}\end{aligned}$$

The derivatives of the inverse trigonometric functions are summarized in Table 3.3.

TABLE 3.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1}u)}{dx} = \dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1}u)}{dx} = -\dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1}u)}{dx} = \dfrac{1}{1 + u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\cot^{-1}u)}{dx} = -\dfrac{1}{1 + u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\sec^{-1}u)}{dx} = \dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1}u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

Exercises 3.8

Finding Derivatives

In Exercises 1–22, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$
2. $y = \cos^{-1}(1/x)$
3. $y = \sin^{-1}\sqrt{2}\,t$
4. $y = \sin^{-1}(1 - t)$
5. $y = \sec^{-1}(2s + 1)$
6. $y = \sec^{-1} 5s$
7. $y = \csc^{-1}(x^2 + 1),\ x > 0$
8. $y = \csc^{-1}\dfrac{x}{2}$
9. $y = \sec^{-1}\dfrac{1}{t},\ 0 < t < 1$
10. $y = \sin^{-1}\dfrac{3}{t^2}$
11. $y = \cot^{-1}\sqrt{t}$
12. $y = \cot^{-1}\sqrt{t - 1}$
13. $y = \ln(\tan^{-1} x)$
14. $y = \tan^{-1}(\ln x)$
15. $y = \csc^{-1}(e^t)$
16. $y = \cos^{-1}(e^{-t})$
17. $y = s\sqrt{1 - s^2} + \cos^{-1} s$
18. $y = \sqrt{s^2 - 1} - \sec^{-1} s$
19. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x,\quad x > 1$
20. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$
21. $y = x\sin^{-1} x + \sqrt{1 - x^2}$
22. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

Applications and Theory

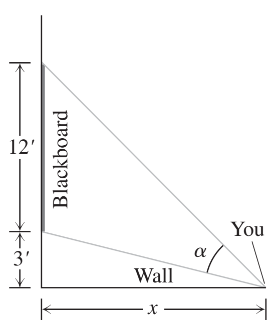
23. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to.

   a. Show that your viewing angle is

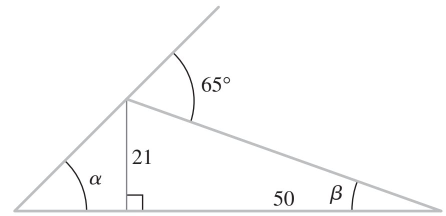
$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

   if you are x ft from the front wall.

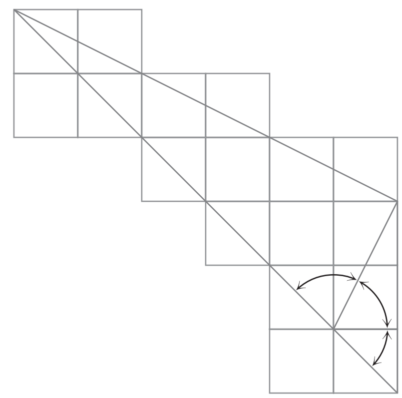
   b. Find x so that α is as large as possible.



24. Find the angle α.

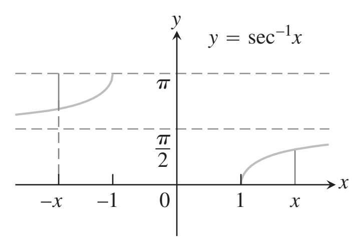


25. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



26. **Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$**

   a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



   b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (3)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (5)}$$

3.9 Derivative of Inverse Function

Derivatives of Inverses of Differentiable Functions

We calculated the inverse of the function $f(x) = (1/2)x + 1$ as $f^{-1}(x) = 2x - 2$ in Example 3. If we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

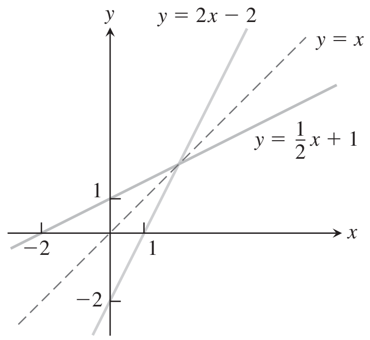
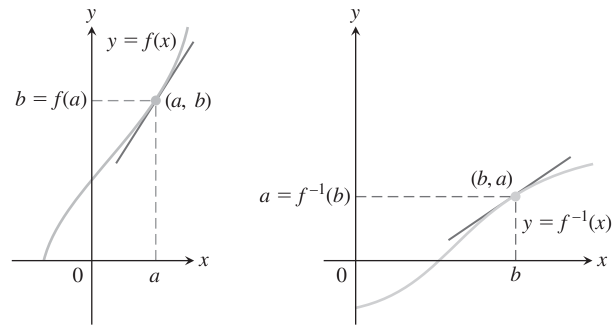


FIGURE 3.41 Graphing the functions $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.41.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 3.42 The graphs of inverse functions have reciprocal slopes at corresponding points.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 3.42). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$, then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable at $f(a)$. Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 1—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Theorem 1 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$
\begin{aligned}
f\left(f^{-1}(x)\right) &= x && \text{Inverse function relationship}\\
\frac{d}{dx} f\left(f^{-1}(x)\right) &= 1 && \text{Differentiating both sides}\\
f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx} f^{-1}(x) &= 1 && \text{Chain Rule}\\
\frac{d}{dx} f^{-1}(x) &= \frac{1}{f'\left(f^{-1}(x)\right)}. && \text{Solving for the derivative}
\end{aligned}
$$

EXAMPLE 1 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/\left(2\sqrt{x}\right)$.

Let's verify that Theorem 1 gives the same formula for the derivative of $f^{-1}(x)$:

$$
\begin{aligned}
\left(f^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)}\\
&= \frac{1}{2\left(f^{-1}(x)\right)} && f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x)\\
&= \frac{1}{2\left(\sqrt{x}\right)}.
\end{aligned}
$$

Theorem 1 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $\left(f^{-1}\right)'(4)$, are reciprocals. It states that

$$
\left(f^{-1}\right)'(4) = \frac{1}{f'\left(f^{-1}(4)\right)} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.
$$

See Figure 3.43. ■

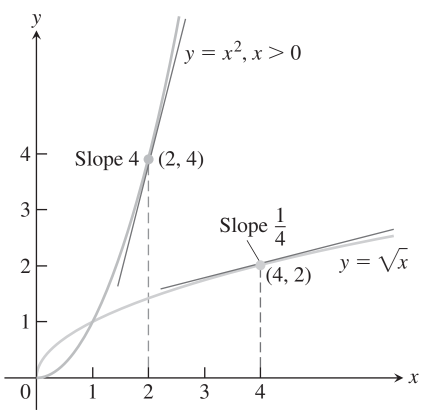


FIGURE 3.43 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 1).

We will use the procedure illustrated in Example 5 to calculate formulas for the derivatives of many inverse functions throughout this chapter. Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

EXAMPLE 2 Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 1 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$
\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12
$$

$$
\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}. \qquad \text{Eq. (1)}
$$

See Figure 3.44. ■

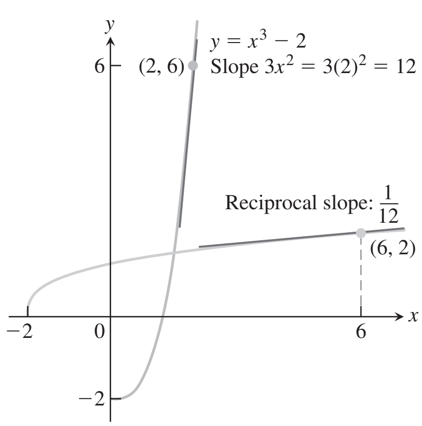


FIGURE 3.44 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 2).

Exercises 3.9

Derivatives of Inverse Functions

Each of Exercises 1–10 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

1. $f(x) = x^5$

2. $f(x) = x^4, \quad x \geq 0$

3. $f(x) = x^3 + 1$

4. $f(x) = (1/2)x - 7/2$

5. $f(x) = 1/x^2, \quad x > 0$

6. $f(x) = 1/x^3, \quad x \neq 0$

7. $f(x) = \dfrac{x + 3}{x - 2}$

8. $f(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 3}$

9. $f(x) = x^2 - 2x, \quad x \leq 1$ (*Hint:* Complete the square.)

10. $f(x) = (2x^3 + 1)^{1/5}$

In Exercises 11–14:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

11. $f(x) = 2x + 3, \quad a = -1$

12. $f(x) = (1/5)x + 7, \quad a = -1$

13. $f(x) = 5 - 4x, \quad a = 1/2$

14. $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

15. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

d. What lines are tangent to the curves at the origin?

16. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

b. Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of h and k at $(2, 2)$ and $(-2, -2)$.

d. What lines are tangent to the curves at the origin?

17. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

18. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

19. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

20. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

3.10 Indeterminate Forms and L'Hôpital's Rule

HISTORICAL BIOGRAPHY
Guillaume François Antoine de l'Hôpital
(1661–1704)
Johann Bernoulli
(1667–1748)

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule,** after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x\to 0}(\sin x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which produces the indeterminant form $0/0$ when we attempt to substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 2—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

We give a proof of Theorem 2 at the end of this section.

Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$$

(b) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

(c) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \qquad \frac{0}{0}\text{; apply l'Hôpital's Rule.}$$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; apply l'Hôpital's Rule again.}$$

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

(d) $$\lim_{x\to 0} \frac{x - \sin x}{x^3} \qquad \frac{0}{0}\text{; apply l'Hôpital's Rule.}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}\text{; apply l'Hôpital's Rule again.}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}\text{; apply l'Hôpital's Rule again.}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

■

Here is a summary of the procedure we followed in Example 1.

> **Using L'Hôpital's Rule**
>
> To find
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)}$$
>
> by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} \qquad \text{Not } \frac{0}{0}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. L'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x \to 0} (\sin x)/(1 + 2x)$ does not give an indeterminate form. Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0. ■

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

Recall that ∞ and $+\infty$ mean the same thing.

(a) $\displaystyle \lim_{x \to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

(b) $\displaystyle \lim_{x \to 0^-} \frac{\sin x}{x^2} \qquad \frac{0}{0}$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$

■

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞, as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\displaystyle \lim_{x\to\pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\displaystyle \lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\displaystyle \lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left so we apply l'Hôpital's Rule.}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

(a) $\displaystyle \lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\displaystyle \lim_{x\to 0^+} \sqrt{x} \ln x$

Solution

(a) $$\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+} \left(\frac{1}{h} \sin h\right) = \lim_{h\to 0^+} \frac{\sin h}{h} = 1 \qquad \infty \cdot 0;\ \text{let } h = 1/x.$$

(b) $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \qquad \infty \cdot 0 \text{ converted to } \infty/\infty$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} \qquad \text{l'Hôpital's Rule applied}$$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$ ■

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x \to 0}\frac{x - \sin x}{x \sin x} && \frac{0}{0}\\
&= \lim_{x \to 0}\frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0}\\
&= \lim_{x \to 0}\frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.
\end{aligned}$$

■

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and Theorem 10 in Section 2.5, and it is formulated as follows. (The formula is also valid for one-sided limits.)

> If $\lim_{x \to a} \ln f(x) = L$, then
>
> $$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$
>
> Here a may be either finite or infinite.

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+}(1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x}\ln (1 + x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned}
\lim_{x \to 0^+} \ln f(x) &= \lim_{x \to 0^+}\frac{\ln (1 + x)}{x} && \frac{0}{0}\\
&= \lim_{x \to 0^+}\frac{\dfrac{1}{1 + x}}{1} && \text{l'Hôpital's Rule applied}\\
&= \frac{1}{1} = 1.
\end{aligned}$$

Therefore, $\lim_{x \to 0^+}(1 + x)^{1/x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} && \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1} && \text{l'Hôpital's Rule applied}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

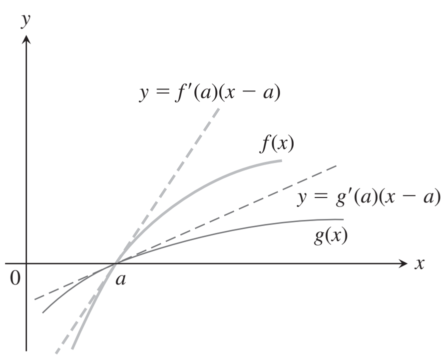


FIGURE 3.45 The two functions in l'Hôpital's Rule, graphed with their linear approximations at $x = a$.

Proof of L'Hôpital's Rule

Before we prove l'Hôpital's Rule, we consider a special case to provide some geometric insight for its reasonableness. Consider the two functions $f(x)$ and $g(x)$ having *continuous* derivatives and satisfying $f(a) = g(a) = 0$, $g'(a) \neq 0$. The graphs of $f(x)$ and $g(x)$, together with their linearizations $y = f'(a)(x - a)$ and $y = g'(a)(x - a)$, are shown in Figure 3.45. We know that near $x = a$, the linearizations provide good approximations to the functions. In fact,

$$f(x) = f'(a)(x - a) + \epsilon_1(x - a) \quad \text{and} \quad g(x) = g'(a)(x - a) + \epsilon_2(x - a)$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $x \to a$. So, as Figure 3.45 suggests,

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f'(a)(x - a) + \epsilon_1(x - a)}{g'(a)(x - a) + \epsilon_2(x - a)}\\ &= \lim_{x\to a} \frac{f'(a) + \epsilon_1}{g'(a) + \epsilon_2} = \frac{f'(a)}{g'(a)} && g'(a) \neq 0\\ &= \lim_{x\to a} \frac{f'(x)}{g'(x)}, && \text{Continuous derivatives}\end{aligned}$$

as asserted by l'Hôpital's Rule. We now proceed to a proof of the rule based on the more general assumptions stated in Theorem 2, which do not require that $g'(a) \neq 0$ and that the two functions have *continuous* derivatives.

HISTORICAL BIOGRAPHY
Augustin-Louis Cauchy
(1789–1857)

The proof of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

When $g(x) = x$, Theorem 6 is the Mean Value Theorem in Chapter 4.

THEOREM 3—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

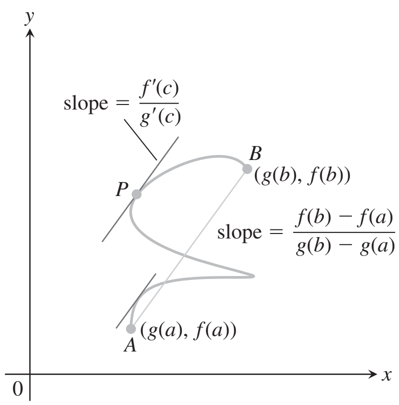


FIGURE 3.46 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Cauchy's Mean Value Theorem has a geometric interpretation for a general winding curve C in the plane joining the two points $A = (g(a), f(a))$ and $B = (g(b), f(b))$. In Chapter 11 you will learn how the curve C can be formulated to show that there is at least one point P on the curve for which the tangent to the curve at P is parallel to the secant line joining the points A and B. The slope of that tangent line turns out to be the quotient f'/g' evaluated at the number c in the interval (a, b), which is the left-hand side of the equation in Theorem 3. Because the slope of the secant line joining A and B is

$$\frac{f(b) - f(a)}{g(b) - g(a)},$$

the equation in Cauchy's Mean Value Theorem says that the slope of the tangent line equals the slope of the secant line. This geometric interpretation is shown in Figure 3.46. Notice from the figure that it is possible for more than one point on the curve C to have a tangent line that is parallel to the secant line joining A and B.

Proof of l'Hôpital's Rule We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. $\blacksquare$

Exercises 3.10

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to-2}\dfrac{x+2}{x^2-4}$
2. $\lim_{x\to0}\dfrac{\sin 5x}{x}$
3. $\lim_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$
4. $\lim_{x\to1}\dfrac{x^3-1}{4x^3-x-3}$
5. $\lim_{x\to0}\dfrac{1-\cos x}{x^2}$
6. $\lim_{x\to\infty}\dfrac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to2}\dfrac{x-2}{x^2-4}$
8. $\lim_{x\to-5}\dfrac{x^2-25}{x+5}$
9. $\lim_{t\to-3}\dfrac{t^3-4t+15}{t^2-t-12}$
10. $\lim_{t\to-1}\dfrac{3t^3+3}{4t^3-t+3}$
11. $\lim_{x\to\infty}\dfrac{5x^3-2x}{7x^3+3}$
12. $\lim_{x\to\infty}\dfrac{x-8x^2}{12x^2+5x}$
13. $\lim_{t\to0}\dfrac{\sin t^2}{t}$
14. $\lim_{t\to0}\dfrac{\sin 5t}{2t}$
15. $\lim_{x\to0}\dfrac{8x^2}{\cos x-1}$
16. $\lim_{x\to0}\dfrac{\sin x-x}{x^3}$
17. $\lim_{\theta\to\pi/2}\dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$
18. $\lim_{\theta\to-\pi/3}\dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$
19. $\lim_{\theta\to\pi/2}\dfrac{1-\sin\theta}{1+\cos 2\theta}$
20. $\lim_{x\to1}\dfrac{x-1}{\ln x-\sin\pi x}$
21. $\lim_{x\to0}\dfrac{x^2}{\ln(\sec x)}$
22. $\lim_{x\to\pi/2}\dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$
23. $\lim_{t\to0}\dfrac{t(1-\cos t)}{t-\sin t}$
24. $\lim_{t\to0}\dfrac{t\sin t}{1-\cos t}$
25. $\lim_{x\to(\pi/2)^-}\left(x-\dfrac{\pi}{2}\right)\sec x$
26. $\lim_{x\to(\pi/2)^-}\left(\dfrac{\pi}{2}-x\right)\tan x$
27. $\lim_{\theta\to0}\dfrac{3^{\sin\theta}-1}{\theta}$
28. $\lim_{\theta\to0}\dfrac{(1/2)^{\theta}-1}{\theta}$
29. $\lim_{x\to0}\dfrac{x2^x}{2^x-1}$
30. $\lim_{x\to0}\dfrac{3^x-1}{2^x-1}$
31. $\lim_{x\to\infty}\dfrac{\ln(x+1)}{\log_2 x}$
32. $\lim_{x\to\infty}\dfrac{\log_2 x}{\log_3(x+3)}$
33. $\lim_{x\to0^+}\dfrac{\ln(x^2+2x)}{\ln x}$
34. $\lim_{x\to0^+}\dfrac{\ln(e^x-1)}{\ln x}$
35. $\lim_{y\to0}\dfrac{\sqrt{5y+25}-5}{y}$
36. $\lim_{y\to0}\dfrac{\sqrt{ay+a^2}-a}{y},\quad a>0$
37. $\lim_{x\to\infty}(\ln 2x-\ln(x+1))$
38. $\lim_{x\to0^+}(\ln x-\ln\sin x)$
39. $\lim_{x\to0^+}\dfrac{(\ln x)^2}{\ln(\sin x)}$
40. $\lim_{x\to0^+}\left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$
41. $\lim_{x\to1^+}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$
42. $\lim_{x\to0^+}(\csc x-\cot x+\cos x)$
43. $\lim_{\theta\to0}\dfrac{\cos\theta-1}{e^\theta-\theta-1}$
44. $\lim_{h\to0}\dfrac{e^h-(1+h)}{h^2}$
45. $\lim_{t\to\infty}\dfrac{e^t+t^2}{e^t-t}$
46. $\lim_{x\to\infty}x^2e^{-x}$
47. $\lim_{x\to0}\dfrac{x-\sin x}{x\tan x}$
48. $\lim_{x\to0}\dfrac{(e^x-1)^2}{x\sin x}$
49. $\lim_{\theta\to0}\dfrac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$
50. $\lim_{x\to0}\dfrac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x\to1^+}x^{1/(1-x)}$
52. $\lim_{x\to1^+}x^{1/(x-1)}$
53. $\lim_{x\to\infty}(\ln x)^{1/x}$
54. $\lim_{x\to e^+}(\ln x)^{1/(x-e)}$
55. $\lim_{x\to0^+}x^{-1/\ln x}$
56. $\lim_{x\to\infty}x^{1/\ln x}$
57. $\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}$
58. $\lim_{x\to0}(e^x+x)^{1/x}$
59. $\lim_{x\to0^+}x^x$
60. $\lim_{x\to0^+}\left(1+\dfrac{1}{x}\right)^x$
61. $\lim_{x\to\infty}\left(\dfrac{x+2}{x-1}\right)^x$
62. $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+2}\right)^{1/x}$
63. $\lim_{x\to0^+}x^2\ln x$
64. $\lim_{x\to0^+}x(\ln x)^2$
65. $\lim_{x\to0^+}x\tan\left(\dfrac{\pi}{2}-x\right)$
66. $\lim_{x\to0^+}\sin x\cdot\ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x\to\infty}\dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$
68. $\lim_{x\to0^+}\dfrac{\sqrt{x}}{\sqrt{\sin x}}$
69. $\lim_{x\to(\pi/2)^-}\dfrac{\sec x}{\tan x}$
70. $\lim_{x\to0^+}\dfrac{\cot x}{\csc x}$
71. $\lim_{x\to\infty}\dfrac{2^x-3^x}{3^x+4^x}$
72. $\lim_{x\to-\infty}\dfrac{2^x+4^x}{5^x-2^x}$
73. $\lim_{x\to\infty}\dfrac{e^{x^2}}{xe^x}$
74. $\lim_{x\to0^+}\dfrac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $\lim_{x\to3}\dfrac{x-3}{x^2-3}=\lim_{x\to3}\dfrac{1}{2x}=\dfrac{1}{6}$ **b.** $\lim_{x\to3}\dfrac{x-3}{x^2-3}=\dfrac{0}{6}=0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $$\lim_{x\to0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\frac{2x-2}{2x-\cos x}=\lim_{x\to0}\frac{2}{2+\sin x}=\frac{2}{2+0}=1$$

 b. $$\lim_{x\to0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\frac{2x-2}{2x-\cos x}=\frac{-2}{0-1}=2$$

77. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a. $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = 0$

b. $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = -\infty$

c. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)} = \dfrac{-\infty}{\infty} = -1$

d. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)}$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$$

78. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

b. $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

79. Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

80. For what values of a and b is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = 0?$$

81. Show that

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r.$$

82. Given that $x > 0$, find the maximum value, if any, of

a. $x^{1/x}$ **b.** x^{1/x^2} **c.** x^{1/x^n} (n a positive integer)

d. Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

83. Use limits to find horizontal asymptotes for each function.

a. $y = x \tan\left(\dfrac{1}{x}\right)$ **b.** $y = \dfrac{3x + e^{2x}}{2x + e^{3x}}$

84. Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Chapter 3 Questions to Guide Your Review

1. What is the derivative of a function f? How is its domain related to the domain of f? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? On a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. What rules do you know for calculating derivatives? Give some examples.
9. Explain how the three formulas

   a. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ **b.** $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

   c. $\dfrac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \dfrac{du_1}{dx} + \dfrac{du_2}{dx} + \cdots + \dfrac{du_n}{dx}$

   enable us to differentiate any polynomial.
10. What formula do we need, in addition to the three listed in Question 9, to differentiate rational functions?
11. What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.
12. What is the relationship between a function's average and instantaneous rates of change? Give an example.
13. What do the limits $\lim_{h\to 0}((\sin h)/h)$ and $\lim_{h\to 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?
14. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?
15. At what points are the six basic trigonometric functions continuous? How do you know?
16. What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.
17. If u is a differentiable function of x, how do you find $(d/dx)(u^n)$ if n is an integer? If n is a real number? Give examples.
18. What is implicit differentiation? When do you need it? Give examples.

Chapter 3 Practice Exercises

Implicit Differentiation

In Exercises 1–8, find dy/dx by implicit differentiation.

1. $xy + 2x + 3y = 1$ **2.** $x^2 + xy + y^2 - 5x = 2$

3. $x^3 + 4xy - 3y^{4/3} = 2x$ **4.** $5x^{4/5} + 10y^{6/5} = 15$

5. $\sqrt{xy} = 1$ **6.** $x^2y^2 = 1$

7. $y^2 = \dfrac{x}{x+1}$ **8.** $y^2 = \sqrt{\dfrac{1+x}{1-x}}$

In Exercises 9 and 10, find dp/dq.

9. $p^3 + 4pq - 3q^2 = 2$ **10.** $q = (5p^2 + 2p)^{-3/2}$

11. Find d^2y/dx^2 by implicit differentiation:

a. $x^3 + y^3 = 1$ **b.** $y^2 = 1 - \dfrac{2}{x}$

12. a. By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

b. Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

13. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	$1/2$
1	3	5	$1/2$	-4

Find the first derivatives of the following combinations at the given value of x.

a. $6f(x) - g(x)$, $x = 1$ **b.** $f(x)g^2(x)$, $x = 0$

c. $\dfrac{f(x)}{g(x)+1}$, $x = 1$ **d.** $f(g(x))$, $x = 0$

e. $g(f(x))$, $x = 0$ **f.** $(x + f(x))^{3/2}$, $x = 1$

g. $f(x + g(x))$, $x = 0$

14. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	$1/5$

Find the first derivatives of the following combinations at the given value of x.

a. $\sqrt{x}\, f(x)$, $x = 1$ **b.** $\sqrt{f(x)}$, $x = 0$

c. $f\left(\sqrt{x}\right)$, $x = 1$ **d.** $f(1 - 5\tan x)$, $x = 0$

e. $\dfrac{f(x)}{2 + \cos x}$, $x = 0$ **f.** $10\sin\left(\dfrac{\pi x}{2}\right) f^2(x)$, $x = 1$

15. Find the value of dy/dt at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

16. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

17. Find the value of dw/ds at $s = 0$ if $w = \sin\left(\sqrt{r} - 2\right)$ and $r = 8\sin(s + \pi/6)$.

18. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

19. If $y^3 + y = 2\cos x$, find the value of d^2y/dx^2 at the point $(0, 1)$.

20. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point $(8, 8)$.

Applying the Derivative Definition

21. a. Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

22. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

23. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b. Is f continuous at $x = 1$?

c. Is f differentiable at $x = 1$?

Give reasons for your answers.

24. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a. continuous at $x = 0$?

b. differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

25. Tangents with specified slope Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

26. Horizontal tangents Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

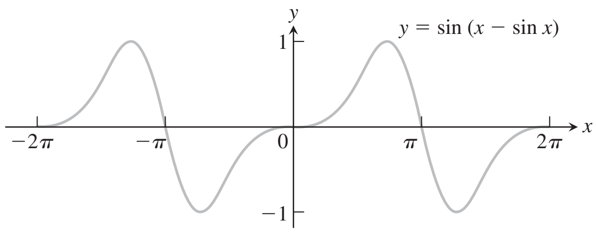
27. Tangents perpendicular or parallel to lines Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a. perpendicular to the line $y = 1 - (x/24)$.

b. parallel to the line $y = \sqrt{2} - 12x$.

28. Normals parallel to a line Find the points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

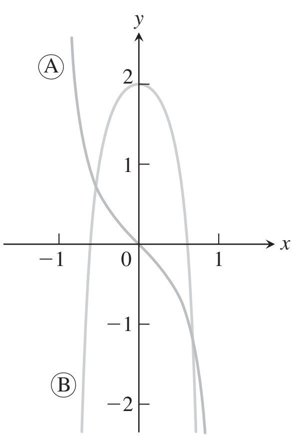
29. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x-axis. Does it? Give reasons for your answer.



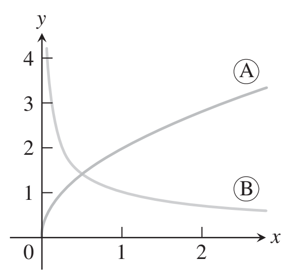
Analyzing Graphs

Each of the figures in Exercises 30 and 31 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

30.



31.

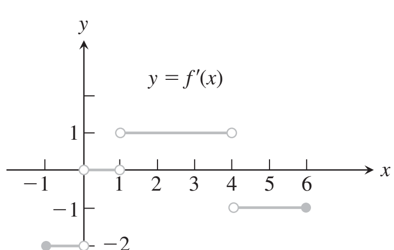


32. Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

i) The graph of f is made of line segments joined end to end.

ii) The graph starts at the point $(-1, 2)$.

iii) The derivative of f, where defined, agrees with the step function shown here.



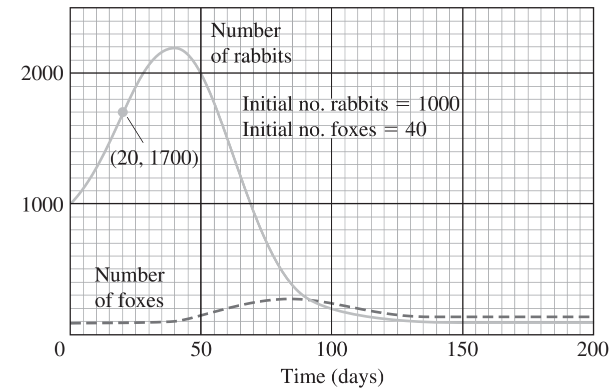
33. Repeat Exercise 32, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 34 and 35 are about the accompanying graphs. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Part (b) shows the graph of the derivative of the rabbit population, made by plotting slopes.

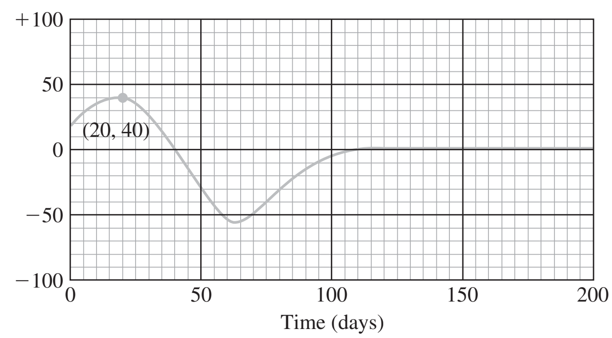
34. a. What is the value of the derivative of the rabbit population when the number of rabbits is largest? Smallest?

b. What is the size of the rabbit population when its derivative is largest? Smallest (negative value)?

35. In what units should the slopes of the rabbit and fox population curves be measured?



(a)



Derivative of the rabbit population

(b)

Source: NCPMF "Differentiation" by W.U. Walton et al., Project CALC. Reprinted by permission of Educational Development Center, Inc.

Trigonometric Limits

Find the limits in Exercises 36–43.

36. $\displaystyle\lim_{x\to 0} \frac{\sin x}{2x^2 - x}$

37. $\displaystyle\lim_{x\to 0} \frac{3x - \tan 7x}{2x}$

38. $\displaystyle\lim_{r\to 0} \frac{\sin r}{\tan 2r}$

39. $\displaystyle\lim_{\theta\to 0} \frac{\sin(\sin\theta)}{\theta}$

40. $\displaystyle\lim_{\theta\to(\pi/2)^-} \frac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

41. $\displaystyle\lim_{\theta\to 0^+} \frac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

42. $\displaystyle\lim_{x\to 0} \frac{x\sin x}{2 - 2\cos x}$

43. $\displaystyle\lim_{\theta\to 0} \frac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 44 and 45 to be continuous at the origin.

44. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

45. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

Chapter 3 Single Choice Questions

1. Set of points where $f(x) = \dfrac{4x}{5+6|x|}$ is differentiable, is

a. $(-\infty, 0) \cup (0, \infty)$ **b.** $(-\infty, -1) \cup (-1, \infty)$
c. $(-\infty, \infty)$ **d.** $(0, \infty)$

2. If $y = x + e^x$, then at $x = 1$, $\dfrac{d^2x}{dy^2}$ is equal to

a. e **b.** $\dfrac{-e}{(1+e)^3}$ **c.** $\dfrac{-e}{(1+e)}$ **d.** $\dfrac{-e}{(1+e)^2}$

3. Let $f(x) = \begin{cases} (x-1)\sin\left(\dfrac{1}{x-1}\right), & x \neq 1 \\ 0, & x = 1 \end{cases}$, which of the following statements is true?

a. f is differentiable at $x = 1$ but not at $x = 0$
b. f is neither differentiable at $x = 0$ nor at $x = 1$
c. f is differentiable at $x = 0$ and at $x = 1$
d. f is differentiable at $x = 0$ but not at $x = 1$

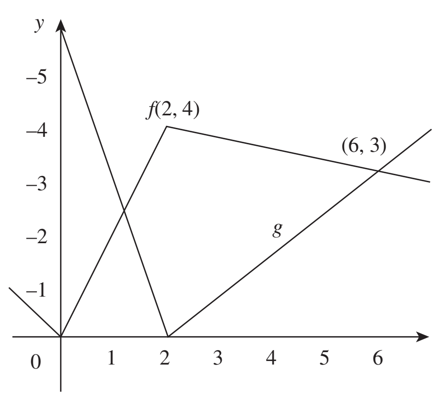
4. Let $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, $g(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. The derivative of $f(x)$ with respect to $g(x)$ at $x = -\dfrac{1}{2}$ is

a. $-\dfrac{1}{2}$ **b.** -1 **c.** $\dfrac{1}{2}$ **d.** 1

5. If $f(x) = \sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{(x-1)}}$, then $f'(x)$ at $x = 1.5$ is

a. 0 **b.** $-\sqrt{2}$ **c.** $-\sqrt{3}$ **d.** -4

6. If f and g are the functions whose graph are as shown, let $u(x) = f(g(x))$; $w(x) = g(g(x))$, then the value of $u'(1) + w'(1)$ is



a. $-\dfrac{1}{2}$ **b.** $-\dfrac{3}{2}$

c. $-\dfrac{5}{4}$ **d.** does not exist

7. Number of points where

$f(x) = \begin{cases} \max.(|x^2 - x - 2|, x^2 - 3x); & x \geq 0 \\ \max.(\ln(-x), e^x); & x < 0 \end{cases}$, is not differentiable are

a. 2 **b.** 3 **c.** 4 **d.** 5

8. Let $g(x) = f(x)\sin x$, where $f(x)$ is a twice differentiable function on $(-\infty, \infty)$ such that $f'(-\pi) = 1$. The value of $g''(-\pi)$ equals

a. 1 **b.** 2 **c.** -2 **d.** 0

9. If the function $f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the value of $g'\left(\dfrac{-7}{6}\right)$ equals

a. $\dfrac{1}{5}$ **b.** $-\dfrac{1}{5}$ **c.** $\dfrac{6}{7}$ **d.** $-\dfrac{6}{7}$

10. If $y = \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\left(\dfrac{a-x}{x-b}\right)}$, then $\dfrac{dy}{dx}$ is equal to

a. $\sqrt{(a-x)(x-b)}$ **b.** $\dfrac{1}{\sqrt{(a-x)(x-b)}}$

c. $\sqrt{\left(\dfrac{a-x}{x-b}\right)}$ **d.** $\sqrt{\left(\dfrac{x-b}{a-x}\right)}$

11. Number of points where $f(x) = |x - \text{sgn}(x)|$ is nondifferentiable, is (sgn(·) denotes signum function)

a. 0 **b.** 1 **c.** 2 **d.** 3

12. $f(x) = |x^2 - a|x| + b|$ is nonderivable at exactly one point, if

a. $a = 0, b > 0$
b. $a = 0, b < 0$
c. both roots of $x^2 - ax + b = 0$ are negative
d. both roots of $x^2 - ax + b = 0$ are positive

13. Let $f(x) = \begin{cases} 3x + 2\sin^{-1}x + p & x \leq 0 \\ x^4 - x^3 + x^2 - qx + r & x > 0 \end{cases}$. If $f(x)$ is differentiable at $x = 0$, ($pr \neq 0$), then

a. $(p-r)^2 + (q+5)^2 = 0$ **b.** $(p+r)^2 + (q+5)^2 = 0$
c. $(p-r)^2 + (q-5)^2 = 0$ **d.** $(p+r)^2 + (q-5)^2 = 0$

14. If $f(x) = (x^2 - 3x + 2)|(\ln x)(2x^2 + 5x - 3)(e^x - e^2)|$, then number of values of x for which $f(x)$ not differentiable is

a. 1 **b.** 2 **c.** 3 **d.** 4

15. Consider $f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

a. $f(x)$ is neither continuous nor differentiable at $x = 0$
b. $f(x)$ is continuous but not differentiable at $x = 0$
c. $f(x)$ is continuous as well as differentiable at $x = 0$
d. Jump of discontinuity for $f(x)$ at $x = 0$ is 2.

16. Number of points at which function $f(x)=\begin{cases}\min(x, x^2) & x<1\\ \min(2x-1, x^2) & x\geq 1\end{cases}$ is not derivable is

a. 0 **b.** 1 **c.** 2 **d.** infinite

17. Consider function $f(x) = 4\sin|x|$ in $x \in [-\pi, \pi]$, then

a. $f(x)$ is discontinuous function

b. $f(x)$ is not differentiable at 3 points

c. $|f(x)| = \pi$ has 4 distinct solutions

d. $|f(x)| = \pi$ has no solution.

18. The point of non-derivability of $f(x)\left[\dfrac{2\sqrt{x}}{1+x}\right]$ is (where $[\cdot]$ denotes greatest integer function)

a. at $x = 1$ **b.** at $x = 0$

c. at $x = 0, 1$ **d.** continuous for all x

19. If $f(x) = a|\sin x| + be^{-|x|} + c|x|^{2015}$, $0 < a, b, c \leq 10$, $a, b, c \in$ integers and if $f(x)$ is differentiable at $x = 0$, then number of ordered triplets of (a, b, c) is

a. 0 **b.** 10 **c.** 100 **d.** 1000

20. A twice differentiable function $f(x)$ is such that $f(x-y)$, $f(x)\cdot f(y)$ and $f(x+y)$ are in A.P. for all $x, y \in R$ and $f(0) \neq 0$, then

(i) $f(x)$ is an even function

(ii) $f''(2) - f''(-2) = 0$

(iii) $f'(2) + f'(-2) \neq 0$

(iv) $f(0) = 2$.

The correct order sequence of above statements is

a. TFTF **b.** TTFT **c.** FFTF **d.** TTFF

(where T and F stands for true and false, respectively)

21. Let $f: R \to R$ be a differentiable function satisfying $f(y)\cdot f(x-y) = f(x)$ $\forall\, x, y \in R$ and $f'(0) = \alpha$, $f'(1) = \beta$, then the value of $f'(1) + f'(-1)$ is equal to (where $\alpha, \beta \neq 0$)

a. $\dfrac{\alpha^2}{\beta}$ **b.** $\dfrac{\beta^2}{\alpha}$

c. $\dfrac{\alpha^2+\beta^2}{\beta}$ **d.** $\dfrac{\alpha^2+\beta^2}{\alpha}$

22. If $f(x) = x^3 + \ln x + 1$ and $g(x)$ is the inverse of $f(x)$, then value of $g''(2)$ is

a. $-\dfrac{98}{625}$ **b.** $-\dfrac{5}{64}$ **c.** $\dfrac{5}{64}$ **d.** $\dfrac{98}{625}$

23. $f(x)$ is a polynomial of degree 4, $f'(1) = f(2) = f(3) = f(4) = 0$, then root of $f(x) = 0$ is

a. $\dfrac{5}{11}$ **b.** $-\dfrac{5}{11}$ **c.** 5 **d.** $\dfrac{11}{5}$

24. If $y = \tan^{-1}\sqrt{\dfrac{1+\sin x}{1-\sin x}}$, $x \in \left(\dfrac{\pi}{2}, \pi\right)$, then $\dfrac{dy}{dx}$ equals

a. -1 **b.** 1 **c.** $\dfrac{1}{2}$ **d.** $-\dfrac{1}{2}$

25. The second-order derivative of a single-valued function parametrically represented by $x = f(t)$ and $y = g(t)$, where $f(t)$, and $g(t)$ are differentiable function and $f'(t) \neq 0$ is given by

a. $\dfrac{d^2y}{dx^2} = \dfrac{\left(\dfrac{dx}{dt}\right)\cdot\left(\dfrac{d^2y}{dt^2}\right) - \left(\dfrac{d^2x}{dt^2}\right)\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)^3}$

b. $\dfrac{d^2y}{dx^2} = \dfrac{\left(\dfrac{dx}{dt}\right)\cdot\left(\dfrac{d^2y}{dt^2}\right) - \left(\dfrac{d^2x}{dt^2}\right)\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)^2}$

c. $\dfrac{d^2y}{dx^2} = \dfrac{\left(\dfrac{d^2x}{dt^2}\right)\cdot\left(\dfrac{dy}{dt}\right) - \left(\dfrac{dx}{dt}\right)\left(\dfrac{d^2y}{dt^2}\right)}{\left(\dfrac{dx}{dt}\right)^3}$

d. $\dfrac{d^2y}{dx^2} = \dfrac{\left(\dfrac{d^2x}{dt^2}\right)\cdot\left(\dfrac{dy}{dt}\right) - \left(\dfrac{d^2y}{dt^2}\right)\left(\dfrac{dx}{dt}\right)}{\left(\dfrac{dx}{dt}\right)^3}$

26. Let f is differentiable function such that $x = f(t^2)$, $y = f(t^3)$ and $f'(1) \neq 0$ if $\left(\dfrac{d^2y}{dx^2}\right)_{t=1}$ is equal to

a. $\dfrac{3}{4}\left(\dfrac{f''(1)+f'(1)}{(f'(1))^2}\right)$ **b.** $\dfrac{3}{4}\left(\dfrac{f'(1)\cdot f''(1)-f''(1)}{(f'(1))^2}\right)$

c. $\dfrac{4}{3}\dfrac{f''(1)}{(f'(1))^2}$ **d.** $\dfrac{4}{3}\left(\dfrac{f'(1)f''(1)-f''(1)}{(f'(1))^2}\right)$

27. If $f(x) = \cos^2(x) + \cos^2(x + 120°) + \cos^2(x - 120°)$, then value of $f'\left(\dfrac{\pi}{3}\right)$ is

a. 1 **b.** 1/2 **c.** 0 **d.** -1

28. If $y\sqrt{1+x} + x\sqrt{1+y} = 0$, then value of $\dfrac{dy}{dx}$ at $y = 1$ is

a. $-\dfrac{1}{2}$ **b.** 1 **c.** -4 **d.** 2

29. Let $g(x) = x^5 \ln f(x))$ where $f(x)$ is a differentiable positive function on $(0, \infty)$ satisfying $f(3) = 1$ and $f'(3) = 2$, then $g'(3)$ is equal to

a. 243 **b.** 0 **c.** 1 **d.** 486

30. Number of points at which the function $f(x) = |\cos|x|| + \cos^{-1}\operatorname{sgn} x + |\ln x|$ is not differentiable in $(0, 2\pi)$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

31. Given $f(x)=\begin{cases}2x+1 & \text{if} \quad x\leq 1\\ x^2+2 & \text{if} \quad 1<x\leq 2\\ 4x^2+2 & \text{if} \quad x>2\end{cases}$. Then the number of points where $f(x)$ is not differentiable is

a. 3 **b.** 2 **c.** 1 **d.** 0

32. The function f: $[0, 2] \to$ R defined by $f(x) = [x^2 - x + 1]$ (where $[\cdot]$ denotes greatest integer function). If m is the number of points where $f(x)$ is discontinuous and n is the number of points where $f(x)$ is nondifferentiable then $(m + n)$ is

a. 6 **b.** 7 **c.** 8 **d.** 9

33. For the curve $32x^3y^2 = (x + y)^5$, the value of $\frac{d^2y}{dx^2}$ at P(1, 1) is equal to

a. 0 **b.** 1 **c.** – 1 **d.** 1/2

34. Let $f: R \to R$ is a function which is defined by $f(x) = \max\{x, x^3\}$ set of points on which $f(x)$ is not differentiable is

a. $\{-1, 1\}$ **b.** $\{-1, 0\}$ **c.** $\{0, 1\}$ **d.** $\{-1, 0, 1\}$

35. The left hand derivative of $f(x) = [x]\sin\pi x$ at $x = k$, k an integer is: (where $[\cdot]$ denotes the greatest function)

a. $(-1)^k(k-1)\pi$ **b.** $(-1)^{k-1}(k-1)\pi$

c. $(-1)^k k\pi$ **d.** $(-1)^{k-1}k\pi$

36. Which of the following functions is differentiable at $x = 0$?

a. $\cos(|x|) + |x|$ **b.** $\cos(|x|) - |x|$

c. $\sin(|x|) + |x|$ **d.** $\sin(|x|) - |x|$

37. The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1}x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$ is

a. $R - \{0\}$ **b.** $R - \{1\}$ **c.** $R - \{-1\}$ **d.** $R - \{-1, 1\}$

38. Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$ The $\underset{x\to 0}{\text{Limit}} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals

a. 1 **b.** $e^{1/2}$ **c.** e^2 **d.** e^3

39. The function given by $y = \big||x|-1\big|$ is differentiable for all real numbers except the points

a. $\{0, 1, -1\}$ **b.** ± 1 **c.** 1 **d.** -1

40. If $f(x) = \min.(1, x^2, x^3)$, then

a. $f(x)$ is continuous $\forall x \in R$

b. $f'(x) > 0, \forall x > 1$

c. $f(x)$ is differentiable but not continuous $\forall x \in R$

d. $f(x)$ is not differentiable for two values of x

41. Let $g(x) = \frac{(x-1)^n}{\ln\cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $n > 0$ and let p be the left-hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x\to 1^+} g(x) = p$, then

a. $n = 1, m = 1$ **b.** $n = 1, m = -1$

c. $n = 2, m = 2$ **d.** $n > 2, m = n$

42. If $\ln(x + y) = 2xy$, then $y'(0)$ is equal to

a. 1 **b.** –1 **c.** 2 **d.** 0

43. If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$, then $y''(0)$

a. 1 **b.** –1 **c.** π **d.** $-\pi$

44. If $P(x)$ is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(1) = 1$, $P(0) = 0$ and $P'(x) > 0\ \forall x \in [0, 1]$, then

a. $S = \phi$

b. $S = \{(1-a)x^2 + ax, 0 < a < 2$

c. $(1-a)x^2 + ax$, a $\in (0, \infty)$

d. $S = \{(1-a)x^2 + ax, 0 < a < 1$

45. If $f(x)$ is a continuous and differentiable function and $f(1/n) = 0$, $\forall n \ge 1$ and $n \in$ I, then

a. $f(x) = 0, x \in (0, 1]$

b. $f(0) = 0, f'(0) = 0$

c. $f'(x) = 0 = f''(x), x \in (0, 1]$

d. $f(0) = 0$ and $f'(0)$ need not be zero

46. For $x > 0$, $\lim\limits_{x\to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is

a. 0 **b.** –1 **c.** 1 **d.** 2

47. $\frac{d^2x}{dy^2}$ equals

a. $\left(\frac{d^2y}{dx^2}\right)^{-1}$ **b.** $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

c. $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ **d.** $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

48. Let $g(x) = \ln f(x)$, where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x f(x)$. Then for $N = 1, 2, 3$

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

a. $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

b. $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

c. $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

d. $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

49. Let f and g be real-valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \ne 0$, $g'(0) = 0$, $g''(0) \ne 0$, and $f(x) = g(x)\sin x$.

Statement-1: $\lim\limits_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

and

Statement-2: $f'(0) = g(0)$

a. Statement-1 is True, Statement-2 is True; statement-2 is a correct explanation for statement-1

b. Statement-1 is True, Statement-2 is True; statement-2 is NOT a correct explanation for statement-1

c. Statement-1 is True, Statement-2 is False

d. Statement-1 is False, Statement-2 is True

50. If $y = \frac{ax+b}{cx+d}$, then value of $\frac{y_2^2}{y_1 y_3}$, wherever exists, is (where y_n denotes n^{th}-order derivative of differentiable function y and a, b, c, and d are constants)

a. $\frac{3}{2}$ **b.** $\frac{2}{3}$ **c.** $\frac{4}{5}$ **d.** $\frac{5}{4}$

Chapter 3 Multiple Choice Questions

1. Which of the following statements is/are correct?

a. $\{\text{sgn}(x)\}$ is differentiable at $x = 0$, where $\{\cdot\}$ is fractional part of function.

b. If $(f(x)$ is differentiable at $x = a$, the $f(x)\,|f(x)|$ is also differentiable at $x = a$

c. If $f(x)$ is differentiable at $x = a$, then $f^{-1}(x)$ is also differentiable at $x = a$

d. $[|\sin^{-1}x| + |\cos^{-1}x|]$ is differentiable in $0 \le x \le 1$, (where $[\cdot]$ is greatest integer function).

2. Let $f'(x) = \dfrac{1}{\sqrt{1+x^3}}\ \forall\ x \in (-1, \infty)$ and $f(0) = 0$. Suppose $g(x)$ be the inverse of $f(x)$, then which of the following statements is/are true.

a. $(g'(x))^2 = g^3(x)$

b. $g'(0) = 1$

c. $g''(0) = 0$

d. $3(g(x))^2 = 2g''(x)$

3. If f: $R \to [0, \infty)$, $f(x) = \min(|x|, e^x)$, then which of the following is/are not true ?

a. f is many-one function

b. f is surjective function

c. f is derivable every where

d. f is an even function

4. If $f(x) = \sqrt{x+\sqrt{x+\sqrt{x+.....}}}$, $x > 0$, then

a. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x+y}-1}$ **b.** $\dfrac{dy}{dx} = \dfrac{1}{2y-1}$

c. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+4x}}$ **d.** $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+2x}}$

5. If $y = \sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+.....\infty}}}$ (where $\cos x > 0$), then which of the following can be the value of $\dfrac{dy}{dx}$?

a. $\dfrac{\sin x}{1-2y}$ **b.** $\dfrac{-\sin x}{\sqrt{1+4\cos x}}$

c. $\dfrac{-y\sin x}{y^2+\cos x}$ **d.** $\dfrac{y\sin x}{y^2-\cos x}$

6. If $f(x)$ be a differentiable function satisfying $f(y)\, f\left(\dfrac{x}{y}\right) = f(x)\ \forall\ x$, $y \in R$, $y \ne 0$ and $f(1) \ne 0$, $f'(1) = 3$, then

a. sgn($f(x)$) is non-differentiable at exactly one point

b. $\lim\limits_{x\to 0} \dfrac{x^2(\cos x - 1)}{f(x)} = 0$

c. $f(x) = x$ has 3 solutions

d. $f(f(x)) - f^3(x) = 0$ has infinitely many solutions

7. If $f(x) = |||x| - 2| + p|$ have more than 3 points of non-derivability then the value of p can be

a. 0 **b.** – 1

c. – 2 **d.** 2

8. If $y = e^{x\sin(x^3)} + (\tan x)^x$ then $\dfrac{dy}{dx}$ may be equal to

a. $e^{x\sin(x^3)}\,[3x^3\cos(x^3) + \sin(x^3)] + (\tan x)^x\,[\ln \tan x + 2x \operatorname{cosec} 2x]$

b. $e^{x\sin(x^3)}\,[x^3\cos(x^3) + \sin(x^3)] + (\tan x)^x\,[\ln \tan x + 2x \operatorname{cosec} 2x]$

c. $e^{x\sin(x^3)}\,[x^3\sin(x^3) + \cos(x^3)] + (\tan x)^x\,[\ln \tan x + 2x \operatorname{cosec} 2x]$

d. $e^{x\sin(x^3)}\,[3x^3\cos(x^3) + \sin(x^3)] + (\tan x)^x\left[\ln \tan x + \dfrac{x\sec^2 x}{\tan x}\right]$

9. If $e^{\sin(x^2+y^2)} = \tan\dfrac{y^2}{4} + \sin^{-1}x$, then $y'(0)$ can be

a. $\dfrac{1}{3\sqrt{\pi}}$

b. Error! Not a valid embedded object.

c. $-\dfrac{1}{5\sqrt{\pi}}$

d. $-\dfrac{1}{3\sqrt{5\pi}}$

10. Let $f(x) = \begin{cases} |x|^x & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$. Which of the following statements is/are correct?

a. f is continuous at $x = 0$

b. f is not differentiable at $x = 0$

c. f is derivable at $x = 0$

d. f is discontinuous at $x = 0$

11. A function $f(x) = \max(\sin x, \cos x, 1 - \cos x)$ is not derivable for some $x \in [0, 2\pi]$, which may lie in the interval

a. $\left[0, \dfrac{\pi}{2}\right)$ **b.** $\left[\pi, \dfrac{3\pi}{2}\right)$

c. $\left[\dfrac{\pi}{2}, \pi\right)$ **d.** $\left[\dfrac{3\pi}{2}, 2\pi\right)$

12. Let $f(x) = \begin{cases} x^3 + x - 1 & ,\ x \ge 0, x \notin N \\ \left(\displaystyle\sum_{r=0}^{x} (-1)^r\, {}^xC_r \left(\frac{1}{2}\right)^r\right) + k & ,\ x \in N \\ -x^2 + x - 1 & ,\ x < 0 \end{cases}$

Then which of the following is (are) true?

a. $f(x)$ is derivable at $x = 0$

b. If $f(x)$ is derivable at $x = 2$ then $k = 9$

c. If $f(x)$ is continuous at $x = 1$ then $k = \frac{1}{2}$

d. If $k = 5$ then $f(x)$ has removable type of discontinuity at all positive integral value of x.

13. $f(x) = \begin{cases} x^2 + [3x]x, & \frac{1}{3} \le x < 0 \\ ax^2 + bx, & 0 \le x \le \frac{1}{3} \end{cases}$, (where $[\cdot]$ denotes greatest integer function.)

a. $f(x)$ is continuous in $\left(-\frac{1}{3}, \frac{1}{3}\right)$ iff $a = 1$ and $b = 0$

b. $f(x)$ is continuous in $\left(-\frac{1}{3}, \frac{1}{3}\right)$ for all a and b

c. $f(x)$ is differentiable in $\left(-\frac{1}{3}, \frac{1}{3}\right)$ for all a, provided $b = -1$

d. $f(x)$ is continuous in $\left[-\frac{1}{3}, \frac{1}{3}\right]$ for all choice of a and b.

14. Let $f(x) = \dfrac{x^6 + x^3 f(0) + f'(1)}{x^3 - xf(0) + f'(1)}$ be a differentiable function ($f'(1) \ne 0$) then correct statement/s is/are

a. $f(0) = 1$ **b.** $f'(1) \cdot f'(0) = 1$

c. $7f'(1) - (f'(1))3 = 4$ **d.** $f(0) = f(-1)$

15. If $f(x) = |x| \sin x$, then f is

a. differentiable everywhere

b. not differentiable at $x = n\pi$, $n \in I$

c. not differentiable at $x = 0$

d. continuous at $x = 0$

16. Let f be a differentiable function satisfying $f'(x) = f'(-x)\ \forall\ x \in R$. Then

a. If $f(1) = f(2)$, then $f(-1) = f(-2)$

b. $\frac{1}{2} f(x) + \frac{1}{2} f(y) = f\left(\frac{1}{2}(x+y)\right)$ for all real values of x, y

c. Let $f(x)$ be an even function, then $f(x) = 0\ \forall\ x \in R$

d. $f(x) + f(-x) = 2f(0)\ \forall\ x \in R$

17. Let $f(x) = \begin{cases} \dfrac{\ln(1+2x)}{x}, & \dfrac{-1}{2} < x < 0 \\ 2\cos x, & x = 0 \\ \dfrac{e^{2x} - 1}{x}, & 0 < x < 1 \\ e^2 - 1, & x \ge 1 \end{cases}$ then

a. $f(x)$ is continuous at $x = 0$

b. $f(x)$ is not differentiable at $x = 0$

c. $f(x)$ is discontinuous at $x = 0$

d. $\lim_{x \to 0^+} [f(x)] = 1$

[**Note:** $[k]$ denotes greatest integer less than or equal to k.]

18. Let $f(x) = \begin{cases} \max.\{e^x, e^{-x}, 2\}, & x \le 0 \\ \min.\{e^x, e^{-x}, 2\}, & x > 0 \end{cases}$

Which of the following statements is/are correct?

a. $f(x)$ is discontinuous at $x = 0$

b. $f(x)$ is non-derivable at exactly two points

c. $f(x)$ has non-removable type of discontinuity at $x = 0$ with jump of discontinuity equals to 2.

d. $f(x)$ is continuous but not derivable at $x = \ln$

Chapter 3 Passage Type Questions

Passage 1

$$f(x) = \begin{cases} 2 + (x-1)^2 & \text{if} \quad x < 1 \\ 2 & \text{if} \quad x \in [1,\ 3] \\ 2 - (x-3)^2 & \text{if} \quad x > 3 \end{cases}$$

$$g(x) = \begin{cases} 2 + \sqrt{-x} & \text{if} \quad x < 0 \\ x + 2 & \text{if} \quad x \in [0,\ 4] \\ 3x - 6 & \text{if} \quad x \in (4, \infty) \end{cases}$$

$$h(x) = \begin{cases} 4 + ae^x & \text{if} \quad x < 0 \\ x + 2 & \text{if} \quad x \in [0,\ 3] \\ b^2 - 7b + 18 - \dfrac{3}{x} & \text{if} \quad x > 3 \end{cases}$$

$$K(x) = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)(x+4)}}}$$

On the basis of above information, answer the following.

1. Which of the following is continuous at each point of its domain?

a. $f(x)$ **b.** $g(x)$ **c.** $k(x)$ **d.** all there f, g, k

2. Value of (a, b) for which $h(x)$ is continuous $\forall\ x \in R$:

a. (4, 3) **b.** (–2, 3) **c.** (3, 4) **d.** None of these

3. Which of the following function is not differentiable at exactly two points of its domain?

a. $f(x)$ **b.** $g(x)$ **c.** $k(x)$ **d.** None of these

Passage 2

Consider the following function

$$f(x) = \begin{cases} (\sin^{-1} x) + \dfrac{\pi}{2}, & -1 \le x \le 0 \\ \ln|x + 1 - \pi|, & 0 < x < \pi, x \ne \pi - 1 \\ |\sin x|, & \pi \le x \le 4\pi \end{cases} \quad \text{and } g(x) = \sqrt{8f(|x|)}$$

On the basis of above information, answer the following.

1. Number of point where $f(x)$ is non-derivable in its domain
   a. 4 **b.** 5 **c.** 3 **d.** 6
2. The domain of function $g(x)$ is
   a. $[-2\pi, 1-\pi] \cup [\pi-1, \pi]$
   b. $[\pi-1, 2\pi]$
   c. $[-4\pi, -2\pi] \cup [-\pi, 1-\pi] \cup [\pi-1, \pi] \cup [2\pi, 4\pi]$
   d. $[-4\pi, -\pi] \cup [2-\pi, \pi-2] \cup [\pi, 4\pi]$
3. The number of solution of the equation $g^2(x) = 1$
   a. 10 **b.** 14
   c. 16 **d.** None of these

Passage 3

An operator Δ is defined to operate on differentiable functions defined as follows:

If $f(x)$ is a differentiable function then $\Delta(f(x)) = \lim\limits_{h\to 0} \dfrac{f^3(x+h) - f^3(x)}{h}$

$g(x)$ is a differentiable function such that the slope of the tangent to the curve $y = g(x)$ at any point $(a, g(a))$ is equal to $2e^a\,(a+1)$ also $g(0) = 0$

On the basis of above information, answer the following.

1. $\Delta(g(x))$ at $x = \ln 2$ is
   a. $24 \ln 2\,\{2\ln 2 + 2\}$ **b.** $\ln\{4e^2 \ln^2 2\}$
   c. $96 \ln(4e^2) \ln^2 2$ **d.** $192 \ln(4e) \ln^2 2$
2. $\Delta(\Delta(x+2))_{x=0}$
   a. $2^5 \times 3^9$ **b.** $2^9 \times 3^5$
   c. $2^4 \times 3^5$ **d.** $2^6 \times 3^4$
3. $\lim\limits_{x\to 0} \dfrac{\Delta g(x)}{\ln(\cos 2x)}$
   a. -12 **b.** 12 **c.** 24 **d.** -24

Passage 4

A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$ $a, b > 0$ and $a \neq 1, b \neq 1$ where $t \in R$

1. Which of the following is not a correct expression for $\dfrac{dy}{dx}$?
   a. $\dfrac{-1}{f(t)^2}$ **b.** $-(g(t))2$ **c.** $\dfrac{-g(t)}{f(t)}$ **d.** $\dfrac{-f(t)}{g(t)}$
2. The value of $\dfrac{d^2y}{dx^2}$ at the point where $f(t) = g(t)$ is
   a. 0 **b.** $\dfrac{1}{2}$ **c.** 1 **d.** 2
3. The value of $\dfrac{f(t)}{f'(t)} \cdot \dfrac{f''(-t)}{f'(-t)} + \dfrac{f(-t)}{f'(-t)} \cdot \dfrac{f''(t)}{f'(t)} \forall\ t \in R$, is equal to
   a. -2 **b.** 2 **c.** -4 **d.** 4

Passage 5

Let $f: R \to [0, \infty)$ be a derivable function which satisfies $f(x^2+y^2) = f(2xy) + f(x^2-y^2)\ \forall\ x, y \in R$. Also $f'(1) = 2$

On the basis of above information, answer the following equations:

1. $f(3) + f(4)$ is equal to
   a. $f(3)$ **b.** $f(4)$ **c.** $f(5)$ **d.** $f(25)$
2. Number of solutions of $f(\cos\theta) = f(\sin\theta)$ in $[0, 2\pi]$
   a. 4 **b.** 6 **c.** 7 **d.** 8

Passage 6

Let $f(x) = \dfrac{\cos^2 x}{1+\cos x+\cos^2 x}$ and $g(x) = \lambda \tan x + (1-\lambda)\sin x - x$, where $\lambda \in R$ and $x \in [0, \pi/2)$.

1. $g'(x)$ equals
   a. $\dfrac{(1-\cos x)(f(x)-\lambda)}{\cos x}$ **b.** $\dfrac{(1-\cos x)(\lambda - f(x))}{\cos x}$
   c. $\dfrac{(1-\cos x)(\lambda - f(x))}{f(x)}$ **d.** $\dfrac{(1-\cos x)(\lambda - f(x))}{(f(x))^2}$
2. The exhaustive set of values of λ such that $g'(x) \geq 0$ for any $x \in [0, \pi/2]$
   a. $[1, \infty]$ **b.** $[0, \infty)$ **c.** $\left[\dfrac{1}{2}, \infty\right)$ **d.** $\left[\dfrac{1}{3}, \infty\right)$

Passage 7

Consider a function defined in $[-2, 2]$

$$f(x) = \begin{cases} \{x\} & -2 \le x - 1 \\ |\operatorname{sgn} x| & -1 \le x \le 1 \\ \{-x\} & 1 < x \le 2 \end{cases}$$

where $\{\cdot\}$ denotes the fractional part function.

1. The total number of points of discontinuity of $f(x)$ for $x \in [-2, 2]$
   a. 0 **b.** 1 **c.** 2 **d.** 4
2. The number of points for $x \in [-2, 2]$ where $f(x)$ is non-differentiable is
   a. 0 **b.** 1 **c.** 2 **d.** 3

Passage 8

Let $f(x) = \begin{cases} \lim\limits_{n\to\infty}(\sqrt{n^2+n+1} - \sqrt{n^2-n+1})x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

and $g(x) = |x| + |x^2 - 1|$, for all $x \in R$.

1. Which one of the following statement is correct?
   a. $f(x)$ is continuous at $x = 0$
   b. $f(x)$ is non-differentiable at $x = 0$
   c. $f(x)$ has non-removable type of discontinuity at $x = 0$
   d. $f(x)$ has removable type of discontinuity at $x = 0$
2. Number of points of non-differentiability of $g(f(x))$, is
   a. 0 **b.** 1 **c.** 2 **d.** 3

Passage 9

Consider two real-valued functions defined on R (the set of all real numbers) as

$$f(x) = \tan^{-1}|x-1| \text{ and } g(x) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

1. The derivative of $f(x)$ with respect to $g(x)$ when $x = 1$, is equal to
   a. -1 **b.** 0 **c.** 1 **d.** non-existent
2. The value of $\dfrac{d^2 g}{dx^2}$ when $x = -1$, is equal to
   a. -1 **b.** 0 **c.** 1 **d.** non-existent

Chapter 3 Matrix Match Type Questions

1. Let $f(x) = \begin{cases} [x], & -2 \le x < 0 \\ |x|, & 0 \le x \le 2 \end{cases}$

(where [·] denotes the greatest integer function) $g(x) = \sec x$, $x \in R - (2n+1)\dfrac{\pi}{2}, n \in I$

Match the following statements is **column-I** with their values in **column-II** in the interval $\left(-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

Column-I	Column-II
(a) Abscissa of points where Limit of $f \circ g(x)$ exist is/are	(p) -1
(b) Abscissa of points in domain of $g \circ f(x)$	(q) π
where limit of $g \circ f(x)$ does not exist is/are	(r) $\dfrac{5\pi}{6}$
(c) Abscissa of points of discontinuity of $f \circ g(x)$ is/are	(s) $-\pi$
(d) Abscissa of points of differentiability of $f \circ g(x)$ is/are	(t) 0

2.

Column-I	Column-II
(a) If $f(x) = \sqrt{\dfrac{1+\sin^{-1}x}{1-\tan^{-1}x}}$; then $f'(0)$ equal to	(p) 0
(b) Let $g(x) = \lim\limits_{t\to\infty} t \ln\left(\dfrac{\tan\left(x+\frac{1}{t}\right)}{\tan x}\right)$ then $\dfrac{1}{2\sqrt{2}}$ $\lvert g'(\pi/8)\rvert$ equals to	(q) 1
(c) Let $h(x) = 2\sin^{-1}\sqrt{1-x} + \sin^{-1}(2\sqrt{x(1-x)})$ then $h'\left(\dfrac{1}{4}\right)$ equals to	(r) 2
(d) Let $k(x) = x + \tan x + 2$ and $l(x)$ be the inverse of $k(x)$ then $2l'(2)$ equals to	(s) 3
	(t) 4

3. List-I contains the function and **List-II** contains their derivatives at $x = 0$. Select the correct answer using the codes given below the list.

List-I	List-II
(p) $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$	(1) 2
(q) $g(x) = \cos^{-1}(2x^2 - 1)$	(2) 3
(r) $h(x) = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$	(3) -2
(s) $k(x) = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$	(4) non-existent

4.

Column-I	Column-II
(a) If $y = \left(\dfrac{1}{x}\right)^x$ then $y''(1)$ is	(p) -1
(b) If $x = 2\cos t - \cos 2t$ and $y = 2\sin t - \sin 2t$ then value of $\left\lvert\dfrac{d^2y}{dx^2}\right\rvert$ at $t = \dfrac{\pi}{2}$ is	(q) $\dfrac{1}{2}$
(c) If $y = a\cos t$ and $x = a(t - \sin t)$ then the value of $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{2}$.	(r) 0
(d) If $f(x) = e^{\tan^{-1}\left(\sin\frac{x}{2}\right)}$, then $f'(0)$ is	(s) $\dfrac{3}{2}$

5.

Column-I	Column-II
(a) $f(x) = \begin{cases} 1-x & ; \quad x<1 \\ (1-x)(2-x) & ; \quad 1 \le x \le 2 \\ 3-x & ; \quad x>2 \end{cases}$	(p) $f(x)$ is differentiable at $x = 1$
(b) $f(x) = [x] + \sqrt{x-[x]}$, where $[x]$ is greatest integer function	(q) $f(x)$ is non-differentiable at $x = 2$
(c) $y = f(x)$, where $x = 2t - \lvert t-1\rvert$ and $y = 2t^2 + t\lvert t\rvert$	(r) $f(x)$ is non-differentiable at $x = 1$
(d) $f(x) = \lvert x-1\rvert \cos\left(\dfrac{\pi x}{2}\right)$	(s) $f(x)$ is differentiable at $x = 2$

Chapter 3 Integer Type Questions

1. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cdot \cot y - 1 = 0$. Then $(f'(1))^2$ equals

2. Let $f(x)$ and $g(x)$ are polynomials of degree 3, $g(\alpha) = g'(\alpha) = 0$. If $\lim\limits_{x\to\alpha} \dfrac{f(x)}{g(x)} = 0$, then find the number of different real solutions of equation $f(x)\, g'(x) + f'(x)\, g(x) = 0$.

3. Let $y = (a \sin x + (b + c) \cos x)e^{x+d}$, where a, b, c, d are parameters represents a family of curves, then differential equation for the given family of curves is given by $y'' - \alpha y' + \beta y = 0$, then $\alpha + \beta$ is equal to

4. Let A_n and B_n be square matrices of order 3, which are defined as $A_n = [a_{ij}]$ and $B_n = [b_{ij}]$ where $a_{ij} = \dfrac{2i+j}{3^{2n}}$ and $b_{ij} = \dfrac{3i+j}{2^{2n}}$ for all i and j, $1 \le i, j \le 3$. If $l = \lim\limits_{n\to\infty} \text{Tr} \times (3A_1 + 3^2A_2 + 3^3A_3 + \ldots + 3^nA_n)$ and $m = \lim\limits_{n\to\infty} \text{Tr} \times (2B_1 + 2^2B_2 + 2^3B_3 + \ldots + 2^nB_n)$

Then find the value of $\dfrac{1}{3}(l + m)$

[**Note:** $\text{Tr} \times (P)$ denotes the trace of matrix P.]

5. Number of values in $[0, 6\pi]$ where $f(x) = \{x\} \sin^2 x + x^3 - \{x\} [\sin^2 x] + [x] \sin^2 x - [x] [\sin^2 x]$ is non-derivable (where $[\cdot]$ denotes greatest integer function and $\{\times\}$ denotes fractional part function)

6. Number of points, where $f(x) = |x| + \left|\dfrac{x}{3} - 1\right| + \left||x| - \left|\dfrac{x}{3} - 1\right|\right|$ is not derivable, is

7. If $y = \tan^{-1}\left(\dfrac{1}{4x^2+6x+3}\right) + \tan^{-1}\left(\dfrac{1}{4x^2+10x+7}\right) + \tan^{-1}\left(\dfrac{1}{4x^2+14x+13}\right) + \tan^{-1}\left(\dfrac{1}{4x^2+22x+31}\right) + \ldots$ up to 10 terms, then find the value of $\left| 61\left(\dfrac{dy}{dx}\right)_{x=0} \right|$.

8. Let $f(x) = x^3 + x$ and $g(x)$ be the inverse of $f(x)$. The value of $g''(-2)$ is of the form $\dfrac{a}{b}$, (where a and b are coprime number), then $(b - 9a)$ is

9. If $f(x) = (\sin x)^{1/3}.\ (\sin 2x)^{1/2}.\ \sin 3x$, then value of $f'\left(\dfrac{\pi}{3}\right)$ is of the form $-\dfrac{3^a}{2^b}$, then value of $\left(a - \dfrac{b}{2}\right)$ is

10. If $\dfrac{d}{dx}(f(x)) = g(x)$ and $\dfrac{d}{dx}(g(x)) = f(2x^2)$, then $\dfrac{d^2}{dx^2} f(3x^3)$ can be expressed as $axg(bx^c) + dx^e f(hx^i)$, then value of $d^{\frac{1+a+b+c-i-h}{e}}$ is equal to (where $a, b, c, d, e, h, i \in N$ and $f(x)$ is a non-zero function)

11. Let f is a differentiable function and $f'(3) = 4$, $f'(2) = -2$, then $\lim\limits_{h\to 0} \dfrac{f(h^3 + 3h^2 + 3) - f(3)}{f(2h^3 - h^2 + 2) - f(2)}$ is equal to

12. Let $f: R^+ \to R$ be a differentiable function satisfying

$$f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x}\ \forall\ x, y \in R^+ \text{ also } f(1) = 0; f'(1) = 1$$

find $\lim\limits_{x\to e}\left[\dfrac{1}{f(x)}\right]$ (where $[\cdot]$ denotes greatest integer functions)

13. Let $\alpha(x) = f(x) - f(2x)$ and $\beta(x) = f(x) - f(4x)$ and $\alpha'(1) = 5$ $\alpha'(2) = 7$ then find the value of $(\beta'(1) - 10)$

14. Let $f(x)$ denotes derivative of $p(x) = 3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ with respect to $\cos x$, then value of $\left(4f\left(\dfrac{\pi}{6}\right) + 2\right)$ is

15. If $f(x) = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x \ldots\ldots\infty}}}$ and $g(x) = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x \ldots\ldots\infty}}}$; $x \in \left(0, \dfrac{\pi}{2}\right)$, then the value of $(f'(x)\,(2\,f(x) - 1))^2 + (g'(x))^2$ equals

16. If $\dfrac{x^3}{2} \operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{y}\right)\right) + \dfrac{y^3}{2} \sec^2\left(\dfrac{1}{2}\tan^{-1}\left(\dfrac{y}{x}\right)\right) = 2\sqrt{2}xy^2$, then value of at $(1, -1)$ is

17. Let $f(x) = \begin{cases} x^2 + 1 & , x < 0 \\ (x-1)^3 + 2b & , x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x + 2a, & \text{if } x < 0 \\ |x - 2|, & \text{if } x \ge 0 \end{cases}$

(where $a \in R^-$ and $b \in R$). If $f \circ g(x)$ is continuous for all real x and differentiable at $x = 0$, then find the value of $8(b - a)$.

18. If $f(x) = \sin x + \cfrac{1}{2\sin x + \cfrac{1}{2\sin x + \cfrac{1}{2\sin x + \ldots\ldots\infty}}}$ then the value of $\dfrac{3}{f\left(\dfrac{\pi}{4}\right) f'\left(\dfrac{\pi}{4}\right)}$ is

19. If $y = x^5(\cos(\ln x) + \sin(\ln x))$, then find the value of $(a + b)$ in the relation $x^2y_2 + axy_1 + by = 0$.

20. Number of values of x in $[-4, 4]$ where $f(x) = [3x + 4] + |4x^2 - 1|\ (2x^2 + 3x - 2) + \sin\left(\dfrac{\pi x}{2}\right)$ (where $[\cdot]$ denotes greatest integer function) is non-derivable.

21. If $f(x) = \begin{cases} \dfrac{a\cos x + bx\sin x + ce^x - 2x}{x^2} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$ is derivable at $x = 0$, then (abc) equals

22. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 5ky$, then find the value of k.

23. Let $f(x)$ be a polynomial of degree 6 with leading coefficient 2009. Suppose further, that $f(1) = 1$, $f(2) = 3$, $f(3) = 5$, $f(4) = 7$, $f(5) = 9$, $f'(2) = 2$, then the sum of all the digits of $f(6)$ is

24. If $y = \sin(8\sin^{-1}x)$ then $(1 - x^2) - \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - ky$, where k is equal to

25. Let $f(x) = \dfrac{\left(\dfrac{\pi}{2} - \cos^{-1}(1 - \{x\}^2)\right)\cdot\left(\cos^{-1}(1 - \{x\})\right)^2}{2(\{x\} - \{x\}^3)}$. If $f(0^+) = p$ and $f(0^-) = q$, then find the value of $\left(\dfrac{p\pi}{q}\right)$.

[**Note:** $\{k\}$ denote the fractional part of k.]

26. If number of points of discontinuity of the function $f(x) = [2 + 10 \sin x]$, in $x \in \left[0, \dfrac{\pi}{2}\right]$ is same as number of points of non-differentiability of the function $g(x) = (x - 1)(x - 2)\,|(x - 1)(x - 2) \ldots (x - 2m)|$, $(m \in N)$ in $x \in (-\infty, \infty)$ then find the value of m.

[**Note:** $[k]$ denotes largest integer less than or equal to k.]

27. Consider, $f: R \to R$, $f(x) = xe^{\tan^{-1}(x+2)} + 6e^{(x+2)}$ and g be the inverse of f. If $g(x) = \left(\dfrac{x}{g(x)}\right)^2$ then find the absolute value of $5G'(4)$

28. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

29. If a function $f: [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left-hand derivative at $x = a$ is 0, then find the left-hand derivative at $x = -a$.

Chapter 3 Additional and Advanced Exercises

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of θ. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of θ, not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ, the resulting new equation will also be an identity.

   Differentiate the following to show that the resulting equations hold for all θ.

   a. $\sin 2\theta = 2\sin\theta\cos\theta$

   b. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$ is differentiated with respect to x, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. **a.** Find values for the constants a, b, and c that will make

   $$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

   satisfy the conditions

   $$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

   b. Find values for b and c that will make

   $$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

   satisfy the conditions

   $$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

   c. For the determined values of a, b, and c, what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. **Solutions to differential equations**

   a. Show that $y = \sin x$, $y = \cos x$, and $y = a\cos x + b\sin x$ (a and b constants) all satisfy the equation

   $$y'' + y = 0.$$

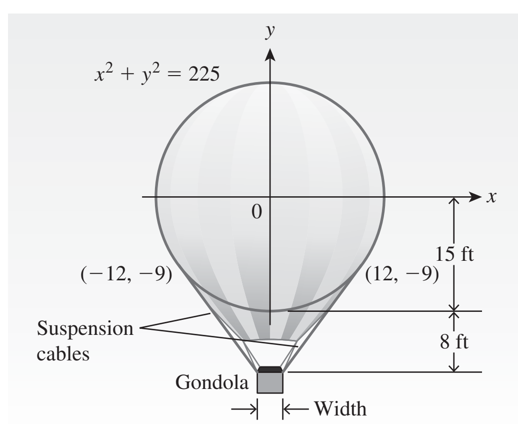
   b. How would you modify the functions in part (a) to satisfy the equation

   $$y'' + 4y = 0?$$

   Generalize this result.

5. **An osculating circle** Find the values of h, k, and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning "to kiss"). We encounter them again in Chapter 13.

6. **Designing a gondola** The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?



NOT TO SCALE

7. **Pisa by parachute** On August 5, 1988, Mike McCarthy of London jumped from the top of the Tower of Pisa. He then opened his parachute in what he said was a world record low-level parachute jump of 179 ft. Make a rough sketch to show the shape of the graph of his speed during the jump. (*Source: Boston Globe*, Aug. 6, 1988.)

8. Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$$

is

   a. continuous at $x = \pi$.

   b. differentiable at $x = \pi$.

9. Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

10. **a.** For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x?

   b. Discuss the geometry of the resulting graph of f.

11. **a.** For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x?

   b. Discuss the geometry of the resulting graph of g.

12. **Odd differentiable functions** Is there anything special about the derivative of an odd differentiable function of x? Give reasons for your answer.

13. **Even differentiable functions** Is there anything special about the derivative of an even differentiable function of x? Give reasons for your answer.

14. Suppose that the functions f and g are defined throughout an open interval containing the point x_0, that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0. This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x\,|x|$ *is* differentiable at $x = 0$.

15. (*Continuation of Exercise 14.*) Use the result of Exercise 14 to show that the following functions are differentiable at $x = 0$.

   a. $|x| \sin x$

   b. $x^{2/3} \sin x$

   c. $\sqrt[3]{x}\,(1 - \cos x)$

   d. $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

16. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

17. Suppose that a function f satisfies the following conditions for all real values of x and y:

   i) $f(x + y) = f(x) \cdot f(y)$.

   ii) $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

   Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

18. **The generalized product rule** Use mathematical induction to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx}u_2 \cdots u_n + u_1 \frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1}\frac{du_n}{dx}.$$

19. **Leibniz's rule for higher-order derivatives of products** Leibniz's rule for higher-order derivatives of products of differentiable functions says that

   a. $\dfrac{d^2(uv)}{dx^2} = \dfrac{d^2u}{dx^2}v + 2\dfrac{du}{dx}\dfrac{dv}{dx} + u\dfrac{d^2v}{dx^2}$.

   b. $\dfrac{d^3(uv)}{dx^3} = \dfrac{d^3u}{dx^3}v + 3\dfrac{d^2u}{dx^2}\dfrac{dv}{dx} + 3\dfrac{du}{dx}\dfrac{d^2v}{dx^2} + u\dfrac{d^3v}{dx^3}$.

   c. $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

   The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

4

Applications of Derivatives

OVERVIEW One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems. Optimization problems abound in mathematics, physical science and engineering, business and economics, and biology and medicine. For example, what are the height and diameter of the cylinder of largest volume that can be inscribed in a given sphere? What are the dimensions of the strongest rectangular wooden beam that can be cut from a cylindrical log of given diameter? Based on production costs and sales revenue, how many items should a manufacturer produce to maximize profit? How much does the trachea (windpipe) contract to expel air at the maximum speed during a cough? What is the branching angle at which blood vessels minimize the energy loss due to friction as blood flows through the branches?

In this chapter we use derivatives as rate measure and also to find extreme values of a functions, to determine and analyze the shapes of graphs, and to solve equation numerically. The key to many of these applications is the mean value theorem, which paves the way to integral calculus.

4.1 The Derivative as a Rate of Change

In Section 2.1 we introduced average and instantaneous rates of change. In this section we study further applications in which derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x + h) - f(x))/h$ as the average rate of change in f over the interval from x to $x + h$, we can interpret its limit as $h \to 0$ as the rate at which f is changing at the point x.

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi\ \mathrm{m}^2/\mathrm{m} \approx 15.71\ \mathrm{m}^2/\mathrm{m}$. ■

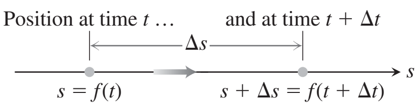


FIGURE 4.1 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 4.1) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

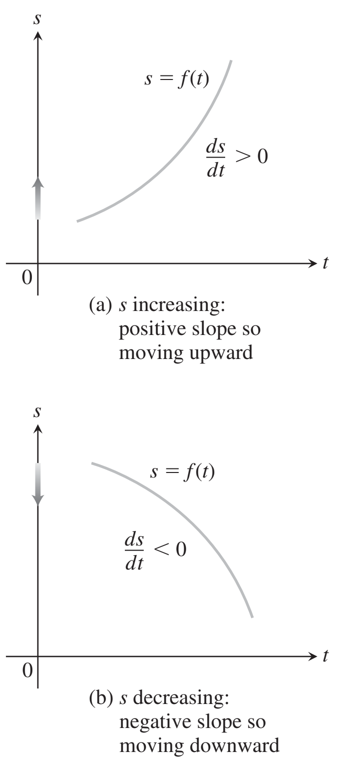


FIGURE 4.2 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

Besides telling how fast an object is moving along the horizontal line in Figure 4.1, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 4.2 represent position along the line over time; they do not portray the path of motion, which lies along the vertical s-axis.

If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 2 Figure 4.3 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line (as opposed to showing a position function $s = f(t)$ such as in Figure 4.2). In the graph of the velocity function, it's not the slope of the curve that tells us if the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Looking at Figure 4.3, we see that the particle moves forward for the first 3 sec (when the velocity is positive), moves backward for the next 2 sec (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ sec (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ sec, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ sec, speeding up during the final second of the forward motion indicated in the velocity graph. ■

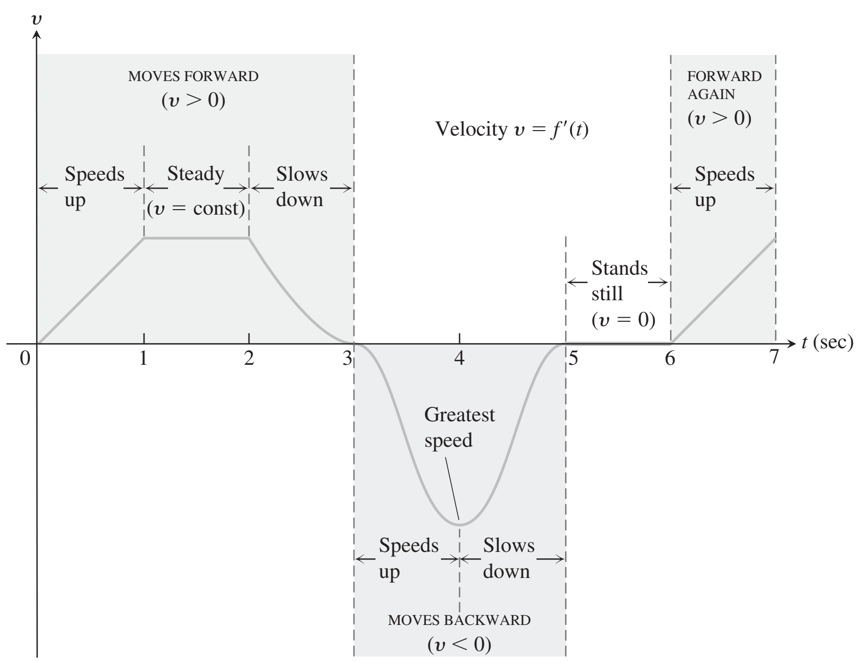


FIGURE 4.3 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

HISTORICAL BIOGRAPHY
Bernard Bolzano
(1781–1848)

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of Earth all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

where s is the distance fallen and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s. With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately $32\ \text{ft/sec}^2$ (feet per second squared) in English units, and $g = 9.8\ \text{m/sec}^2$ (meters per second squared) in metric units. (These gravitational constants depend on the distance from Earth's center of mass, and are slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 32\ \text{ft/sec}^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

EXAMPLE 3 Figure 4.4 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 3 sec?

(b) What is its velocity, speed, and acceleration when $t = 3$?

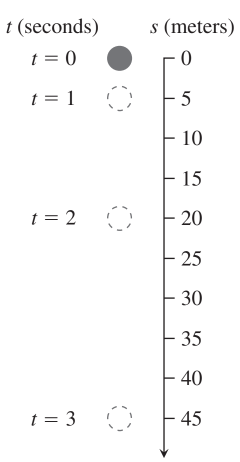


FIGURE 4.4 A ball bearing falling from rest (Example 3).

Solution

(a) The metric free-fall equation is $s = 4.9t^2$. During the first 3 sec, the ball falls

$$s(3) = 4.9(3)^2 = 44.1\ \text{m}.$$

(b) At any time t, *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}\left(4.9t^2\right) = 9.8t.$$

At $t = 3$, the velocity is

$$v(3) = 29.4\ \text{m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 3$ is

$$\text{speed} = |v(3)| = 29.4\ \text{m/sec}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8\ \text{m/sec}^2.$$

At $t = 3$, the acceleration is $9.8\ \text{m/sec}^2$. ■

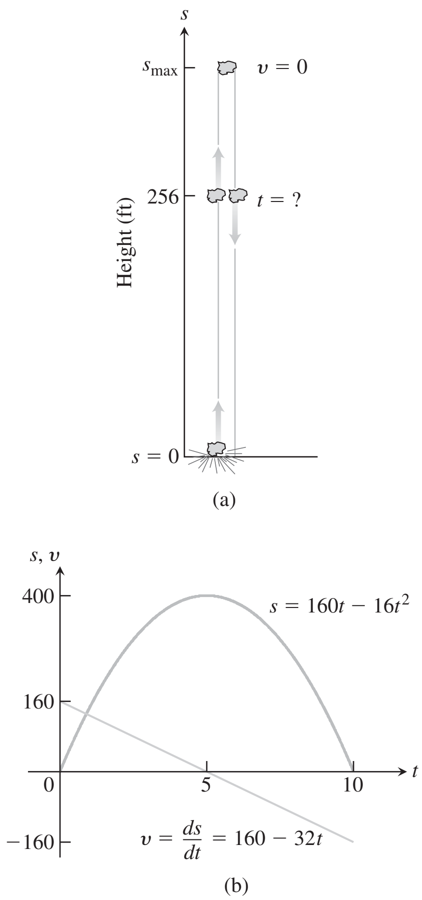


FIGURE 4.5 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 4.5a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution

(a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Figure 4.5b.

(b) To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t &= 2 \text{ sec}, t = 8 \text{ sec}. \end{aligned}$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$\begin{aligned} v(2) &= 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec}. \\ v(8) &= 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}. \end{aligned}$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

(c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward and is the effect of gravity on the rock. As the rock rises, it slows down; as it falls, it speeds up.

(d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

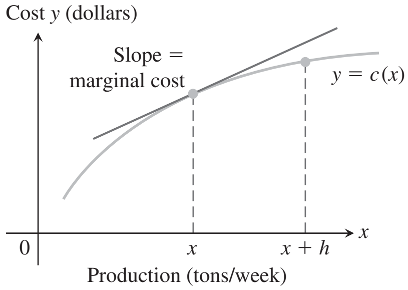


FIGURE 4.6 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

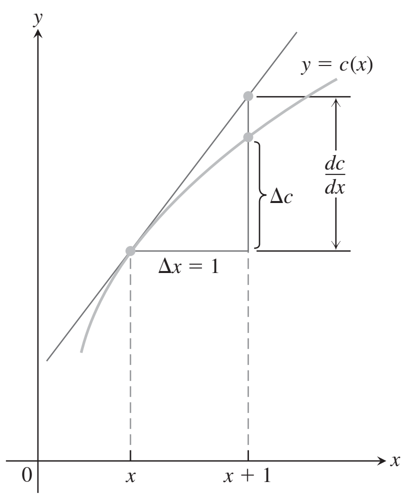


FIGURE 4.7 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx.

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference, divided by h, is the average cost of producing each additional ton:

$$\frac{c(x + h) - c(x)}{h} = \begin{array}{l}\text{average cost of each of the additional}\\ h \text{ tons of steel produced.}\end{array}$$

The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel per week when the current weekly production is x tons (Figure 4.6):

$$\frac{dc}{dx} = \lim_{h\to 0}\frac{c(x + h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x + 1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if the slope of the graph of c does not change quickly near x. Then the difference quotient will be close to its limit dc/dx, which is the rise in the tangent line if $\Delta x = 1$ (Figure 4.7). The approximation works best for large values of x.

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where δ represents *fixed costs*, such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs*, such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

EXAMPLE 5 Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 15x\right) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. The marginal revenue is

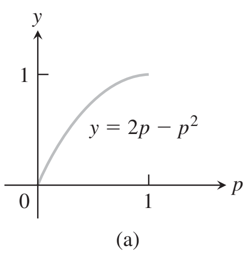
$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. ■

EXAMPLE 6 To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ■



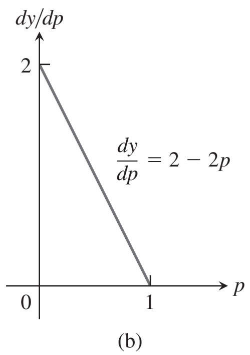


FIGURE 4.8 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas in the next generation. (b) The graph of dy/dp (Example 7).

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

EXAMPLE 7 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 4.8a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 4.8b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth skin peas. ■

Exercises 4.1

Free-Fall Applications

1. **Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?

2. **Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in t sec.
   a. Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
   b. How long does it take the rock to reach its highest point?
   c. How high does the rock go?
   d. How long does it take the rock to reach half its maximum height?
   e. How long is the rock aloft?

3. **Finding g on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was g_s m/sec^2, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m t sec later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of g_s?

4. **Speeding bullet** A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long will the bullet be aloft in each case? How high will the bullet go?

5. **Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 179 ft above the ground, the ball's height above the ground t sec into the fall would have been $s = 179 - 16t^2$.

   a. What would have been the ball's velocity, speed, and acceleration at time t?

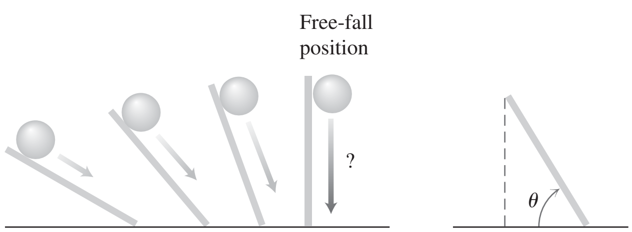
   b. About how long would it have taken the ball to hit the ground?

   c. What would have been the ball's velocity at the moment of impact?

6. **Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t sec into motion was a constant multiple of t. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.

   In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ, the ball's velocity t sec into the roll was

$$v = 9.8(\sin\theta)t \text{ m/sec}.$$



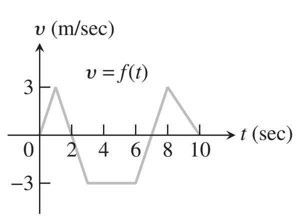
(a) (b)

   a. What is the equation for the ball's velocity during free fall?

   b. Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

Understanding Motion from Graphs

7. The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



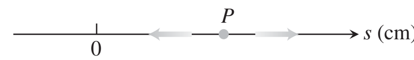
   a. When does the body reverse direction?

   b. When (approximately) is the body moving at a constant speed?

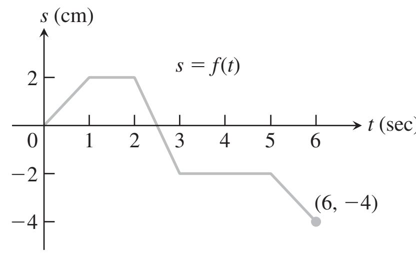
   c. Graph the body's speed for $0 \le t \le 10$.

   d. Graph the acceleration, where defined.

8. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)



(b)

   a. When is P moving to the left? Moving to the right? Standing still?

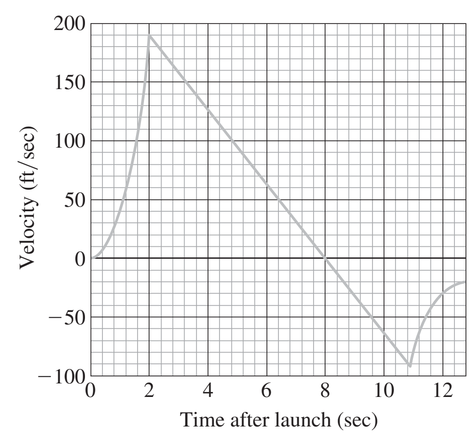
   b. Graph the particle's velocity and speed (where defined).

9. **Launching a rocket** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

   The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

   a. How fast was the rocket climbing when the engine stopped?

   b. For how many seconds did the engine burn?



   c. When did the rocket reach its highest point? What was its velocity then?

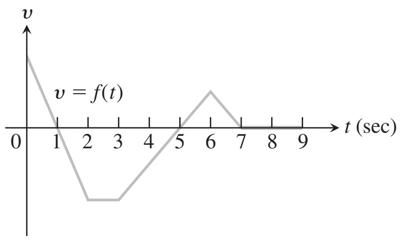
   d. When did the parachute pop out? How fast was the rocket falling then?

   e. How long did the rocket fall before the parachute opened?

   f. When was the rocket's acceleration greatest?

   g. When was the acceleration constant? What was its value then (to the nearest integer)?

10. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



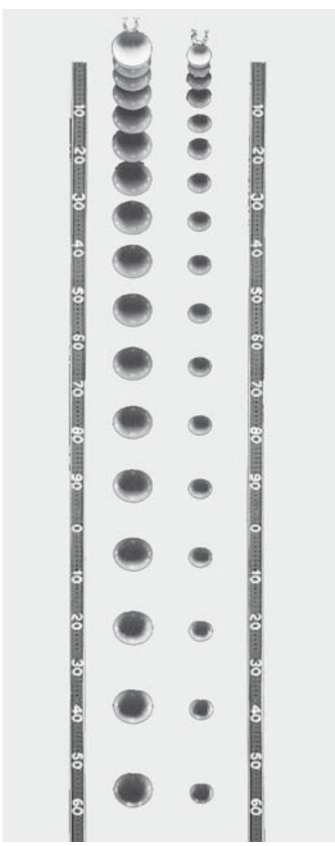
a. When does the particle move forward? Move backward? Speed up? Slow down?

b. When is the particle's acceleration positive? Negative? Zero?

c. When does the particle move at its greatest speed?

d. When does the particle stand still for more than an instant?

11. Two falling balls The multiflash photograph in the accompanying figure shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Education Development Center, Inc.)



a. How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

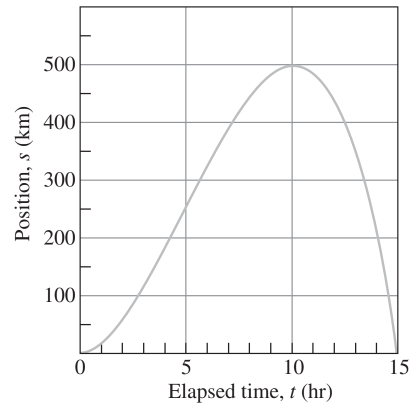
b. How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

c. About how fast was the light flashing (flashes per second)?

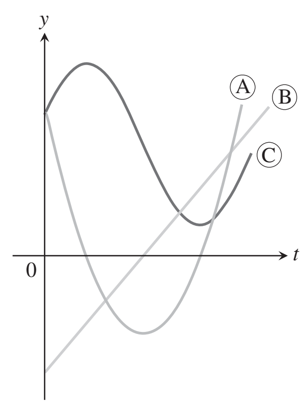
12. A traveling truck The accompanying graph shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 h later at $t = 15$.

a. Use the technique described in Section 3.2, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

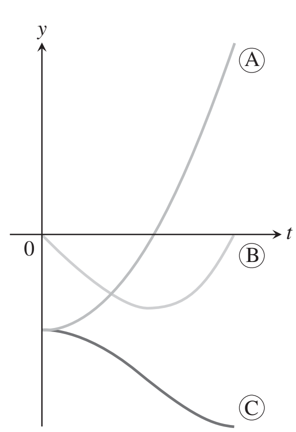
b. Suppose that $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2 and compare your graphs with those in part (a).



13. The graphs in the accompanying figure show the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.



14. The graphs in the accompanying figure show the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.



Economics

15. Marginal cost Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a. Find the average cost per machine of producing the first 100 washing machines.

b. Find the marginal cost when 100 washing machines are produced.

c. Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

16. Marginal revenue Suppose that the revenue from selling x washing machines is

$$r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$$

dollars.

a. Find the marginal revenue when 100 machines are produced.

b. Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.

c. Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Additional Applications

17. Bacterium population When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at

a. $t = 0$ hours.

b. $t = 5$ hours.

c. $t = 10$ hours.

18. Body surface area A typical male's body surface area S in square meters is often modeled by the formula $S = \frac{1}{60}\sqrt{wh}$, where h is the height in cm, and w the weight in kg, of the person. Find the rate of change of body surface area with respect to weight for males of constant height $h = 180$ cm (roughly 5′9″). Does S increase more rapidly with respect to weight at lower or higher body weights? Explain.

T **19. Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

a. Find the rate dy/dt (m/h) at which the tank is draining at time t.

b. When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?

c. Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

20. Draining a tank The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

21. Vehicular stopping distance Based on data from the U.S. Bureau of Public Roads, a model for the total stopping distance of a moving car in terms of its speed is

$$s = 1.1v + 0.054v^2,$$

where s is measured in ft and v in mph. The linear term $1.1v$ models the distance the car travels during the time the driver perceives a need to stop until the brakes are applied, and the quadratic term $0.054v^2$ models the additional braking distance once they are applied. Find ds/dv at $v = 35$ and $v = 70$ mph, and interpret the meaning of the derivative.

22. Inflating a balloon The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

a. At what rate (ft^3/ft) does the volume change with respect to the radius when $r = 2$ ft?

b. By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

23. Airplane takeoff Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

24. Volcanic lava fountains Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a Hawaiian record). What was the lava's exit velocity in feet per second? In miles per hour? (*Hint:* If v_0 is the exit velocity of a particle of lava, its height t sec later will be $s = v_0t - 16t^2$ ft. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

4.2 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

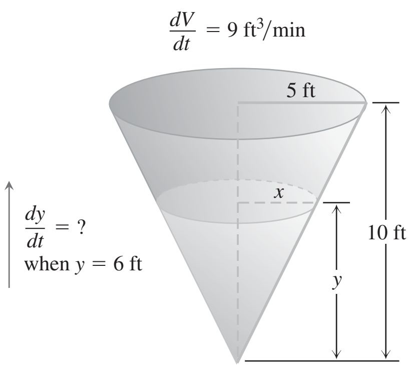


FIGURE 4.9 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 4.9 shows a partially filled conical tank. The variables in the problem are

$V =$ volume (ft^3) of the water in the tank at time t (min)

$x =$ radius (ft) of the surface of the water at time t

$y =$ depth (ft) of the water in the tank at time t.

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \qquad \text{and} \qquad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 4.9 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore, we find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

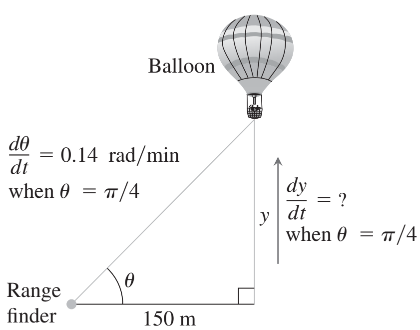


FIGURE 4.10 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in the six strategy steps.

1. *Draw a picture and name the variables and constants* (Figure 4.10). The variables in the picture are

   $\theta =$ the angle in radians the range finder makes with the ground.

   $y =$ the height in meters of the balloon above the ground.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give it a special symbol.

2. *Write down the additional numerical information*.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find*. We want dy/dt when $\theta = \pi/4$.
4. *Write an equation that relates the variables y and θ*.

$$\frac{y}{150} = \tan\theta \qquad \text{or} \qquad y = 150\tan\theta$$

5. *Differentiate with respect to t using the Chain Rule*. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 150\,(\sec^2\theta)\frac{d\theta}{dt}$$

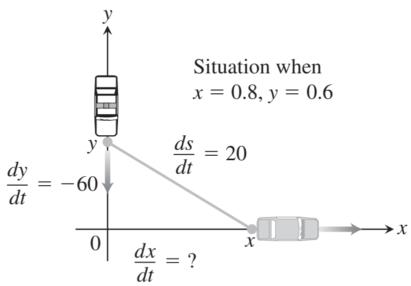


FIGURE 4.11 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance s between them (Example 3).

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

$$\frac{dy}{dt} = 150\left(\sqrt{2}\right)^2(0.14) = 42 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 42 m/min. ■

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 4.11). We let t represent time and set

$$\begin{aligned} x &= \text{position of car at time } t \\ y &= \text{position of cruiser at time } t \\ s &= \text{distance between car and cruiser at time } t. \end{aligned}$$

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation between the car and the cruiser,

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \\ &= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right). \end{aligned}$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$\begin{aligned} 20 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) \\ \frac{dx}{dt} &= \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70 \end{aligned}$$

At the moment in question, the car's speed is 70 mph. ■

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is (0, 10) on the y-axis, and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?

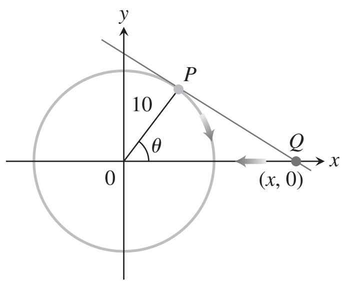


FIGURE 4.12 The particle P travels clockwise along the circle (Example 4).

Solution We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 4.12). We let t represent time and let θ denote the angle from the x-axis to the radial line joining the origin to P. Since the particle travels from start to finish in 30 sec, it is traveling along the circle at a constant rate of $\pi/2$ radians in $1/2$ min, or π rad/min. In other words, $d\theta/dt = -\pi$, with t being measured in minutes. The negative sign appears because θ is decreasing over time.

Setting $x(t)$ to be the distance at time t from the point Q to the origin, we want to find dx/dt when

$$x = 20 \text{ m} \qquad \text{and} \qquad \frac{d\theta}{dt} = -\pi \text{ rad/min}.$$

To relate the variables x and θ, we see from Figure 4.12 that $x \cos\theta = 10$, or $x = 10 \sec\theta$. Differentiation of this last equation gives

$$\frac{dx}{dt} = 10 \sec\theta \tan\theta \frac{d\theta}{dt} = -10\pi \sec\theta \tan\theta.$$

Note that dx/dt is negative because x is decreasing (Q is moving toward the origin).

When $x = 20$, $\cos\theta = 1/2$ and $\sec\theta = 2$. Also, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{3}$. It follows that

$$\frac{dx}{dt} = (-10\pi)(2)\left(\sqrt{3}\right) = -20\sqrt{3}\pi.$$

At the moment in question, the point Q is moving toward the origin at the speed of $20\sqrt{3}\pi \approx 109$ m/min. ■

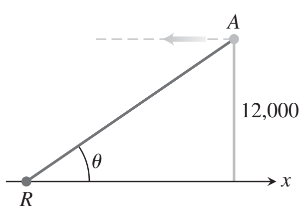


FIGURE 4.13 Jet airliner A traveling at constant altitude toward radar station R (Example 5).

EXAMPLE 5 A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of $2/3$ deg/sec in order to keep the aircraft within its direct line of sight?

Solution The aircraft A and radar station R are pictured in the coordinate plane, using the positive x-axis as the horizontal distance at sea level from R to A, and the positive y-axis as the vertical altitude above sea level. We let t represent time and observe that $y = 12{,}000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 4.13. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

From Figure 4.13, we see that

$$\frac{12{,}000}{x} = \tan\theta \qquad \text{or} \qquad x = 12{,}000 \cot\theta.$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280} \cot\theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -\frac{1200}{528} \csc^2\theta \frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$, so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3$ deg/sec to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600) \text{ rad/hr}. \qquad 1 \text{ hr} = 3600 \text{ sec}, 1 \text{ deg} = \pi/180 \text{ rad}$$

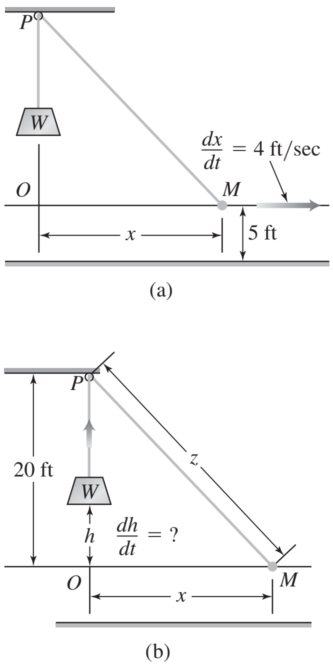


FIGURE 4.14 A worker at M walks to the right, pulling the weight W upward as the rope moves through the pulley P (Example 6).

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380.$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar. ■

EXAMPLE 6 Figure 4.14a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line PW at the rate of 4 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW?

Solution We let OM be the horizontal line of length x ft from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 4.14). Let h be the height of the weight W above O, and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 4$. Note that the height of P above O is 20 ft because O is 5 ft above the ground. We assume the angle at O is a right angle.

At any instant of time t we have the following relationships (see Figure 4.14b):

$$20 - h + z = 45 \qquad \text{Total length of rope is 45 ft.}$$

$$20^2 + x^2 = z^2. \qquad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \tag{1}$$

Differentiating both sides with respect to t gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{25 + h}\frac{dx}{dt}. \tag{2}$$

Since we know dx/dt, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \qquad \text{or} \qquad 25 + h = 29.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{21}{29}\cdot 4 = \frac{84}{29} \approx 2.9 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft. ■

Exercises 4.2

1. **Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

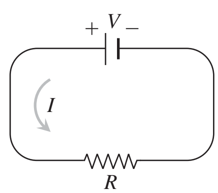
2. **Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.

3. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.

4. **Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

5. **Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

6. **Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in seconds.



   a. What is the value of dV/dt?
   b. What is the value of dI/dt?
   c. What equation relates dR/dt to dV/dt and dI/dt?
   d. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps. Is R increasing, or decreasing?

7. **Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
   a. How are dP/dt, dR/dt, and dI/dt related if none of P, R, and I are constant?
   b. How is dR/dt related to dI/dt if P is constant?

8. **Distance** Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.
   a. How is ds/dt related to dx/dt if y is constant?
   b. How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
   c. How is dx/dt related to dy/dt if s is constant?

9. **Diagonals** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
   a. Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt?
   b. How is ds/dt related to dy/dt and dz/dt if x is constant?
   c. How are dx/dt, dy/dt, and dz/dt related if s is constant?

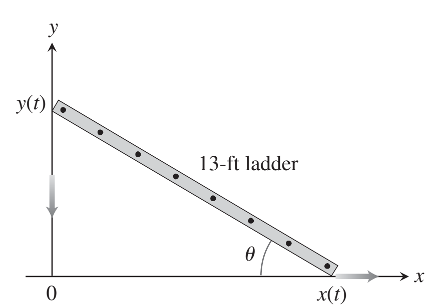
10. **Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is
$$A = \frac{1}{2}ab\sin\theta.$$
   a. How is dA/dt related to $d\theta/dt$ if a and b are constant?
   b. How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
   c. How is dA/dt related to $d\theta/dt$, da/dt, and db/dt if none of a, b, and θ are constant?

11. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

12. **Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

13. **Changing dimensions in a rectangular box** Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:
$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$
Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

14. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
   a. How fast is the top of the ladder sliding down the wall then?
   b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
   c. At what rate is the angle θ between the ladder and the ground changing then?



15. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

16. **Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

17. **Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

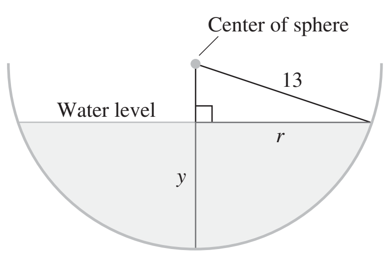
18. **A growing sand pile** Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

19. **A draining conical reservoir** Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.

    a. How fast (centimeters per minute) is the water level falling when the water is 5 m deep?

    b. How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

20. **A draining hemispherical reservoir** Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.



    a. At what rate is the water level changing when the water is 8 m deep?

    b. What is the radius r of the water's surface when the water is y m deep?

    c. At what rate is the radius r changing when the water is 8 m deep?

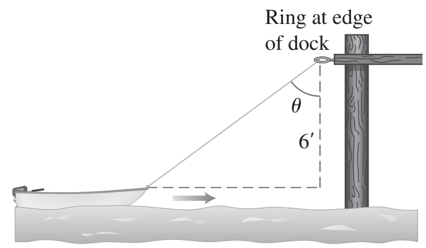
21. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

22. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

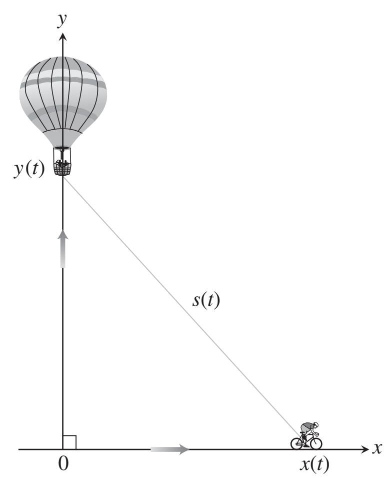
23. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.

    a. How fast is the boat approaching the dock when 10 ft of rope are out?

    b. At what rate is the angle θ changing at this instant (see the figure)?



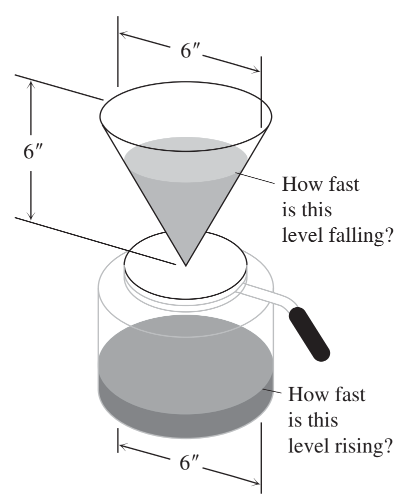
24. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



25. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

    a. How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

    b. How fast is the level in the cone falling then?



26. Cardiac output In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

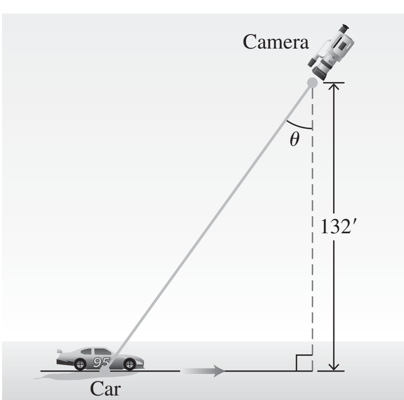
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

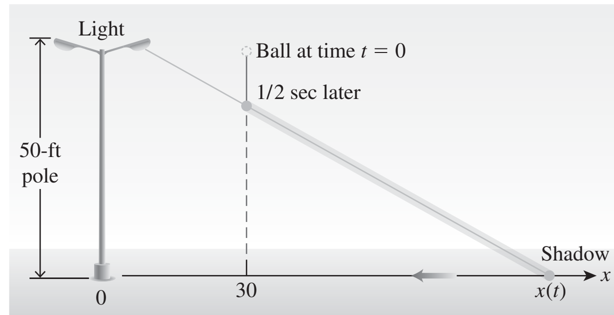
27. Moving along a parabola A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

28. Motion in the plane The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

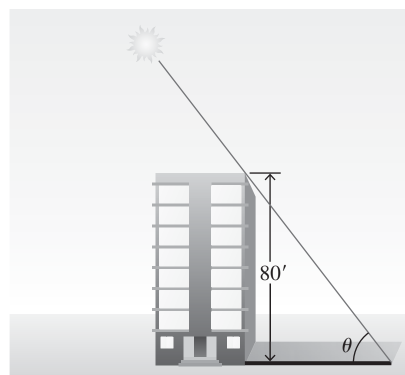
29. Videotaping a moving car You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



30. A moving shadow A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



31. A building's shadow On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



32. A melting ice layer A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

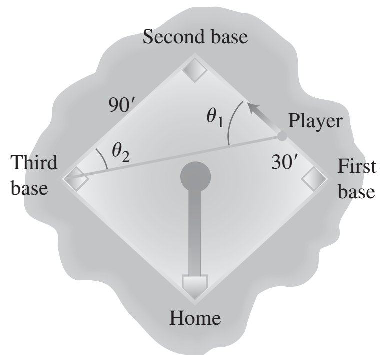
33. Highway patrol A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

34. Baseball players A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

a. At what rate is the player's distance from third base changing when the player is 30 ft from first base?

b. At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

c. The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



35. Ships Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

36. Clock's moving hands At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?

37. Oil spill An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 9 in. thick. After several days, when the major axis of the slick is 2 mi long and the minor axis is 3/4 mi wide, it is determined that its length is increasing at the rate of 30 ft/hr, and its width is increasing at the rate of 10 ft/hr. At what rate (in cubic feet per hour) is oil flowing from the site of the rig at that time?

4.3 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines.

We introduce new variables dx and dy, called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

Linearization

As you can see in Figure 4.15, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line

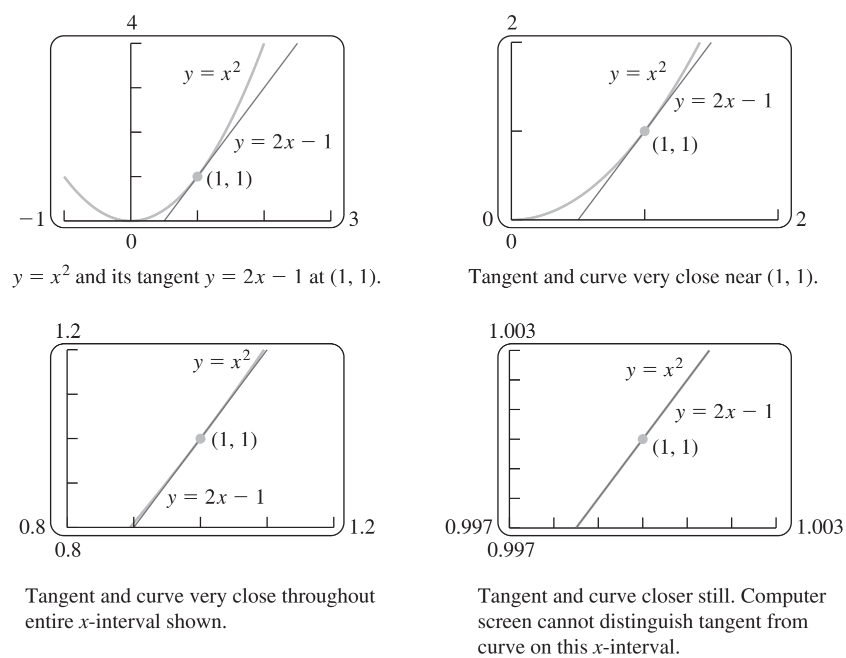


FIGURE 4.15 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

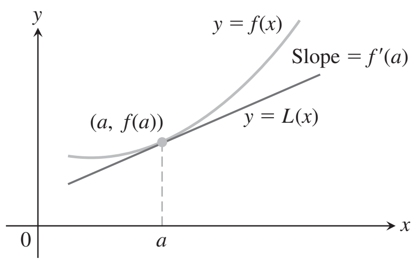


FIGURE 4.16 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

give good approximations to the y-values on the curve. We observe this phenomenon by zooming in on the two graphs at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the x-coordinate of the point of tangency. The phenomenon is true not just for parabolas; every differentiable curve behaves locally like its tangent line.

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable (Figure 4.16), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of f as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.

DEFINITIONS **If f is differentiable at $x = a$, then the approximating function**

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 4.17).

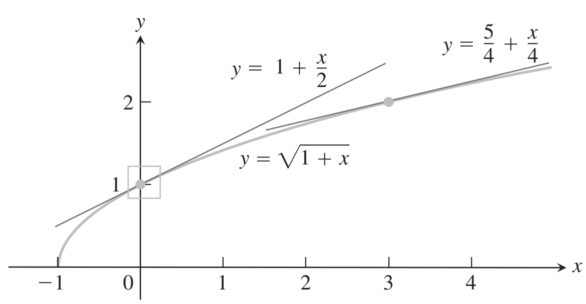


FIGURE 4.17 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 4.18 shows a magnified view of the small window about 1 on the y-axis.

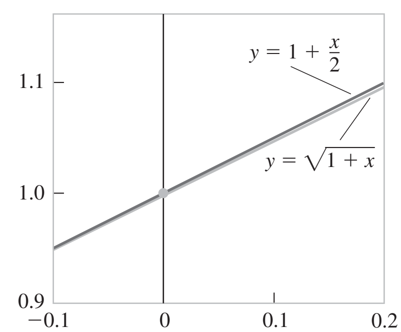


FIGURE 4.18 Magnified view of the window in Figure 4.17.

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 4.18. ■

The following table shows how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for some values of x near 0. As we move away from zero, we lose

accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	True value	$\lvert$True value $-$ approximation$\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1 + x}$ for x close to 0 and can tolerate the small amount of error involved, we can work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is.

A linear approximation normally loses accuracy away from its center. As Figure 4.17 suggests, the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}. \qquad \blacksquare$$

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

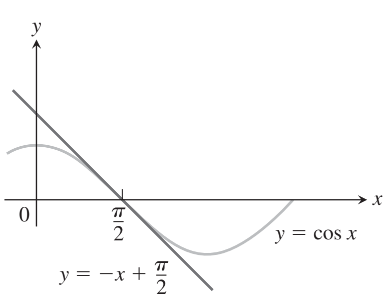


FIGURE 4.19 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 4.19).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned} \qquad \blacksquare$$

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \qquad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

Approximations Near $x = 0$

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

$$\frac{1}{1-x} \approx 1 + x$$

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2}$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1;\ \text{replace } x \text{ by } -x.$$

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \qquad k = 1/3;\ \text{replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2;\ \text{replace } x \text{ by } -x^2.$$

Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then these values determine the numerical value of dy. Often the variable dx is chosen to be Δx, the change in x.

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

■

The geometric meaning of differentials is shown in Figure 4.20. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

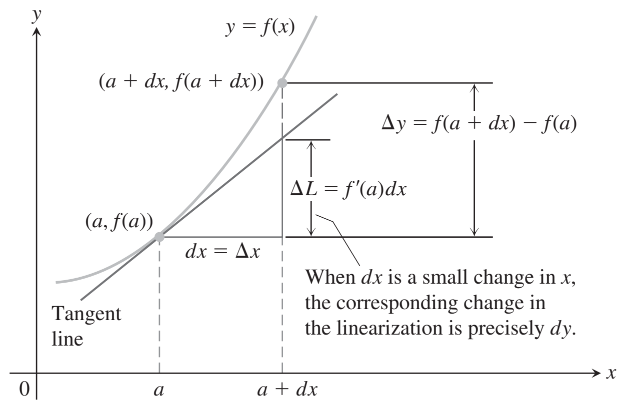


FIGURE 4.20 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

That is, the change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f**. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{or} \qquad \frac{d(\sin u)}{dx} = \cos u\,\frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \qquad \text{or} \qquad d(\sin u) = \cos u\,du.$$

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

(b) $d\left(\dfrac{x}{x + 1}\right) = \dfrac{(x + 1)\,dx - x\,d(x + 1)}{(x + 1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x + 1)^2} = \dfrac{dx}{(x + 1)^2}$ ■

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 4.20 that Δy is approximately equal to the differential dy. Since

$$f(a + dx) = f(a) + \Delta y, \qquad \Delta x = dx$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a)\,dx$.

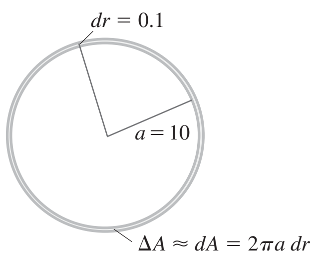


FIGURE 4.21 When dr is small compared with a, the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 4.21). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately $102\pi \text{ m}^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is $0.01\pi \text{ m}^2$, which is the difference $\Delta A - dA$. ■

EXAMPLE 7 Use differentials to estimate

(a) $7.97^{1/3}$

(b) $\sin(\pi/6 + 0.01)$.

Solution

(a) The differential associated with the cube root function $y = x^{1/3}$ is

$$dy = \frac{1}{3x^{2/3}}\,dx.$$

We set $a = 8$, the closest number near 7.97 where we can easily compute $f(a)$ and $f'(a)$. To arrange that $a + dx = 7.97$, we choose $dx = -0.03$. Approximating with the differential gives

$$\begin{aligned} f(7.97) = f(a + dx) &\approx f(a) + dy \\ &= 8^{1/3} + \frac{1}{3(8)^{2/3}}(-0.03) \\ &= 2 + \frac{1}{12}(-0.03) = 1.9975 \end{aligned}$$

This gives an approximation to the true value of $7.97^{1/3}$, which is 1.997497 to 6 decimals.

(b) The differential associated with $y = \sin x$ is

$$dy = \cos x\,dx.$$

To estimate $\sin(\pi/6 + 0.01)$, we set $a = \pi/6$ and $dx = 0.01$. Then

$\sin(a + dx) \approx \sin a + (\cos a)\,dx$

$$\begin{aligned} f(\pi/6 + 0.01) = f(a + dx) &\approx f(a) + dy \\ &= \sin\frac{\pi}{6} + \left(\cos\frac{\pi}{6}\right)(0.01) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}(0.01) \approx 0.5087 \end{aligned}$$

For comparison, the true value of $\sin(\pi/6 + 0.01)$ to 6 decimals is 0.508635. ■

The method in part (b) of Example 7 is used by some calculator and computer algorithms to give values of trigonometric functions. The algorithms store a large table of sine and cosine values between 0 and $\pi/4$. Values between these stored values are computed using differentials as in Example 7b. Values outside of $[0, \pi/4]$ are computed from values in this interval using trigonometric identities.

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. We have two ways to describe the change in f as x changes from a to $a + \Delta x$:

The true change:	$\Delta f = f(a + \Delta x) - f(a)$
The differential estimate:	$df = f'(a)\,\Delta x.$

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned} \text{Approximation error} &= \Delta f - df \\ &= \Delta f - f'(a)\Delta x \\ &= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\ &= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right)}_{\text{Call this part } \epsilon.} \cdot \Delta x \\ &= \epsilon \cdot \Delta x. \end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it ϵ). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When Δx is small, the approximation error $\epsilon\,\Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true}\\\text{change}}} = \underbrace{f'(a)\Delta x}_{\substack{\text{estimated}\\\text{change}}} + \epsilon\,\underbrace{\Delta x}_{\text{error}}$$

Although we do not know the exact size of the error, it is the product $\epsilon \cdot \Delta x$ of two small quantities that both approach zero as $\Delta x \to 0$. For many common functions, whenever Δx is small, the error is still smaller.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\text{ m}^2$$

so the approximation error is $\Delta A - dA = \epsilon\Delta r = 0.01\pi$ and $\epsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Proof of the Chain Rule

Equation (1) enables us to prove the Chain Rule correctly. Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composite $y = f(g(x))$ is a differentiable function of x. Since a function is differentiable if and only if it has a derivative at each point in its domain, we must show that whenever g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composite is differentiable at x_0 and the derivative of the composite satisfies the equation

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y. Applying Equation (1) we have

$$\Delta u = g'(x_0)\Delta x + \epsilon_1\,\Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2\,\Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since ϵ_1 and ϵ_2 go to zero as Δx goes to zero, the last three terms on the right vanish in the limit, leaving

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0). \qquad \blacksquare$$

Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx. As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

EXAMPLE 8 You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t\,dt$$

depends on how big t is. If $t = 2$ sec, the change caused by $dt = 0.1$ is about

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the change caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

For a fixed error in the time measurement, the error in using ds to estimate the depth is larger when it takes a longer time before the stone splashes into the water. That is, the estimate is more sensitive to the effect of the error for larger values of t. ■

EXAMPLE 9 Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of an object increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" m_0 represents the mass of an object that is not moving and c is the speed of light, which is about 300,000 km/sec. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \tag{2}$$

to estimate the increase Δm in mass resulting from the added velocity v.

Solution When v is very small compared with c, v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \qquad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{3}$$

Equation (3) expresses the increase in mass that results from the added velocity v. ■

Converting Mass to Energy

Equation (3) derived in Example 9 has an important interpretation. In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the object, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy.

Exercises 4.3

Linearization for Approximation

In Exercises 1–3, find a linearization at a suitably chosen integer near a at which the given function and its derivative are easy to evaluate.

1. $f(x) = 1 + x, \quad a = 8.1$

2. $f(x) = \sqrt[3]{x}, \quad a = 8.5$

3. $f(x) = \dfrac{x}{x + 1}, \quad a = 1.3$

4. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

5. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

a. $f(x) = (1 - x)^6$ **b.** $f(x) = \dfrac{2}{1 - x}$

c. $f(x) = \dfrac{1}{\sqrt{1 + x}}$

6. Faster than a calculator Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

7. Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

Derivatives in Differential Form

In Exercises 8–10, find dy.

8. $y = x^3 - 3\sqrt{x}$

9. $y = x\sqrt{1 - x^2}$

10. $y = \dfrac{2x}{1 + x^2}$

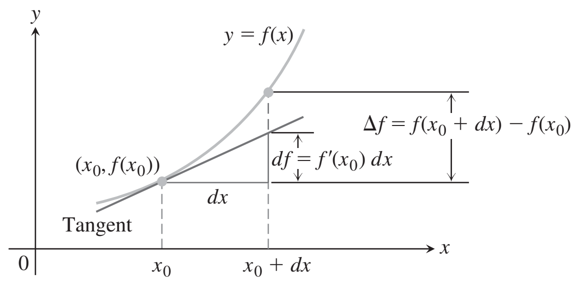
Approximation Error

In Exercises 11–12, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b. the value of the estimate $df = f'(x_0)\,dx$; and

c. the approximation error $|\Delta f - df|$.



11. $f(x) = x^2 + 2x, \quad x_0 = 1, \quad dx = 0.1$

12. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

Differential Estimates of Change

In Exercises 13–15, write a differential formula that estimates the given change in volume or surface area.

13. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

14. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

15. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

Applications

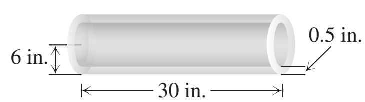
16. The radius of a circle is increased from 2.00 to 2.02 m.

a. Estimate the resulting change in area.

b. Express the estimate as a percentage of the circle's original area.

17. The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-sectional area?

18. Estimating volume Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.



19. The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's

a. surface area? **b.** volume?

20. Tolerance The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

21. Tolerance

a. About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

b. About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

22. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

23. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

24. The effect of flight maneuvers on the heart The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the weight density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \ (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2\ \text{ft/sec}^2$, with the effect the same change dg would have on Earth, where $g = 32\ \text{ft/sec}^2$. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}.

25. Drug concentration The concentration C in milligrams per milliliter (mg/ml) of a certain drug in a person's bloodstream t hrs after a pill is swallowed is modeled by the approximation

$$C(t) = \frac{4t}{1 + t^3} + 0.06t.$$

Estimate the change in concentration when t changes from 20 to 30 min.

26. Unclogging arteries The formula $V = kr^4$, discovered by the physiologist Jean Poiseuille (1797–1869), allows us to predict how much the radius of a partially clogged artery has to be expanded in order to restore normal blood flow. The formula says that the volume V of blood flowing through the artery in a unit of time at a fixed pressure is a constant k times the radius of the artery to the fourth power. How will a 10% increase in r affect V?

27. Measuring acceleration of gravity When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

a. With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).

b. If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

c. A clock with a 100-cm pendulum is moved from a location where $g = 980\ \text{cm/sec}^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

28. Quadratic approximations

a. Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i) $Q(a) = f(a)$

ii) $Q'(a) = f'(a)$

iii) $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

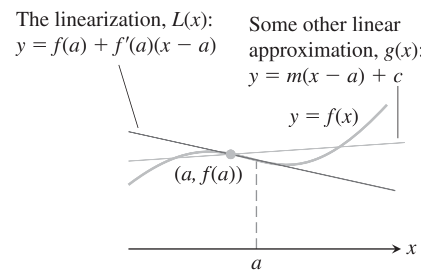
b. Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

29. The linearization is the best linear approximation Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.

2. $\lim_{x \to a} \dfrac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



4.4 Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of optimization problems (see Section 4.5). The domains of the functions we consider are intervals or unions of separate intervals.

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

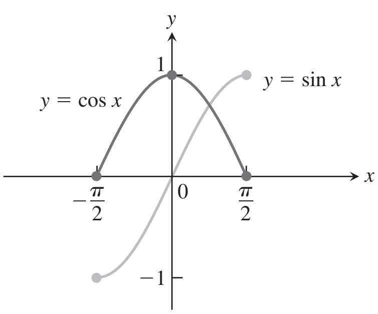


FIGURE 4.22 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.22).

Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain. We see this in the following example.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.23. Each function has the same defining equation, $y = x^2$, but the domains vary. Notice that a function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

Function rule	**Domain** D	**Absolute extrema on** D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

■

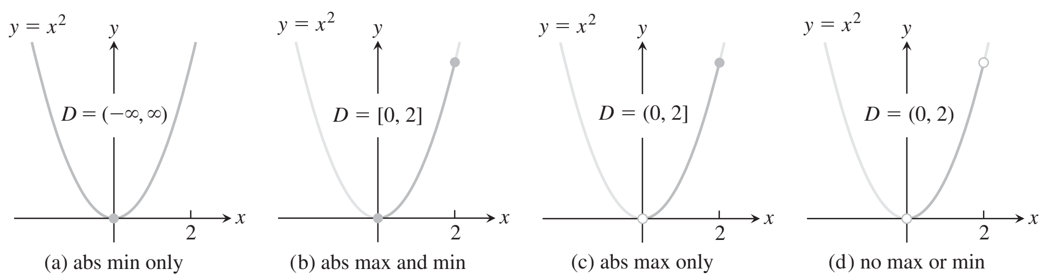


FIGURE 4.23 Graphs for Example 1.

HISTORICAL BIOGRAPHY
Daniel Bernoulli
(1700–1789)

Some of the functions in Example 1 did not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

> **THEOREM 1—The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system and we will not give it here. Figure 4.24 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are key ingredients. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. The function $y = x$ over $(-\infty, \infty)$ shows that neither extreme value need exist on an infinite interval. Figure 4.25 shows that the continuity requirement cannot be omitted.

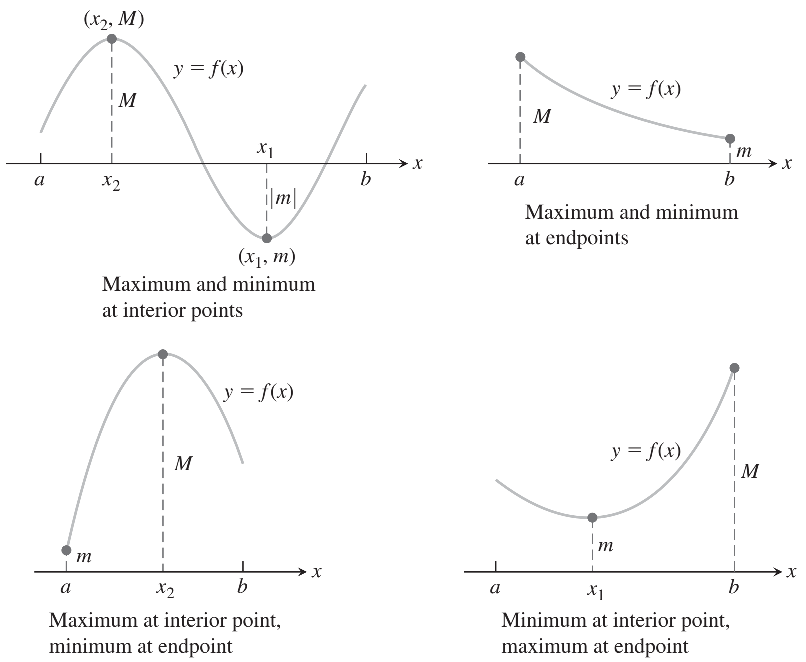


FIGURE 4.24 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

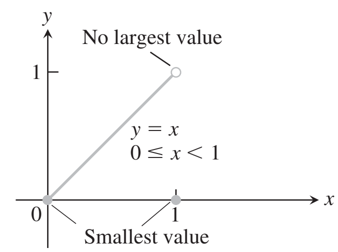


FIGURE 4.25 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

Local (Relative) Extreme Values

Figure 4.26 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d. We now define what we mean by local extrema.

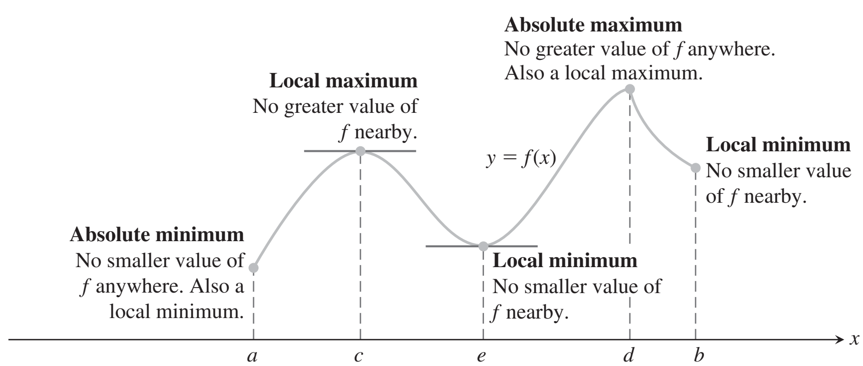


FIGURE 4.26 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$, if $f(x) \le f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \le f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \le f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.26, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.40.)

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

> **THEOREM 2—The First Derivative Theorem for Local Extreme Values** If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then
>
> $$f'(c) = 0.$$

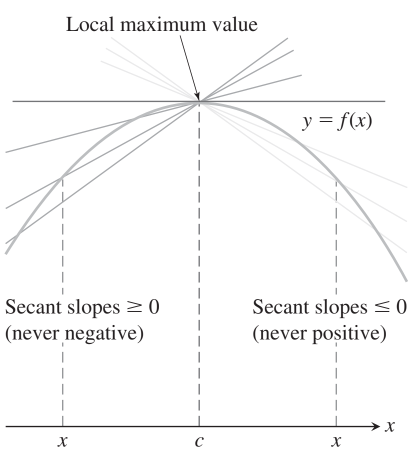


FIGURE 4.27 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.27) so that $f(x) - f(c) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

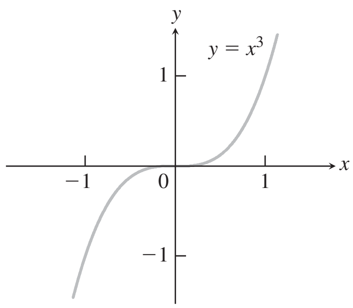
$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \le f(c) \qquad (1)$$

Similarly,

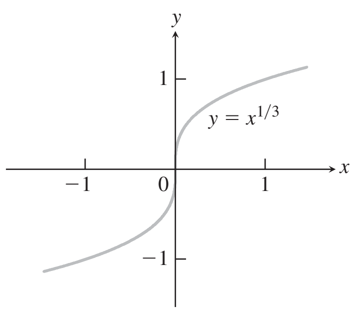
$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \le f(c) \qquad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \ge f(c)$, which reverses the inequalities in Equations (1) and (2). ■



(a)



(b)

FIGURE 4.28 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there.
(b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$, — At $x = c$ and $x = e$ in Fig. 4.26
2. interior points where f' is undefined, — At $x = d$ in Fig. 4.26
3. endpoints of the domain of f. — At $x = a$ and $x = b$ in Fig. 4.26

The following definition helps us to summarize these results.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints. However, be careful not to misinterpret what is being said here. A function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin, but neither function has a local extreme value at the origin. Instead, each function has a *point of inflection* there (see Figure 4.28). We define and explore inflection points in Section 4.4.

Most problems that ask for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located. Of course, if the interval is not closed or not finite (such as $a < x < b$ or $a < x < \infty$), we have seen that absolute extrema need not exist. If an absolute maximum or minimum value does exist, it must occur at a critical point or at an included right- or left-hand endpoint of the interval.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

$$\text{Critical point value:} \quad f(0) = 0$$

$$\text{Endpoint values:} \quad f(-2) = 4$$

$$f(1) = 1.$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

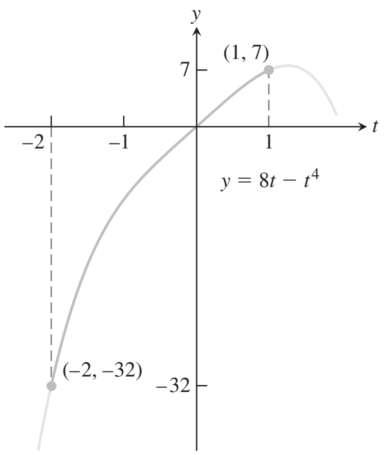


FIGURE 4.29 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \qquad \text{or} \qquad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See Figure 4.29. ■

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

$$\begin{aligned} &\text{Critical point value:} && f(0) = 0 \\ &\text{Endpoint values:} && f(-2) = (-2)^{2/3} = \sqrt[3]{4} \\ &&& f(3) = (3)^{2/3} = \sqrt[3]{9}. \end{aligned}$$

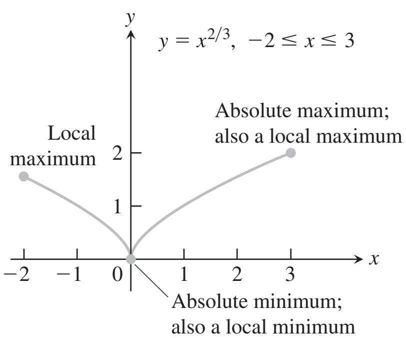


FIGURE 4.30 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

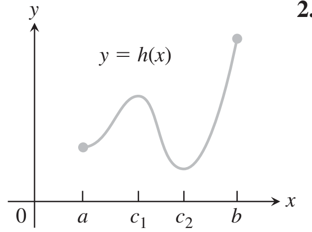
We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.30). ■

Exercises 4.4

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with .Theorem 1

1.



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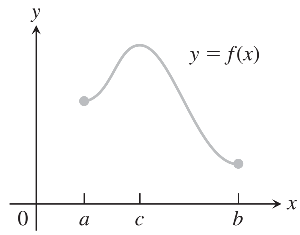

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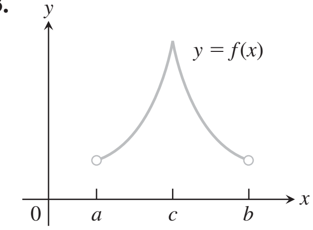

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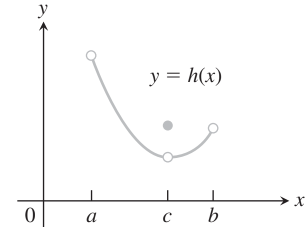

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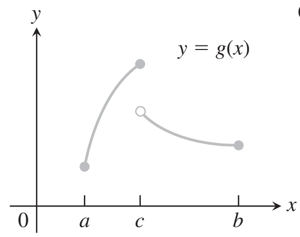
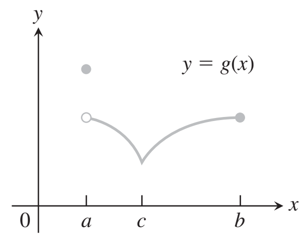
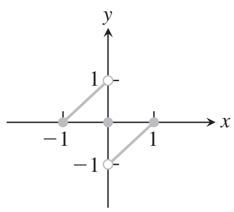


6.



In Exercises 7–10, find the absolute extreme values and where they occur.

7.



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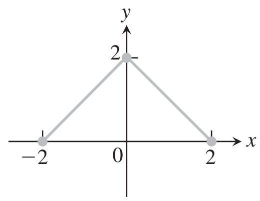
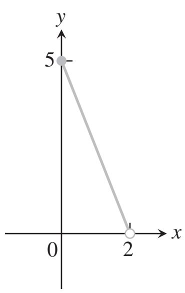
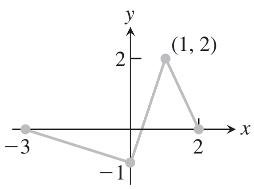


9.



10.



In Exercises 11–14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

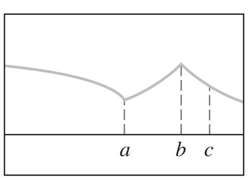
x	$f'(x)$
a	0
b	0
c	-5

13.

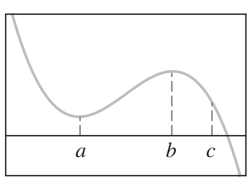
x	$f'(x)$
a	does not exist
b	0
c	-2

14.

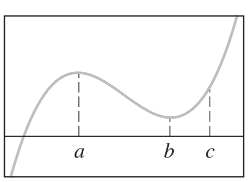
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



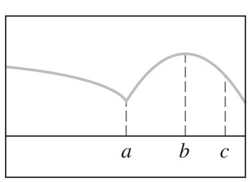
(a)



(b)



(c)



(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

15. $f(x) = |x|, \quad -1 < x < 2$

16. $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

17. $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

18. $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

19. $y = 3 \sin x, \quad 0 < x < 2\pi$

20. $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 < x \le \dfrac{\pi}{2} \end{cases}$

Absolute Extrema on Finite Closed Intervals

In Exercises 21–27, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \dfrac{2}{3}x - 5, \quad -2 \le x \le 3$

22. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

23. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 37–40, find the function's absolute maximum and minimum values and say where they are assumed.

24. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

25. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

26. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

27. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Finding Critical Points

In Exercises 28–35, determine all critical points for each function.

28. $y = x^2 - 6x + 7$

29. $f(x) = 6x^2 - x^3$

30. $f(x) = x(4 - x)^3$

31. $g(x) = (x - 1)^2(x - 3)^2$

32. $y = x^2 + \dfrac{2}{x}$

33. $f(x) = \dfrac{x^2}{x - 2}$

34. $y = x^2 - 32\sqrt{x}$

35. $g(x) = \sqrt{2x - x^2}$

Finding Extreme Values

In Exercises 36–45, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

36. $y = 2x^2 - 8x + 9$

37. $y = x^3 - 2x + 4$

38. $y = x^3 + x^2 - 8x + 5$

39. $y = x^3(x - 5)^2$

40. $y = \sqrt{x^2 - 1}$

41. $y = x - 4\sqrt{x}$

42. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

43. $y = \sqrt{3 + 2x - x^2}$

44. $y = \dfrac{x}{x^2 + 1}$

45. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

Local Extrema and Critical Points

In Exercises 46–53, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

46. $y = x^{2/3}(x + 2)$

47. $y = x^{2/3}(x^2 - 4)$

48. $y = x\sqrt{4 - x^2}$

49. $y = x^2\sqrt{3 - x}$

50. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

51. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

52. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

53. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 54 and 55, give reasons for your answers.

54. Let $f(x) = (x - 2)^{2/3}$.

a. Does $f'(2)$ exist?

b. Show that the only local extreme value of f occurs at $x = 2$.

c. Does the result in part (b) contradict the Extreme Value Theorem?

d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

55. Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

Theory and Examples

56. A minimum with no derivative The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

57. Even functions If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

58. Odd functions If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

59. No critical points or endpoints exist We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

60. The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

a. Find the extreme values of V.

b. Interpret any values found in part (a) in terms of the volume of the box.

61. Cubic functions Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

a. Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

b. How many local extreme values can f have?

62. Maximum height of a vertically moving body The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \qquad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

63. Peak alternating current Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 64–67. Then find the extreme values of the function on the interval and say where they occur.

64. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

65. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

66. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

67. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

4.5 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.31). We now state and prove this result.

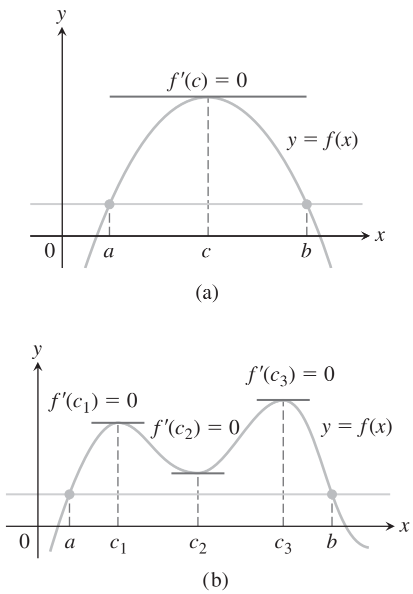


FIGURE 4.31 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

HISTORICAL BIOGRAPHY
Michel Rolle
(1652–1719)

> **THEOREM 3—Rolle's Theorem** Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b). ■

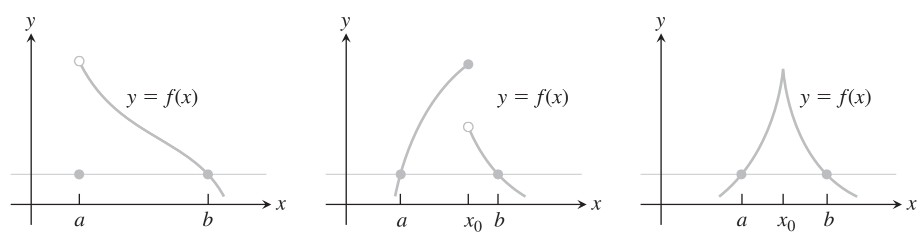
The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.32).

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.32 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. (See Figure 4.33.) Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. However, the derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Therefore, f has no more than one zero. ■

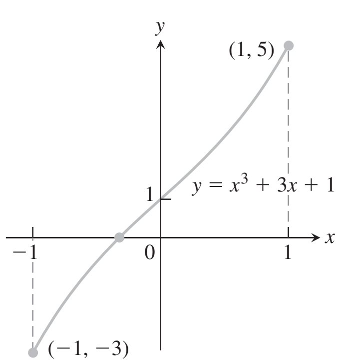


FIGURE 4.33 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

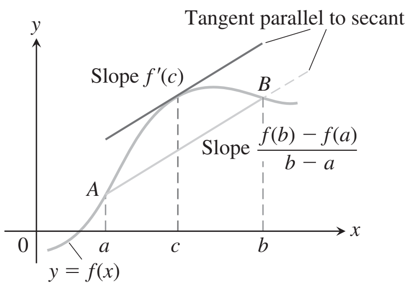


FIGURE 4.34 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to the secant joining A and B.

Our main use of Rolle's Theorem is in proving the Mean Value Theorem.

The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.34). The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant joining A and B.

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Proof We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. (See Figure 4.35.) The secant line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

HISTORICAL BIOGRAPHY
Joseph-Louis Lagrange
(1736–1813)

(point-slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 4.36 shows the graphs of f, g, and h together.

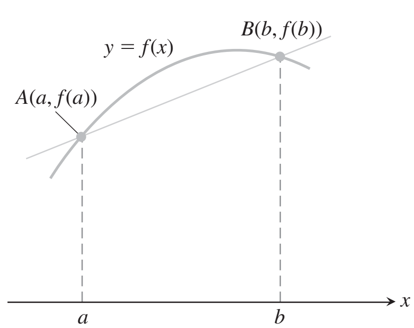


FIGURE 4.35 The graph of f and the secant AB over the interval $[a, b]$.

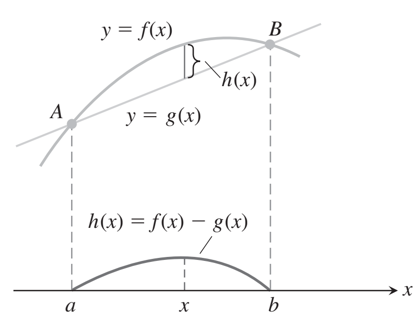


FIGURE 4.36 The secant AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1) in the theorem.

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$
\begin{aligned}
h'(x) &= f'(x) - \frac{f(b) - f(a)}{b - a} && \text{Derivative of Eq. (3) . . .}\\
h'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} && \text{. . . with } x = c\\
0 &= f'(c) - \frac{f(b) - f(a)}{b - a} && h'(c) = 0\\
f'(c) &= \frac{f(b) - f(a)}{b - a}, && \text{Rearranged}
\end{aligned}
$$

which is what we set out to prove. ■

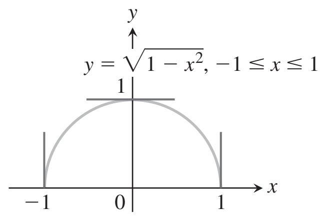


FIGURE 4.37 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. One-sided continuity at a and b is enough (Figure 4.37).

EXAMPLE 2 The function $f(x) = x^2$ (Figure 4.38) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■

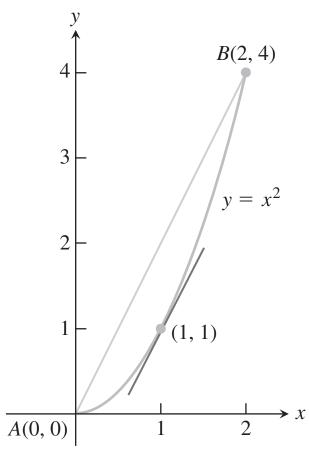


FIGURE 4.38 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.39). ■

Mathematical Consequences

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer that only constant functions have zero derivatives.

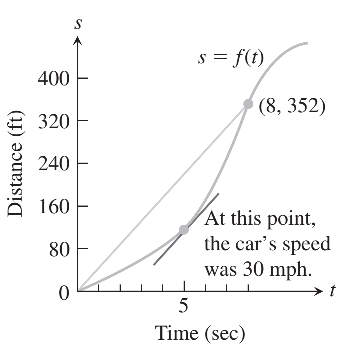


FIGURE 4.39 Distance versus elapsed time for the car in Example 3.

> **COROLLARY 1** If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Proof We want to show that f has a constant value on the interval (a, b). We do so by showing that if x_1 and x_2 are any two points in (a, b) with $x_1 < x_2$, then $f(x_1) = f(x_2)$. Now f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$ and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout (a, b), this equation implies successively that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \qquad f(x_2) - f(x_1) = 0, \qquad \text{and} \qquad f(x_1) = f(x_2). \qquad \blacksquare$$

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

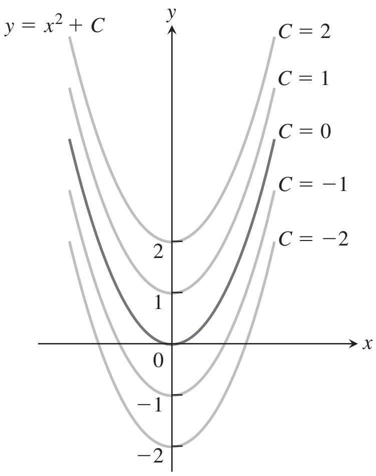


FIGURE 4.40 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

> **COROLLARY 2** If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

Proof At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on (a, b) by Corollary 1. That is, $f(x) - g(x) = C$ on (a, b), so $f(x) = g(x) + C$. $\blacksquare$

Corollaries 1 and 2 are also true if the open interval (a, b) fails to be finite. That is, they remain true if the interval is (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

Corollary 2 plays an important role when we discuss antiderivatives in Section 4.7. It tells us, for instance, that since the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$, any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Figure 4.40).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant C. Since the graph of f passes through the point $(0, 2)$, the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. $\blacksquare$

Finding Velocity and Position from Acceleration

We can use Corollary 2 to find the velocity and position functions of an object moving along a vertical line. Assume the object or body is falling freely from rest with acceleration 9.8 m/sec^2. We assume the position $s(t)$ of the body is measured positive downward from the rest position (so the vertical coordinate line points *downward*, in the direction of the motion, with the rest position at 0).

We know that the velocity $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. What about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \quad \text{and} \quad C = 0.$$

The position function is $s(t) = 4.9t^2$ until the body hits the ground.

The ability to find functions from their rates of change is one of the very powerful tools of calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 5.

Exercises 4.5

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–6.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \dfrac{1}{x}, \quad \left[\dfrac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$
5. $f(x) = x^3 - x^2, \quad [-1, 2]$
6. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 7–12 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

7. $f(x) = x^{2/3}, \quad [-1, 8]$
8. $f(x) = x^{4/5}, \quad [0, 1]$
9. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
10. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
11. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$
12. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$
13. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

14. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

15. **a.** Plot the zeros of each polynomial on a line together with the zeros of its first derivative.
    - **i)** $y = x^2 - 4$
    - **ii)** $y = x^2 + 8x + 15$
    - **iii)** $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$
    - **iv)** $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

    b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

16. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.
17. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
18. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 19–26 have exactly one zero in the given interval.

19. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$
20. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$
21. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$
22. $g(t) = \dfrac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$
23. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$
24. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$
25. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$
26. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

Finding Functions from Derivatives

27. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

28. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

29. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

30. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 31–36, find all possible functions with the given derivative.

31. a. $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$

32. a. $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$

33. a. $y' = -\dfrac{1}{x^2}$ **b.** $y' = 1 - \dfrac{1}{x^2}$ **c.** $y' = 5 + \dfrac{1}{x^2}$

34. a. $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

35. a. $y' = \sin 2t$ **b.** $y' = \cos\dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos\dfrac{t}{2}$

36. a. $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 37–40, find the function with the given derivative whose graph passes through the point P.

37. $f'(x) = 2x - 1, \quad P(0, 0)$

38. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

39. $r'(\theta) = 8 - \csc^2\theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

40. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

Finding Position from Velocity or Acceleration

Exercises 41–44 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time t.

41. $v = 9.8t + 5, \quad s(0) = 10$

42. $v = 32t - 2, \quad s(0.5) = 4$

43. $v = \sin \pi t, \quad s(0) = 0$

44. $v = \dfrac{2}{\pi}\cos\dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 45–48 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time t.

45. $a = 32, \quad v(0) = 20, \quad s(0) = 5$

46. $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

47. $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

48. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

Theory and Examples

49. The geometric mean of *a* and *b* The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

50. The arithmetic mean of *a* and *b* The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **51.** Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

52. Rolle's Theorem

a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2.

b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

53. Unique solution Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

54. Parallel tangents Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

55. Suppose that $f'(x) \le 1$ for $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

56. Suppose that $0 < f'(x) < 1/2$ for all x-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

57. Show that $|\cos x - 1| \le |x|$ for all x-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

58. Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \le |b - a|$ is true.

59. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

60. If $|f(w) - f(x)| \le |w - x|$ for all values w and x and f is a differentiable function, show that $-1 \le f'(x) \le 1$ for all x-values.

61. Assume that f is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

62. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \le \frac{f(b) - f(a)}{b - a} \le \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

T **63.** Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4\cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

T **64.** Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

65. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

a. Show that $f(x) \ge 1$ for all x.

b. Must $f'(1) = 0$? Explain.

66. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

4.6 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ■

Corollary 3 tells us that $f(x) = \sqrt{x}$ is increasing on the interval $[0, b]$ for any $b > 0$ because $f'(x) = 1/\sqrt{x}$ is positive on $(0, b)$. The derivative does not exist at $x = 0$, but Corollary 3 still applies. The corollary is valid for infinite as well as finite intervals, so $f(x) = \sqrt{x}$ is increasing on $[0, \infty)$.

To find the intervals where a function f is increasing or decreasing, we first find all of the critical points of f. If $a < b$ are two critical points for f, and if the derivative f' is continuous but never zero on the interval (a, b), then by the Intermediate Value Theorem applied to f', the derivative must be everywhere positive on (a, b), or everywhere negative there. One way we can determine the sign of f' on (a, b) is simply by evaluating the derivative at a single point c in (a, b). If $f'(c) > 0$, then $f'(x) > 0$ for all x in (a, b) so f is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then f is decreasing on $[a, b]$. The next example illustrates how we use this procedure.

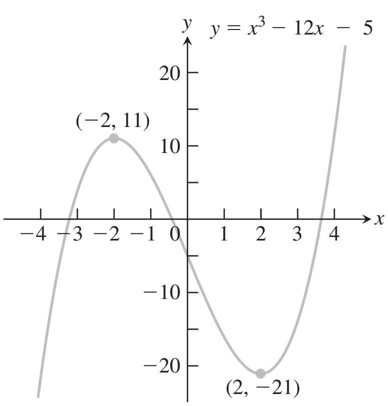


FIGURE 4.41 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.41.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' **evaluated**	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

■

We used "strict" less-than inequalities to identify the intervals in the summary table for Example 1, since open intervals were specified. Corollary 3 says that we could use $\leq$ inequalities as well. That is, the function f in the example is increasing on $-\infty < x \leq -2$, decreasing on $-2 \leq x \leq 2$, and increasing on $2 \leq x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

HISTORICAL BIOGRAPHY
Edmund Halley
(1656–1742)

First Derivative Test for Local Extrema

In Figure 4.42, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

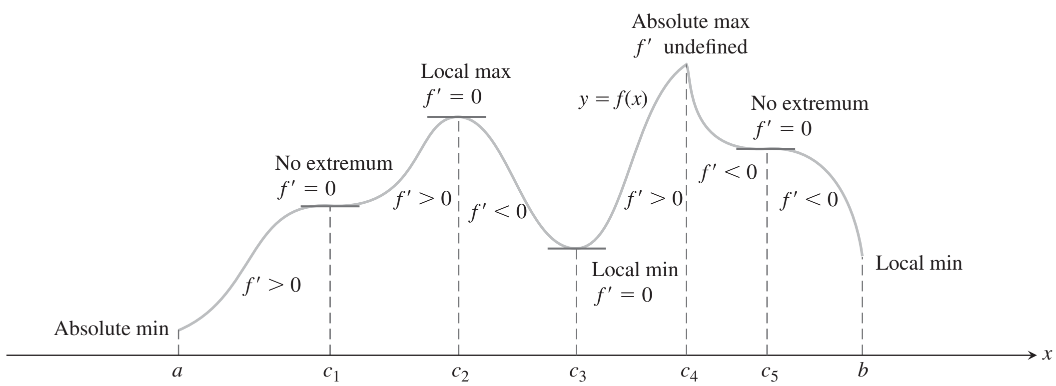


FIGURE 4.42 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

> **First Derivative Test for Local Extrema**
>
> Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,
>
> **1.** if f' changes from negative to positive at c, then f has a local minimum at c;
> **2.** if f' changes from positive to negative at c, then f has a local maximum at c;
> **3.** if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f'.

Proof of the First Derivative Test Part (1). Since the sign of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c.

Parts (2) and (3) are proved similarly. ■

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}} \end{aligned}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into open intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points, as summarized in the following table.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

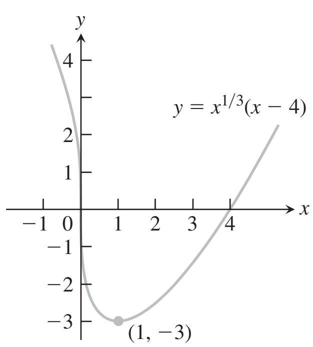


FIGURE 4.43 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Corollary 3 to the Mean Value Theorem implies that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Figure 4.43 shows this value in relation to the function's graph.

Note that $\lim_{x \to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin. ■

EXAMPLE 3 Within the interval $0 \leq x \leq 2\pi$, find the critical points of

$$f(x) = \sin^2 x - \sin x - 1.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous over $[0, 2\pi]$ and differentiable over $(0, 2\pi)$, so the critical points occur at the zeros of f' in $(0, 2\pi)$. We find

$$f'(x) = 2 \sin x \cos x - \cos x = (2 \sin x - 1)(\cos x).$$

The first derivative is zero if and only if $\sin x = \frac{1}{2}$ or $\cos x = 0$. So the critical points of f in $(0, 2\pi)$ are $x = \pi/6$, $x = 5\pi/6$, $x = \pi/2$, and $x = 3\pi/2$. They partition $[0, 2\pi]$ into open intervals as follows.

Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing

$0 \quad \frac{\pi}{6} \quad \frac{\pi}{2} \quad \frac{5\pi}{6} \quad \frac{3\pi}{2} \quad 2\pi \quad x$

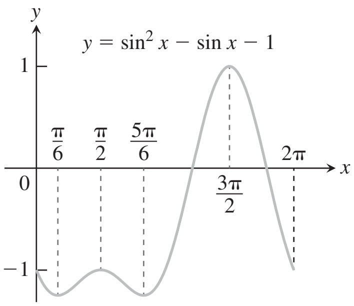


FIGURE 4.44 The graph of the function in Example 3.

The table displays the open intervals on which f is increasing and decreasing. We can deduce from the table that there is a local minimum value of $f(\pi/6) = \frac{1}{4} - \frac{1}{2} - 1 = -\frac{5}{4}$, a local maximum value of $f(\pi/2) = 1 - 1 - 1 = -1$, another local minimum value of $f(5\pi/6) = -\frac{5}{4}$, and another local maximum value of $f(3\pi/2) = 1 - (-1) - 1 = 1$. The endpoint values are $f(0) = f(2\pi) = -1$. The absolute minimum in $[0, 2\pi]$ is $-\frac{5}{4}$ occurring at $x = \pi/6$ and $x = 5\pi/6$; the absolute maximum is 1 occurring at $x = 3\pi/2$. Figure 4.44 shows the graph. ■

Exercises 4.6

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$ **2.** $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$ **4.** $f'(x) = (x - 1)^2(x + 2)^2$

5. $f'(x) = (x - 1)(x + 2)(x - 3)$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

8. $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$

9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$ **10.** $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

11. $f'(x) = x^{-1/3}(x + 2)$ **12.** $f'(x) = x^{-1/2}(x - 3)$

13. $f'(x) = (\sin x - 1)(2\cos x + 1), 0 \le x \le 2\pi$

14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x), 0 \le x \le 2\pi$

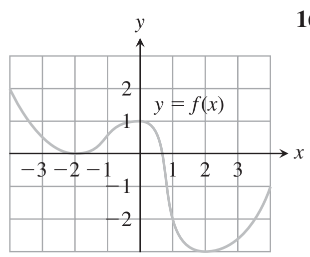
Identifying Extrema

In Exercises 15–40:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

15.



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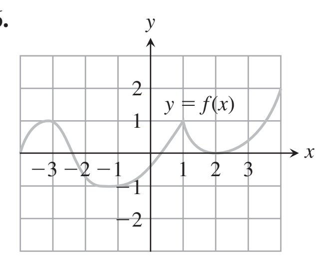

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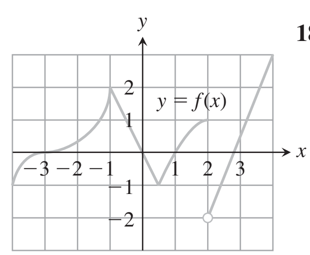

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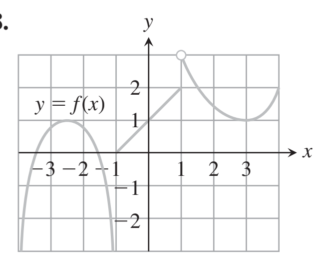


19. $g(t) = -t^2 - 3t + 3$ **20.** $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$ **22.** $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$ **24.** $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$ **26.** $h(r) = (r + 7)^3$

27. $f(x) = x^4 - 8x^2 + 16$ **28.** $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \dfrac{3}{2}t^4 - t^6$ **30.** $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x - 1}$ **32.** $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$ **34.** $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$ **36.** $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$ **38.** $g(x) = x^{2/3}(x + 5)$

39. $h(x) = x^{1/3}(x^2 - 4)$ **40.** $k(x) = x^{2/3}(x^2 - 4)$

Theory and Examples

Show that the functions in Exercises 41 and 42 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

41. $h(\theta) = 3 \cos \dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad \text{at } \theta = 0 \text{ and } \theta = 2\pi$

42. $h(\theta) = 5 \sin \dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad \text{at } \theta = 0 \text{ and } \theta = \pi$

43. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c. $f'(x) > 0$ for $x \ne 1$;

d. $f'(x) < 0$ for $x \ne 1$.

44. Sketch the graph of a differentiable function $y = f(x)$ that has

a. a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

b. a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

c. local maxima at $(1, 1)$ and $(3, 3)$;

d. local minima at $(1, 1)$ and $(3, 3)$.

45. Sketch the graph of a continuous function $y = g(x)$ such that

a. $g(2) = 2, 0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b. $g(2) = 2, g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

46. Sketch the graph of a continuous function $y = h(x)$ such that

a. $h(0) = 0, -2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b. $h(0) = 0, -2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

47. Discuss the extreme-value behavior of the function $f(x) = x \sin(1/x)$, $x \ne 0$. How many critical points does this function have? Where are they located on the x-axis? Does f have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

48. Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

49. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

50. Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

4.7 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior studied in Sections 1.1 and 2.6, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

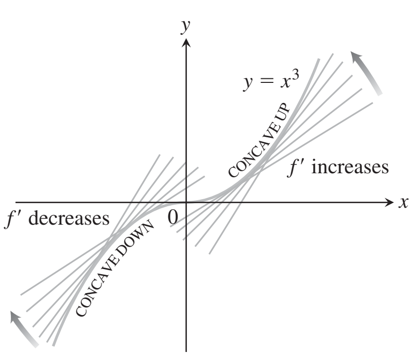


FIGURE 4.45 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

Concavity

As you can see in Figure 4.45, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

> **The Second Derivative Test for Concavity**
> Let $y = f(x)$ be twice-differentiable on an interval I.
> 1. If $f'' > 0$ on I, the graph of f over I is concave up.
> 2. If $f'' < 0$ on I, the graph of f over I is concave down.

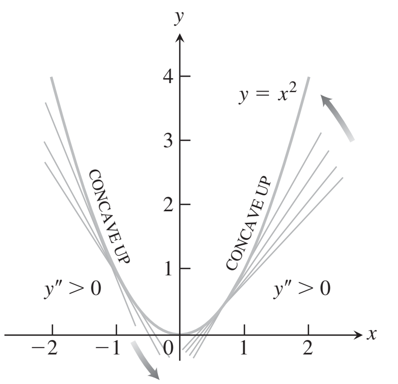


FIGURE 4.46 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.45) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.46) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

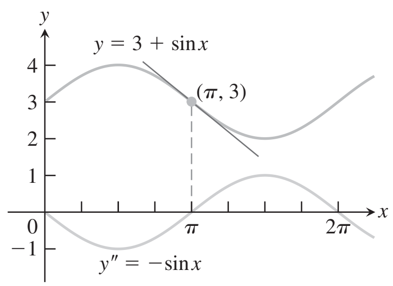


FIGURE 4.47 Using the sign of y'' to determine the concavity of y (Example 2).

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.47). ■

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a *point of inflection* of the curve. Notice from Figure 4.47 that the graph crosses its tangent line at this point and that the second derivative $y'' = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

> **DEFINITION** A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection $(c, f(c))$, then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever f'' is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced argument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or the graph has a vertical tangent at the point. At a vertical tangent neither the first nor second derivative exists. In summary, we conclude the following result.

> At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

The next example illustrates a function having a point of inflection where the first derivative exists, but the second derivative fails to exist.

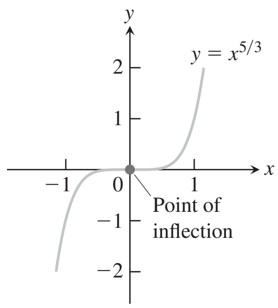


FIGURE 4.48 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.48. ■

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

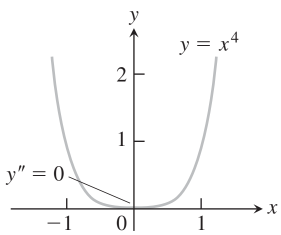


FIGURE 4.49 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

EXAMPLE 4 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.49). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

As our final illustration, we show a situation in which a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

EXAMPLE 5 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.50. ■

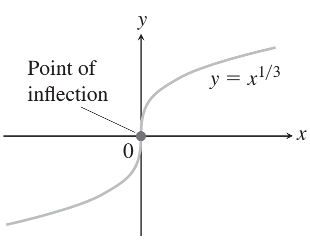


FIGURE 4.50 A point of inflection where y' and y'' fail to exist (Example 5).

Caution Example 4 in Section 4.1 (Figure 4.30) shows that the function $f(x) = x^{2/3}$ does not have a second derivative at $x = 0$ and does not have a point of inflection there (there is no change in concavity at $x = 0$). Combined with the behavior of the function in Example 5 above, we see that when the second derivative does not exist at $x = c$, an inflection point may or may not occur there. So we need to be careful about interpreting functional behavior whenever first or second derivatives fail to exist at a point. At such points the graph can have vertical tangents, corners, cusps, or various discontinuities.

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

EXAMPLE 6 A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

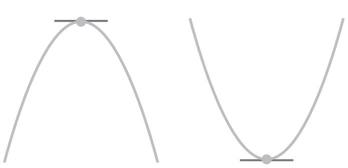
The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $[0, 7/3)$. The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration, so it is speeding up. ■

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$ $\Rightarrow$ local max $f' = 0, f'' > 0$ $\Rightarrow$ local min

Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph—that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$\boldsymbol{x < 0}$	$\boldsymbol{0 < x < 2}$	$\boldsymbol{2 < x < 3}$	$\boldsymbol{3 < x}$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

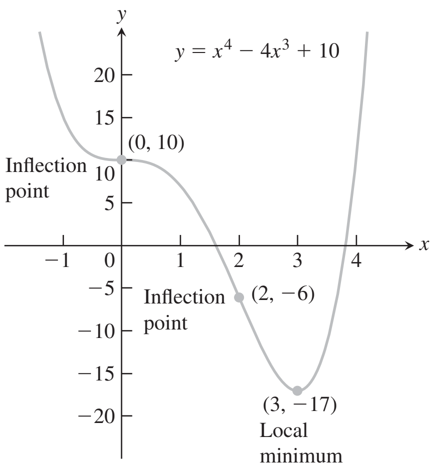
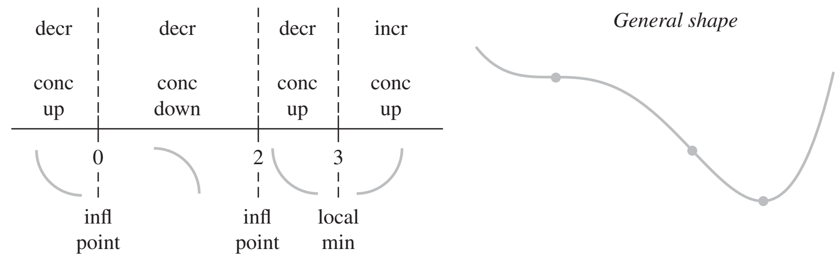


FIGURE 4.51 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

The general shape of the curve is shown in the accompanying figure.



(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.51 shows the graph of f. ■

The steps in Example 7 give a procedure for graphing the key features of a function. Asymptotes were defined and discussed in Section 2.6. We can find them for rational functions, and the methods in the next section give tools to help find them for more general functions.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2} \qquad \text{x-intercept at $x=-1$, y-intercept ($y=1$) at $x=0$}$$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2} \qquad \text{Critical points: } x=-1,\ x=1$$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3} \qquad \text{After some algebra}$$

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$\begin{aligned} f(x) &= \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} && \text{Expanding numerator} \\ &= \frac{1 + (2/x) + (1/x^2)}{(1/x^2)+1}. && \text{Dividing by } x^2 \end{aligned}$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).
7. The graph of f is sketched in Figure 4.52. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

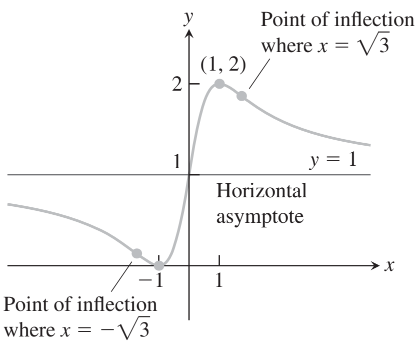


FIGURE 4.52 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$\begin{aligned} f(x) &= \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x} && \text{Function simplified for differentiation} \\ f'(x) &= \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} && \text{Combine fractions to solve easily } f'(x) = 0. \\ f''(x) &= \frac{4}{x^3} && \text{Exists throughout the entire domain of } f \end{aligned}$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

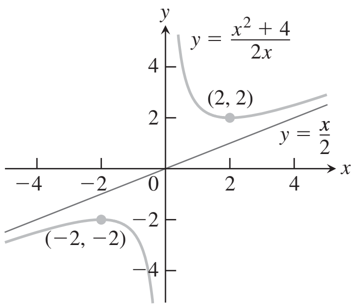


FIGURE 4.53 The graph of $y = \dfrac{x^2+4}{2x}$ (Example 9).

4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \quad \text{and} \quad \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the y-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.
7. The graph of f is sketched in Figure 4.53. ■

EXAMPLE 10 Sketch the graph of $f(x) = \cos x - \dfrac{\sqrt{2}}{2}x$ over $0 \le x \le 2\pi$.

Solution The derivatives of f are

$$f'(x) = -\sin x - \frac{\sqrt{2}}{2} \qquad \text{and} \qquad f''(x) = -\cos x.$$

Both derivatives exist everywhere over the interval $(0, 2\pi)$. Within that open interval, the first derivative is zero when $\sin x = -\sqrt{2}/2$, so the critical points are $x = 5\pi/4$ and $x = 7\pi/4$. Since $f''(5\pi/4) = -\cos(5\pi/4) = \sqrt{2}/2 > 0$, the function has a local minimum value of $f(5\pi/4) \approx -3.48$ (evaluated with a calculator) by The Second Derivative Test. Also, $f''(7\pi/4) = -\cos(7\pi/4) = -\sqrt{2}/2 < 0$, so the function has a local maximum value of $f(7\pi/4) \approx -3.18$.

Examining the second derivative, we find that $f'' = 0$ when $x = \pi/2$ or $x = 3\pi/2$. We conclude that $(\pi/2, f(\pi/2)) \approx (\pi/2, -1.11)$ and $(3\pi/2, f(3\pi/2)) \approx (3\pi/2, -3.33)$ are points of inflection.

Finally, we evaluate f at the endpoints of the interval to find $f(0) = 1$ and $f(2\pi) \approx -3.44$. Therefore, the values $f(0) = 1$ and $f(5\pi/4) \approx -3.48$ are the absolute maximum and absolute minimum values of f over the closed interval $[0, 2\pi]$. The graph of f is sketched in Figure 4.54. ■

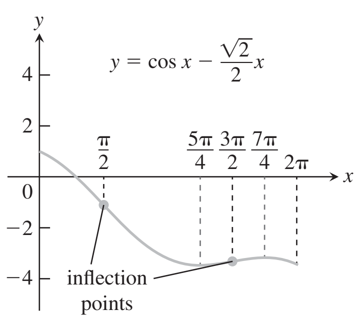
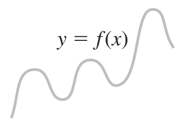
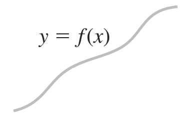
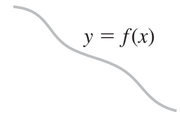
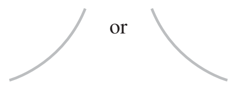
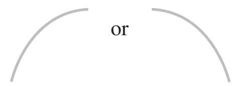
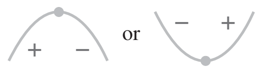
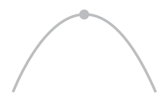
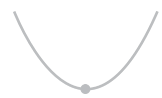


FIGURE 4.54 The graph of the function in Example 10.

Graphical Behavior of Functions from Derivatives

As we saw in Examples 7–10, we can learn much about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are located. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. Information we cannot get from the derivative is how to place the graph in the xy-plane. But, as we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point. Information about the asymptotes is found using limits (Section 2.6). The following figure summarizes how the first derivative and second derivative affect the shape of a graph.

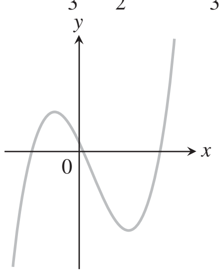
 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Exercises 4.7

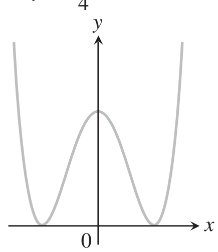
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

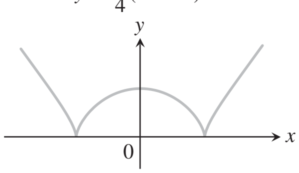
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



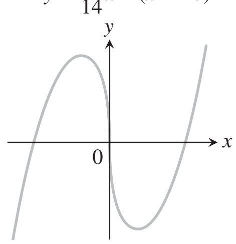
2. $y = \frac{x^4}{4} - 2x^2 + 4$



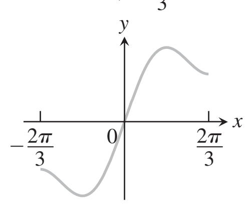
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



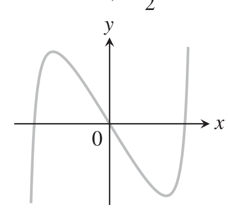
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



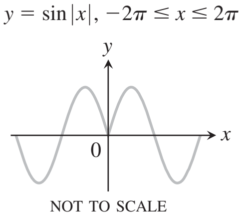
5. $y = x + \sin 2x,\ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



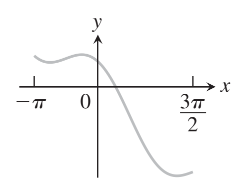
6. $y = \tan x - 4x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



7. $y = \sin |x|,\ -2\pi \le x \le 2\pi$



8. $y = 2\cos x - \sqrt{2}x,\ -\pi \le x \le \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–48, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\dfrac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \dfrac{4}{3}x - \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$ **30.** $y = x^{2/5}$

31. $y = \dfrac{x}{\sqrt{x^2 + 1}}$ **32.** $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$ **34.** $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\dfrac{5}{2} - x\right)$ **36.** $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$ **38.** $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$ **40.** $y = x^2 + \dfrac{2}{x}$

41. $y = \dfrac{x^2 - 3}{x - 2}$ **42.** $y = \sqrt[3]{x^3 + 1}$

43. $y = \dfrac{8x}{x^2 + 4}$ **44.** $y = \dfrac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$ **46.** $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

Sketching the General Shape, Knowing y'

Each of Exercises 49–70 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 209 to sketch the general shape of the graph of f.

49. $y' = 2 + x - x^2$ **50.** $y' = x^2 - x - 6$

51. $y' = x(x - 3)^2$ **52.** $y' = x^2(2 - x)$

53. $y' = x(x^2 - 12)$ **54.** $y' = (x - 1)^2(2x + 3)$

55. $y' = (8x - 5x^2)(4 - x)^2$ **56.** $y' = (x^2 - 2x)(x - 5)^2$

57. $y' = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

58. $y' = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

59. $y' = \cot\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$ **60.** $y' = \csc^2\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

61. $y' = \tan^2\theta - 1, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

62. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

63. $y' = \cos t, \quad 0 \le t \le 2\pi$

64. $y' = \sin t, \quad 0 \le t \le 2\pi$

65. $y' = (x + 1)^{-2/3}$ **66.** $y' = (x - 2)^{-1/3}$

67. $y' = x^{-2/3}(x - 1)$ **68.** $y' = x^{-4/5}(x + 1)$

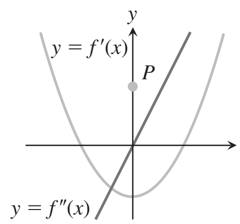
69. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

70. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

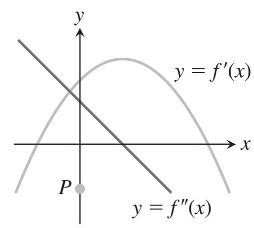
Sketching y from Graphs of y' and y''

Each of Exercises 71–74 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

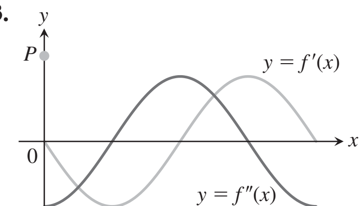
71.



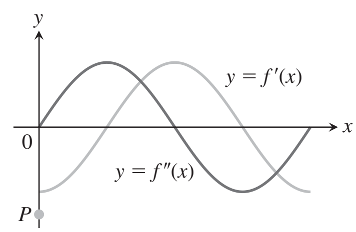
72.



73.



74.



Graphing Rational Functions

Graph the rational functions in Exercises 75–92 using all the steps in the graphing procedure on page 209.

75. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$ **76.** $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

77. $y = \dfrac{x^4 + 1}{x^2}$ **78.** $y = \dfrac{x^2 - 4}{2x}$

79. $y = \dfrac{1}{x^2 - 1}$ **80.** $y = \dfrac{x^2}{x^2 - 1}$

81. $y = -\dfrac{x^2 - 2}{x^2 - 1}$ **82.** $y = \dfrac{x^2 - 4}{x^2 - 2}$

83. $y = \dfrac{x^2}{x + 1}$ **84.** $y = -\dfrac{x^2 - 4}{x + 1}$

85. $y = \dfrac{x^2 - x + 1}{x - 1}$ **86.** $y = -\dfrac{x^2 - x + 1}{x - 1}$

87. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$ **88.** $y = \dfrac{x^3 + x - 2}{x - x^2}$

89. $y = \dfrac{x}{x^2 - 1}$

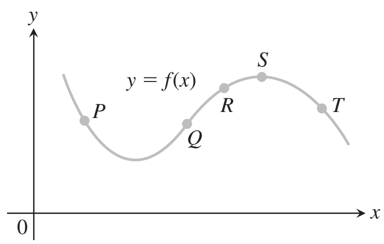
90. $y = \dfrac{x - 1}{x^2(x - 2)}$

91. $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

92. $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

Theory and Examples

93. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



94. Sketch a smooth connected curve $y = f(x)$ with

$$
\begin{array}{ll}
f(-2) = 8, & f'(2) = f'(-2) = 0, \\
f(0) = 4, & f'(x) < 0 \quad \text{for} \quad |x| < 2, \\
f(2) = 0, & f''(x) < 0 \quad \text{for} \quad x < 0, \\
f'(x) > 0 \quad \text{for} \quad |x| > 2, & f''(x) > 0 \quad \text{for} \quad x > 0.
\end{array}
$$

95. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0,\quad y'' > 0$
2	1	$y' = 0,\quad y'' > 0$
$2 < x < 4$		$y' > 0,\quad y'' > 0$
4	4	$y' > 0,\quad y'' = 0$
$4 < x < 6$		$y' > 0,\quad y'' < 0$
6	7	$y' = 0,\quad y'' < 0$
$x > 6$		$y' < 0,\quad y'' < 0$

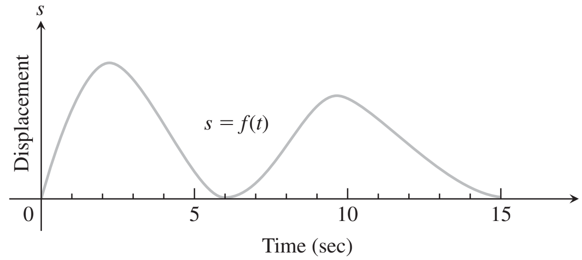
96. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$$
y': \quad \begin{array}{ccccccc} + & & - & & + & & - \\ \hline & -2 & & 0 & & 2 & \end{array}
$$

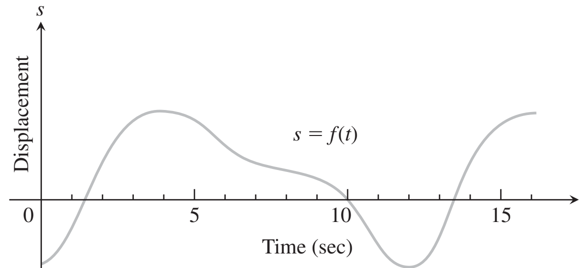
$$
y'': \quad \begin{array}{ccccc} - & & + & & - \\ \hline & -1 & & 1 & \end{array}
$$

Motion Along a Line The graphs in Exercises 97 and 98 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

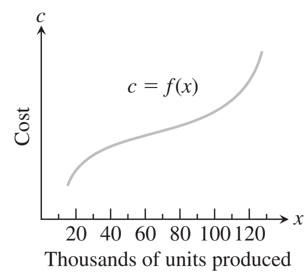
97.



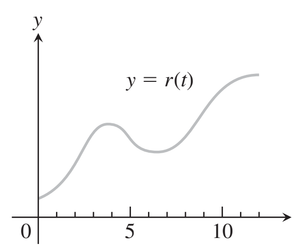
98.



99. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



100. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



101. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

102. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

103. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

104. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

105. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

106. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

107. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

108. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

109. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x-values does the graph of f have an inflection point?

110. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x-values does the graph of f have an inflection point?

111. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

112. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

113. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

114. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.8 Applied Optimization

What are the dimensions of a rectangle with fixed perimeter having *maximum area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? How many items should be produced for the *most profitable* production run? Each of these questions asks for the best, or optimal, value of a given function. In this section we use derivatives to solve a variety of optimization problems in mathematics, physics, economics, and business.

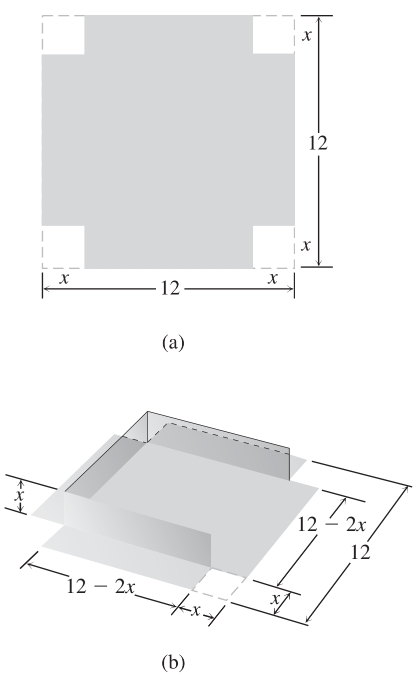


FIGURE 4.55 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.55). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

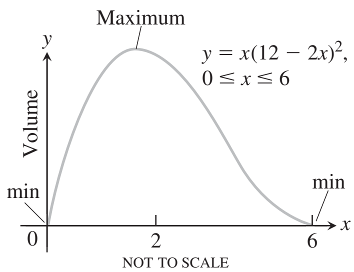


FIGURE 4.56 The volume of the box in Figure 4.34 graphed as a function of x.

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 4.56) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

$$\begin{aligned} &\text{Critical point value:} \quad && V(2) = 128 \\ &\text{Endpoint values:} \quad && V(0) = 0, \qquad V(6) = 0. \end{aligned}$$

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side. ■

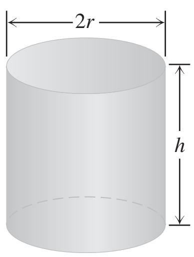


FIGURE 4.57 This one-liter can uses the least material when $h = 2r$ (Example 2).

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.57). What dimensions will use the least material?

Solution *Volume of can:* If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad \text{1 liter} = 1000 \text{ cm}^3$$

Surface area of can: $A = \underbrace{2\pi r^2}_{\substack{\text{circular}\\ \text{ends}}} + \underbrace{2\pi r h}_{\substack{\text{cylindrical}\\ \text{wall}}}$

How can we interpret the phrase "least material"? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$ cm^3.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.58 suggests that such a value exists.

Notice from the graph that for small r (a tall, thin cylindrical container), the term $2000/r$ dominates (see Section 2.6) and A is large. For large r (a short, wide cylindrical container), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

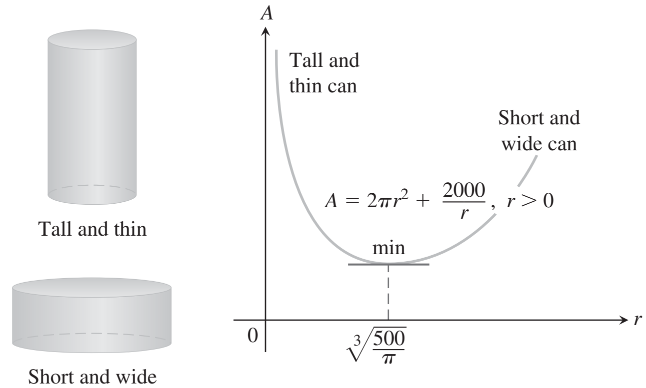


FIGURE 4.58 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

$$0 = 4\pi r - \frac{2000}{r^2} \qquad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \qquad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \qquad \text{Solve for } r.$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ■

Examples from Mathematics and Physics

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $\left(x, \sqrt{4 - x^2}\right)$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.59). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4 - x^2}, \qquad \text{Area: } 2x\sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain $[0, 2]$.

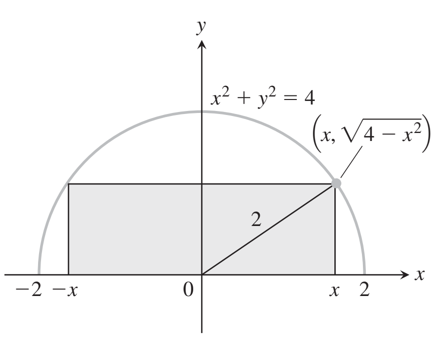


FIGURE 4.59 The rectangle inscribed in the semicircle in Example 3.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned} \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} &= 0 \\ -2x^2 + 2(4 - x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\begin{aligned} &\text{Critical point value:} \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4 - 2} = 4 \\ &\text{Endpoint values:} \qquad A(0) = 0, \quad A(2) = 0. \end{aligned}$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ■

HISTORICAL BIOGRAPHY
Willebrord Snell van Royen
(1580–1626)

EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.

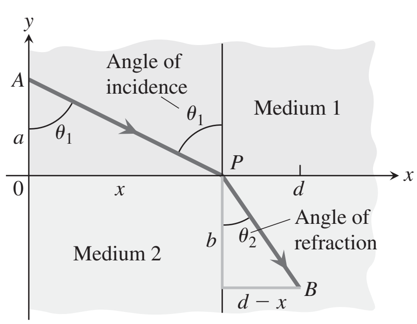


FIGURE 4.60 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

Solution Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis (Figure 4.60).

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. Distance traveled equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

From Figure 4.60, the time required for light to travel from A to P is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B, the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

This equation expresses t as a differentiable function of x whose domain is $[0, d]$. We want to find the absolute minimum value of t on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

and observe that it is continuous. In terms of the angles θ_1 and θ_2 in Figure 4.60,

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}.$$

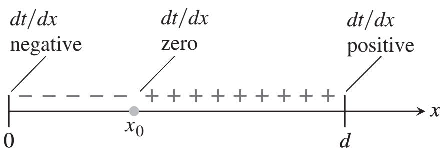


FIGURE 4.61 The sign pattern of dt/dx in Example 4.

The function t has a negative derivative at $x = 0$ and a positive derivative at $x = d$. Since dt/dx is continuous over the interval $[0, d]$, by the Intermediate Value Theorem for continuous functions (Section 2.5), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.61). There is only one such point because dt/dx is an increasing function of x (Exercise 62). At this unique point we then have

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and is an important principle in the theory of optics. It describes the path the ray of light follows. ■

Examples from Economics

Suppose that

$r(x) =$ the revenue from selling x items

$c(x) =$ the cost of producing the x items

$p(x) = r(x) - c(x) =$ the profit from producing and selling x items.

Although x is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms **marginal revenue**, **marginal cost**, and **marginal profit** to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let's consider the relationship of the profit p to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for x in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an endpoint of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

> At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.62).

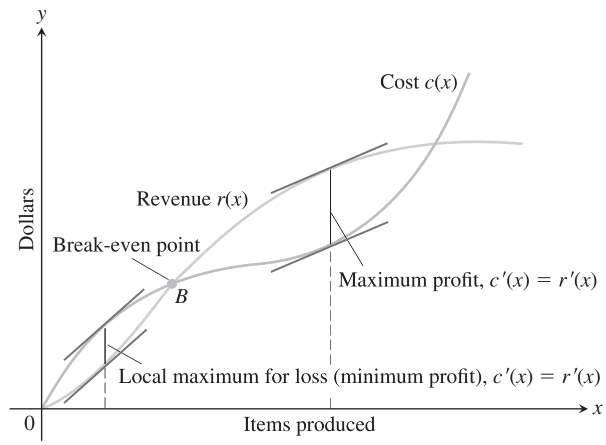


FIGURE 4.62 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

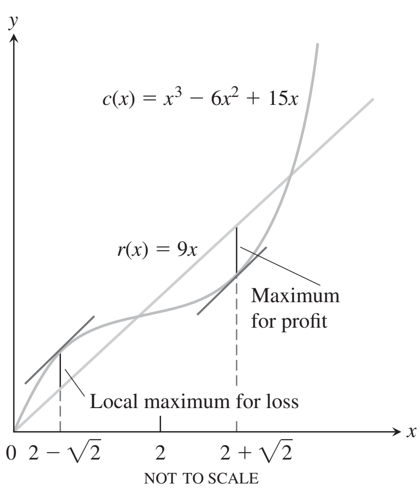


FIGURE 4.63 The cost and revenue curves for Example 5.

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$\begin{aligned} 3x^2 - 12x + 15 &= 9 \qquad \text{Set } c'(x) = r'(x). \\ 3x^2 - 12x + 6 &= 0 \end{aligned}$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \qquad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.63. ■

EXAMPLE 6 A cabinetmaker uses mahogany wood to produce 5 desks each day. Each delivery of one container of wood is \$5000, whereas the storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 desk. How much material should be ordered each time, and how often should the material be delivered, to minimize her average daily cost in the production cycle between deliveries?

Solution If she asks for a delivery every x days, then she must order $5x$ units to have enough material for that delivery cycle. The *average* amount in storage is approximately

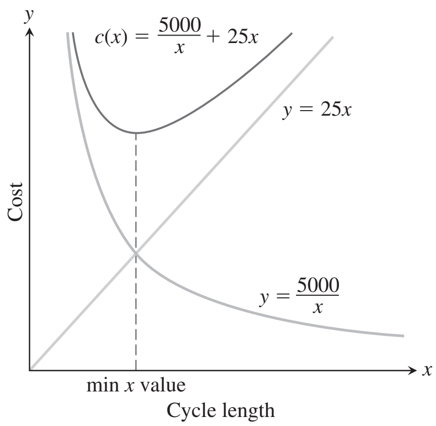


FIGURE 4.64 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

one-half of the delivery amount, or $5x/2$. Thus, the cost of delivery and storage for each cycle is approximately

$$\text{Cost per cycle} = \text{delivery costs} + \text{storage costs}$$

$$\text{Cost per cycle} = \underbrace{5000}_{\substack{\text{delivery}\\ \text{cost}}} + \underbrace{\left(\frac{5x}{2}\right)}_{\substack{\text{average}\\ \text{amount stored}}} \cdot \underbrace{x}_{\substack{\text{number of}\\ \text{days stored}}} \cdot \underbrace{10}_{\substack{\text{storage cost}\\ \text{per day}}}$$

We compute the *average daily cost* $c(x)$ by dividing the cost per cycle by the number of days x in the cycle (see Figure 4.64).

$$c(x) = \frac{5000}{x} + 25x, \qquad x > 0.$$

As $x \to 0$ and as $x \to \infty$, the average daily cost becomes large. So we expect a minimum to exist, but where? Our goal is to determine the number of days x between deliveries that provides the absolute minimum cost.

We find the critical points by determining where the derivative is equal to zero:

$$c'(x) = -\frac{500}{x^2} + 25 = 0$$

$$x = \pm\sqrt{200} \approx \pm 14.14.$$

Of the two critical points, only $\sqrt{200}$ lies in the domain of $c(x)$. The critical point value of the average daily cost is

$$c\left(\sqrt{200}\right) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that $c(x)$ is defined over the open interval $(0, \infty)$ with $c''(x) = 10000/x^3 > 0$. Thus, an absolute minimum exists at $x = \sqrt{200} \approx 14.14$ days.

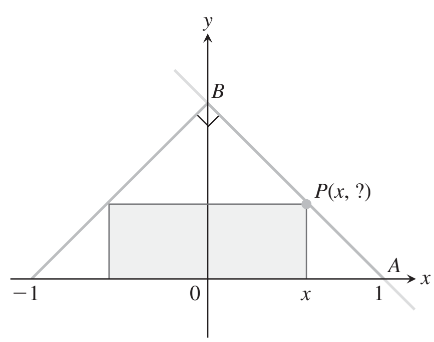
The cabinetmaker should schedule a delivery of $5(14) = 70$ units of the mahogany wood every 14 days. ■

Exercises 4.8

Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?
2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a. Express the y-coordinate of P in terms of x. (*Hint:* Write an equation for the line AB.)
   b. Express the area of the rectangle in terms of x.
   c. What is the largest area the rectangle can have, and what are its dimensions?



4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

5. You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

7. The best fencing plan A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

8. The shortest fence A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

9. Designing a tank Your iron works has contracted to design and build a 500 ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

a. What dimensions do you tell the shop to use?

b. Briefly describe how you took weight into account.

10. Catching rainwater A 1125 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.

a. If the total cost is

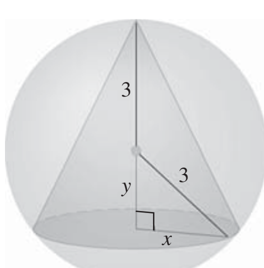
$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

b. Give a possible scenario for the cost function in part (a).

11. Designing a poster You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



13. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)ab\sin\theta$.)

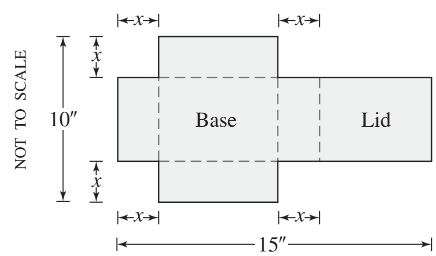
14. Designing a can What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

15. Designing a can You are designing a 1000 cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

T **16. Designing a box with a lid** A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



a. Write a formula $V(x)$ for the volume of the box.

b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

d. Confirm your result in part (c) analytically.

T **17. Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

a. Write a formula $V(x)$ for the volume of the box.

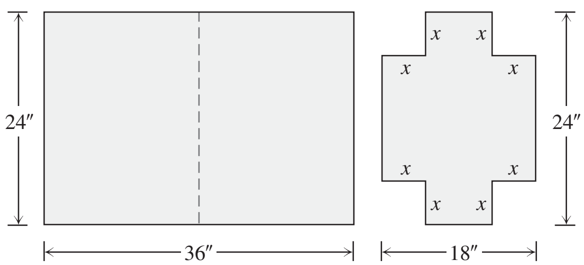
b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

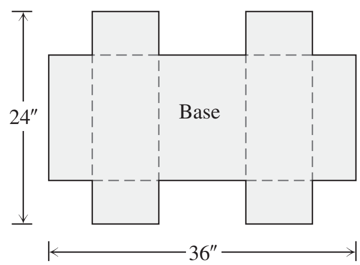
d. Confirm your result in part (c) analytically.

e. Find a value of x that yields a volume of 1120 in^3.

f. Write a paragraph describing the issues that arise in part (b).



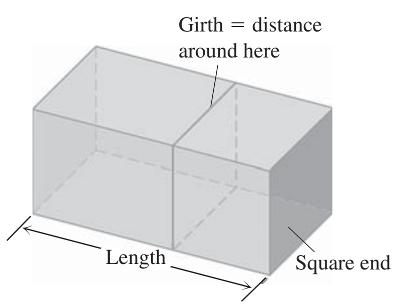
The sheet is then unfolded.



18. A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

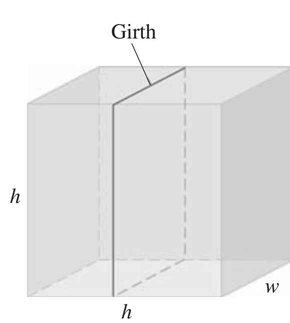
19. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

20. a. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?

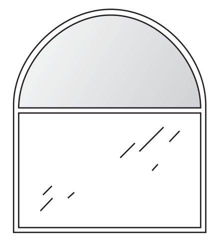


21. (*Continuation of Exercise 20.*)

a. Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?

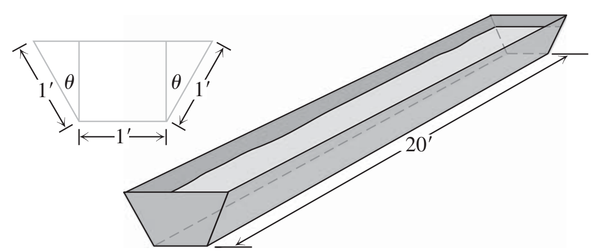


22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



23. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

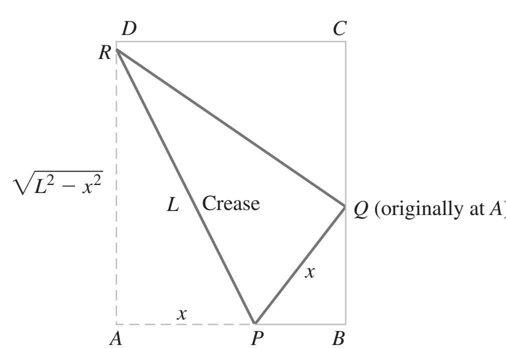


25. Paper folding A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

a. Show that $L^2 = 2x^3/(2x - 8.5)$.

b. What value of x minimizes L^2?

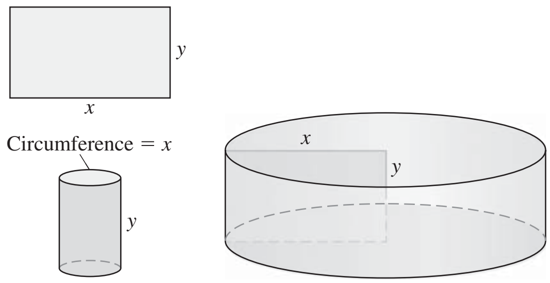
c. What is the minimum value of L?



26. Constructing cylinders Compare the answers to the following two construction problems.

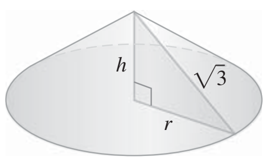
a. A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

b. The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



(a) (b)

27. Constructing cones A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



28. Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

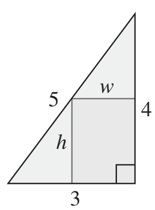
29. Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

30. Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.

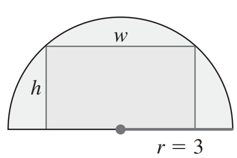
31. A wire b m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?

32. Answer Exercise 31 if one piece is bent into a square and the other into a circle.

33. Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.



34. Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)



35. What value of a makes $f(x) = x^2 + (a/x)$ have

a. a local minimum at $x = 2$?

b. a point of inflection at $x = 1$?

36. What values of a and b make $f(x) = x^3 + ax^2 + bx$ have

a. a local maximum at $x = -1$ and a local minimum at $x = 3$?

b. a local minimum at $x = 4$ and a point of inflection at $x = 1$?

Physical Applications

37. Vertical motion The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in feet and t in seconds. Find

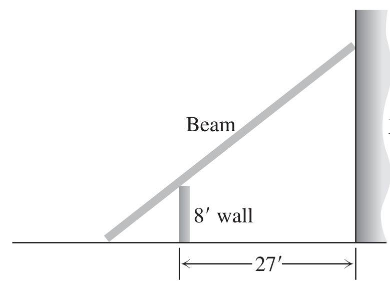
a. the object's velocity when $t = 0$;

b. its maximum height and when it occurs;

c. its velocity when $s = 0$.

38. Quickest route Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

39. Shortest beam The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



40. Motion on a line The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

a. At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

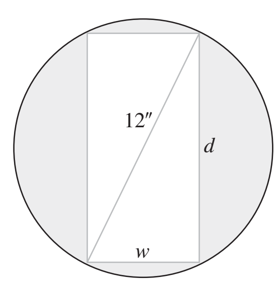
b. What is the farthest apart that the particles ever get?

c. When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

41. The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?

T **42. Strength of a beam** The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

a. Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.



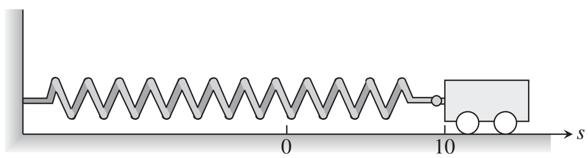
T **43. Stiffness of a beam** The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

a. Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

44. Frictionless cart A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

a. What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

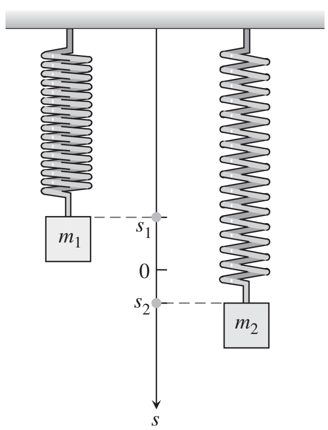
b. Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



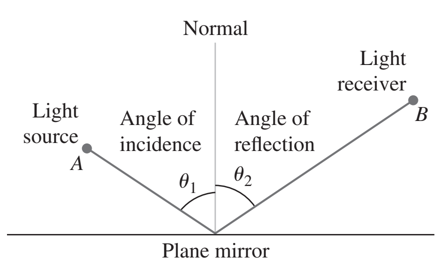
45. Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

a. At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

b. When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)



46. Fermat's principle in optics Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



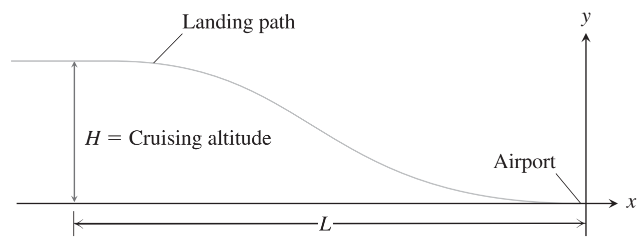
47. Airplane landing path An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

a. What is dy/dx at $x = 0$?

b. What is dy/dx at $x = -L$?

c. Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



Business and Economics

48. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

49. You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

50. Production level Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

51. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

52. Production level Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

53. You are to construct an open rectangular box with a square base and a volume of 48 ft^3. If material for the bottom costs \$6/ft^2 and material for the sides costs \$4/ft^2, what dimensions will result in the least expensive box? What is the minimum cost?

54. The 800-room Mega Motel chain is filled to capacity when the room charge is $50 per night. For each $10 increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

Biology

55. Sensitivity to medicine (*Continuation of Exercise 72, Section 3.3.*) Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and C is a constant.

56. How we cough

a. When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity υ can be modeled by the equation

$$\upsilon = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that υ is greatest when $r = (2/3)r_0$; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

T **b.** Take r_0 to be 0.5 and c to be 1 and graph υ over the interval $0 \le r \le 0.5$. Compare what you see with the claim that υ is at a maximum when $r = (2/3)r_0$.

Theory and Examples

57. An inequality for positive integers Show that if a, b, c, and d are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \ge 16.$$

58. The derivative dt/dx in Example 4

a. Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of x.

b. Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

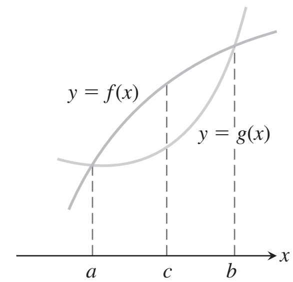
is a decreasing function of x.

c. Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

is an increasing function of x.

59. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



60. You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

a. Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

b. Is f ever negative? Explain.

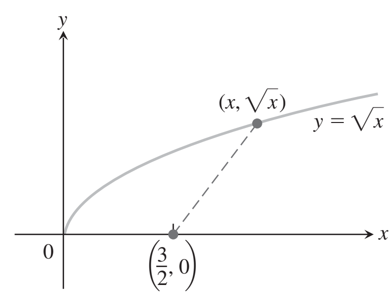
61. a. The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

62. a. The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

63. a. How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)



64. a. How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $\left(1, \sqrt{3}\right)$?

Chapter 4 Questions to Guide Your Review

1. How do derivatives arise in the study of motion? What can you learn about an object's motion along a line by examining the derivatives of the object's position function? Give examples.
2. How can derivatives arise in economics?
3. Give examples of still other applications of derivatives.
4. How do related rates problems arise? Give examples.
5. Outline a strategy for solving related rates problems. Illustrate with an example.
6. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.
7. If x moves from a to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.
8. What can be said about the extreme values of a function that is continuous on a closed interval?
9. What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
10. How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
11. What are the hypotheses and conclusion of Rolle's Theorem? Are the hypotheses really necessary? Explain.
12. What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
13. State the Mean Value Theorem's three corollaries.
14. How can you sometimes identify a function $f(x)$ by knowing f' and knowing the value of f at a point $x = x_0$? Give an example.
15. What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
16. How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
17. What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
18. What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
19. What do the derivatives of a function tell you about the shape of its graph?
20. List the steps you would take to graph a polynomial function. Illustrate with an example.
21. What is a cusp? Give examples.
22. List the steps you would take to graph a rational function. Illustrate with an example.
23. Outline a general strategy for solving max-min problems. Give examples.

Chapter 4 Practice Exercises

Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.
2. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.
3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.
4. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

5. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.
6. **a.** Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

   b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.
7. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?
8. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

The Mean Value Theorem

9. **a.** Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

   b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.
10. **a.** Show that $y = \tan \theta$ increases on every open interval in its domain.

    b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan \pi = 0$ is less than $\tan(\pi/4) = 1$?

11. **a.** Show that $f(x) = x/(x + 1)$ increases on every open interval in its domain.

    b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

12. **Water in a reservoir** As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

13. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

14. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \neq -\frac{1}{x+1}.$$

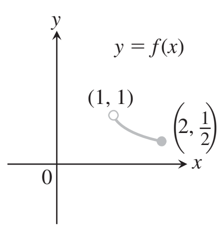
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

15. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Analyzing Graphs

In Exercises 16 and 17, use the graph to answer the questions.

16. Identify any global extreme values of f and the values of x at which they occur.

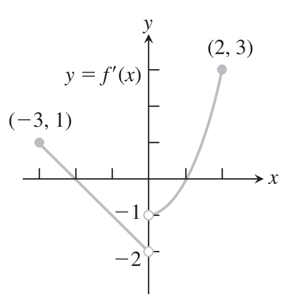


17. Estimate the open intervals on which the function $y = f(x)$ is

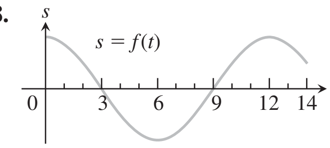
    a. increasing.

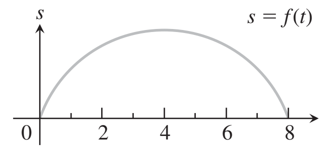
    b. decreasing.

    c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.



Each of the graphs in Exercises 21 and 22 is the graph of the position function $s = f(t)$ of an object moving on a coordinate line (t represents time). At approximately what times (if any) is each object's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the object move **(c)** forward? **(d)** Backward?

18. 

19. 

Graphs and Graphing

Graph the curves in Exercises 21–29.

20. $y = x^2 - (x^3/6)$
21. $y = x^3 - 3x^2 + 3$
22. $y = -x^3 + 6x^2 - 9x + 3$
23. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$
24. $y = x^3(8 - x)$
25. $y = x^2(2x^2 - 9)$
26. $y = x - 3x^{2/3}$
27. $y = x^{1/3}(x - 4)$
28. $y = x\sqrt{3 - x}$
29. $y = x\sqrt{4 - x^2}$

Each of Exercises 30–35 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

30. $y' = 16 - x^2$
31. $y' = x^2 - x - 6$
32. $y' = 6x(x + 1)(x - 2)$
33. $y' = x^2(6 - 4x)$
34. $y' = x^4 - 2x^2$
35. $y' = 4x^2 - x^4$

In Exercises 36–39, graph each function. Then use the function's first derivative to explain what you see.

36. $y = x^{2/3} + (x - 1)^{1/3}$
37. $y = x^{2/3} + (x - 1)^{2/3}$
38. $y = x^{1/3} + (x - 1)^{1/3}$
39. $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the rational functions in Exercises 40–47.

40. $y = \dfrac{x + 1}{x - 3}$
41. $y = \dfrac{2x}{x + 5}$
42. $y = \dfrac{x^2 + 1}{x}$
43. $y = \dfrac{x^2 - x + 1}{x}$
44. $y = \dfrac{x^3 + 2}{2x}$
45. $y = \dfrac{x^4 - 1}{x^2}$
46. $y = \dfrac{x^2 - 4}{x^2 - 3}$
47. $y = \dfrac{x^2}{x^2 - 4}$

Optimization

48. The sum of two nonnegative numbers is 36. Find the numbers if

    a. the difference of their square roots is to be as large as possible.

    b. the sum of their square roots is to be as large as possible.

49. The sum of two nonnegative numbers is 20. Find the numbers

    a. if the product of one number and the square root of the other is to be as large as possible.

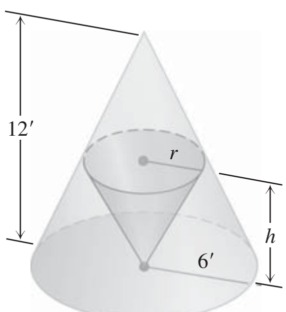
    b. if one number plus the square root of the other is to be as large as possible.

50. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

51. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

52. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

53. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



54. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

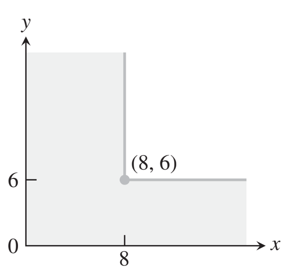
Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

55. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

56. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Related Rates

57. Right circular cylinder The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

d. How is dr/dt related to dh/dt if S is constant?

58. Right circular cone The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

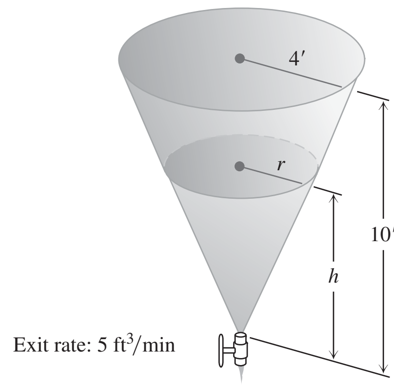
c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

59. Circle's changing area The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

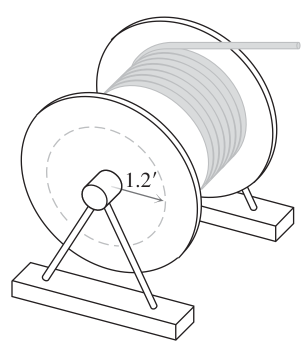
60. Draining a tank Water drains from the conical tank shown in the accompanying figure at the rate of $5 \text{ ft}^3/\text{min}$.

a. What is the relation between the variables h and r in the figure?

b. How fast is the water level dropping when $h = 6$ ft?



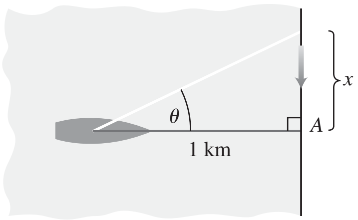
61. Rotating spool As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.



62. Moving searchlight beam The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a. How fast is the light moving along the shore when it reaches point A?

b. How many revolutions per minute is 0.6 rad/sec?



63. Points moving on coordinate axes Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Linearization

64. Find the linearizations of

a. $\tan x$ at $x = -\pi/4$ **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

65. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \qquad \text{and} \qquad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1 - x.$$

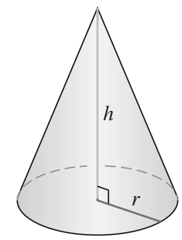
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

66. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

67. Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1+x} - 3.1$ at $x = 0$.

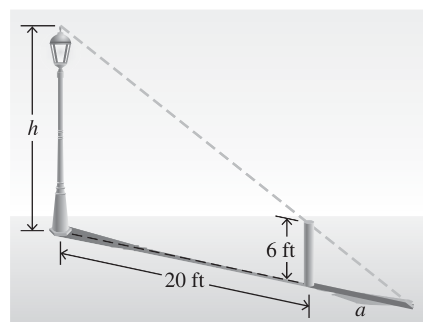
Differential Estimates of Change

68. Surface area of a cone Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



$$V = \frac{1}{3}\pi r^2 h$$
$$S = \pi r\sqrt{r^2 + h^2}$$
(Lateral surface area)

69. Finding height To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



Chapter 4 Single Choice Questions

1. Number of real roots of the equation $\dfrac{e^x}{x^2} = 5 - x$ is

a. 1 **b.** 2 **c.** 3 **d.** cannot be determined

2. Let $y = f(x)$ be a thrice derivable function such that $f(a)\,f(b) < 0$, $f(b)\,f(c) < 0$, $f(c)\,f(d) < 0$, where $a < b < c < d$. Also the equations $f(x) = 0$ and $f''(x) = 0$ have no common roots.

Statement-1: The equation $f(x)\,(f''(x))^2 + f(x)\,f'(x)\,f'''(x) + (f'(x))^2 f''(x) = 0$ has at least 5 real roots.

and

Statement-2: The equation $f(x) = 0$ has at least 3 real distinct roots and if $f(x) = 0$ has k real distinct roots, then

$f'(x) = 0$ has at least $k - 1$ distinct roots.

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

3. Statement-1: If $a, b \in R$ and $a < b$, then there is at least one real number $c \in (a, b)$ such that $\dfrac{c}{a+b} = \dfrac{b^2 + a^2}{4c^2}$

and

Statement-2: If $f(x)$ is continuous in $[a, b]$ and derivable in (a, b) and $f'(c) = 0$ for at least one $c \in (a, b)$, then it necessarily implies that $f(a) = f(b)$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

4. **Statement-1:** If the tangent to the curve $y = \frac{x-2}{x+1}$ at point P $\left(e, \frac{e-2}{e+1}\right)$ makes an angle θ with positive x-axis, then $e^{\cos\theta} < 1$

Statement-2: For the curve $xy = -c^2$, $\left.\frac{dy}{dx}\right|_{x=t} > 0 \; \forall t \in R$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

5. **Statememt-1:** $\frac{2}{\log_3 2} < \frac{e}{\log_3 e} < \frac{5}{\log_3 5}$

and

Statement-2: The function $f(x) = \frac{3^x}{x}$ decrease in $\left(0, \frac{1}{\ln 3}\right)$ and increases in $\left(\frac{1}{\ln 3}, \infty\right)$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

6. A solid rectangular brick is to the made from 1 cu feet of clay. The brick must be 3 times as long s it is wide. The width of brick for which it will have minimum surface area is a. Then a^3 is

a. $\left(\frac{2}{9}\right)^{1/3}$ **b.** $\frac{2}{9}$ **c.** $\frac{8}{729}$ **d.** $\frac{3}{2}$

7. Consider the function $f(x) = \begin{cases} x^2+2x+2 & x<0 \\ x+2 & x \ge 0 \end{cases}$, and identify the correct statement

a. Mean value theorem is applicable in [–2, 2]

b. Mean value theorem is not applicable in [–5, – 1]

c. Mean value theorem is applicable in [–4, –1] and its c is $\frac{-5}{2}$

d. Mean value theorem is applicable in [–4, –1] and its c is –1

8. If f is a differentiable function for all real x and $f'(x) \le 5$, $\forall x \in R$. If $f(2) = 0$ and $f(5) = 15$, then $f(3)$

a. 0 **b.** 15 **c.** 1 **d.** 5

9. Let $f(x)$ be a non-zero function whose all successive derivative exist and are non-zero. If $f(x)$. $f'(x)$ and $f''(x)$ are in G.P. and $f(0) = 0, f'(0) = 1$, then

a. $f'(x) < 0 \; \forall x \in R$ **b.** $f''(x) < 1 \; \forall x \in R$

c. $f''(0) \ne f'''(0)$ **d.** $f''(x) > 0 \; \forall x \in R$

10. The set of values of "a" for which the function $f(x) = (4a-3)(x + \ln 5) + (a-7)\sin x$ does not possess critical points is

a. $\left(-\infty, -\frac{4}{3}\right) \cup (2, \infty)$ **b.** $(-\infty, 2)$

c. $[1, \infty)$ **d.** $(1, \infty)$

11. If $f(x)$ is continuous and increasing function such that domain of $g(x) = \sqrt{f(x)-x}$ be R and $h(x) = \frac{1}{1-x}$, then the domain of $g(x) = \sqrt{f(f(f(x))) - h(h(h(x)))}$ is

a. R **b.** $\{0, 1\}$ **c.** $R - \{0, 1\}$ **d.** $R - (0, 1)$

12. The angle between the tangents drawn to the curve $x^2y = 1 - y$ at the points where it meets with $xy = 1 - y$ is

a. $\tan^{-1}\frac{1}{2}$ **b.** $\frac{\pi}{6}$ **c.** $\frac{\pi}{3}$ **d.** $\tan^{-1}2$

13. Tangents are drawn to $y = \cos x$ from the point $P(0, 0)$. Points of contacts of these tangents will always lie on

a. $\frac{1}{x^2} = \frac{1}{y^2} + 1$ **b.** $\frac{1}{x^2} = \frac{1}{y^2} - 1$

c. $x^2 + y^2 = 1$ **d.** $x^2 - y^2 = 1$

14. let $f(x) = \frac{x^2+2}{[x]}$, $1 \le x \le 3$, where $[\cdot]$ denote greatest integer function, then incorrect statement is

a. $f(x)$ is increasing in [1, 3]

b. least value of $f(x)$ is 3

c. $f(x)$ has no greatest value

d. domain of $f'(x)$ is $(1, 3) - \{2\}$

15. The surface area of a spherical balloon, being inflated, changes at a rate proportional to time t. If initially its radius is 3 units and after 2 seconds it is 5 units, then radius after 3 seconds is

a. $3\sqrt{5}$ units **b.** $5\sqrt{3}$ units

c. 9 units **d.** $\frac{7}{2}$ units

16. Let $f(x) = x^3 + ax + b$ with $a \ne b$ and suppose that the tangent lines to the graph of $f(x)$ at $x = a$ and $x = b$ are parallel. Then $f(1)$ is equal to

a. 1 **b.** 0 **c.** – 1 **d.** 2

17. The interval on which the function $f(x) = \log_{1/3}(x^2 - 5x + 6)$ is monotonic increasing is

a. $\left(2, \frac{5}{2}\right) \cup (3, \infty)$ **b.** $(3, \infty)$

c. $(-\infty, 2)$ **d.** $(-\infty, 2) \cup \left(\frac{5}{2}, 3\right)$

18. Angle of intersection of the curves $y = 2^x \log x$ and $y = 2x^x - 2$ at point with integral coordinate is

a. $\frac{\pi}{2}$ **b.** $\frac{\pi}{3}$ **c.** $\frac{\pi}{6}$ **d.** 0

19. A twice differentiable function $f(x)$ defined over R is such that its second order derivative is always negative. Identify the correct option

a. $f(-1) + f(1) < 2f(0)$ **b.** $f(-1) + f(1) > 2f(0)$

c. $f(-1) - f(1) < 2f(0)$ **d.** $f(-1) - f(1) > 2f(0)$

20. Let f be a twice differentiable function such that $f(1) = 3, f(2) = 2, f(3) = 1$, then identify the correct statement

a. $f'(x) = -1$ for at least two $x \in (1, 3)$

b. $f''(x) = 0$ for at least one $x \in (1, 3)$

c. $f''(x) = 0$ for at least one $x \in (1, 2)$

d. $f''(x) = 0$ for at least one $x \in (2, 3)$

21. If $f(x)$ is a strictly increasing function and $g(x)$ is a strictly decreasing function, then which of following is true for $x_1 > x_2$

a. $g(g(x_1)) < g(g(x_2))$ **b.** $g(f(x_1)) > g(f(x_2))$

c. $g(f(x_1)) < g(f(x_2))$ **d.** $f(g(x_1)) > f(g(x_2))$

22. Statement-1: Consider $f(x) = \sin x,\ x \in \left[\dfrac{\pi}{6}, \dfrac{25\pi}{6}\right]$. The equation $f(x) \cdot f'(x) = 0$ will have 8 roots.

Statement-2: If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) such that equation $f(x) = 0$ has n roots in (a, b), then equation $f'(x) = 0$ will have exactly $(n - 1)$ roots in (a, b)

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

23. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on curve $y = e2^{|x| + x}$ ($x_1 > 0$ and $x_2 < 0$) such that tangents at points A and B are perpendicular, then number of such pair of points A and B is

a. 0 **b.** 1 **c.** 2 **d.** more than 2

24. Let $f(x) = \int_0^x e^x \sin^{-1}\ (x - 1) \ln x\, dx$ $(x > 0)$, then

a. $f(x)$ has one local minima **b.** $f(x)$ is an increasing function

c. $f(x)$ has one local maxima **d.** $f(x)$ is a decreasing function

25. Consider the function $f(x) = \dfrac{ae^x + be^{-x}}{ce^x + de^{-x}}$ $(a, b, c, d > 0)$. Identify the correct options

a. $f(x)$ is decreasing if $bd < ac$

b. $f(x)$ is increasing if $bd > ac$

c. $f(x)$ is decreasing if $ad > bc$

d. $f(x)$ is increasing if $ad > bc$

26. If for $f(x) = x^3 + ax^2 + bx$, Rolle's theorem holds in the interval $[1, 2]$ at $x = \dfrac{4}{3}$, then value of $a + b$ is

a. -3 **b.** 0 **c.** 3 **d.** 5

27. Let $f(x) = \begin{cases} e^{ax} & x \le 0 \\ b\tan x + c & x > 0 \end{cases}$. If $x = 0$ is point of local maxima, then

a. $a < 0, b \ne 0, c \le 1$ **b.** $a > 0, b \ne 0, c \le 1$

c. $a < 0, b \ne 0, c \ge 1$ **d.** $a > 0, b \ne 0, c \ge 1$

28. If X is a five-digit number $abcde$, then maximum value of $\dfrac{X}{a+b+c+d+e}$ will be

a. 10,000 **b.** 11,000 **c.** 9000 **d.** 9999

29. Let $3a + 6c - 4b - 12d = 0$.

Statement-I: Equation $ax^3 + bx^2 + cx + d = 0$ will have at least one roots in $(-1, 0)$

Statement-II: $f(x) = ax^3 + bx^2 + cx + d$ is continuous in $(-1, 0)$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

30. Consider $f(x) = \begin{cases} x(x+3) & -1 \le x < 0 \\ -\sin x & 0 \le x < \dfrac{\pi}{2} \\ -(1+\cos x) & \dfrac{\pi}{2} \le x \le \pi \end{cases}$. The sum of maximum and minimum values of $f(x)$ is

a. -2 **b.** -1 **c.** 0 **d.** 2

31. If in $\left(0, \dfrac{\pi}{2}\right)$, the equation $\dfrac{9}{\cos x} + \dfrac{1}{1 - \cos x} = k$ has at least one roots, then k cannot be

a. 15 **b.** 20 **c.** 25 **d.** 30

32. Consider the following statements:

I. Derivative of differentiable aperiodic function is also aperiodic.

II. If a differentiable function f is increasing in (a, b) then f' will be decreasing in (a, b)

III. A continuous monotonic function defined on R will have R as its range.

Identify the correct options

a. Only one of I, II and III is correct

b. Only two of I, II and III is correct

c. All three are correct

d. None of I, II , III is correct

33. **Statement-I:** For a twice differentiable function f, if 0 and 3 are roots of f and $f(1) = 2$ and $f(2) = -3$, then number of roots of equation $g(x) = (f'(x))^2 + f(x).\,f''(x)$ in $[0, 3]$ is 4.

Statement-II: For a continuous differentiable function f, if $f(x) = 0$ has n roots in $[a, b]$ then $f'(x) = 0$ has $(n - 1)$ roots in $[a, b]$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

34. If $f(x)$ is a differentiable function such that $f'(\alpha) = f'(\beta) = 0$ and $f''(\alpha)\, f''(\beta) < 0$, $f(\alpha) = 3$, $f(\beta) = -1$. $(\alpha < \beta)$. If $f''(\beta) - f''(\alpha) < 0$, then minimum number of roots of $f'(x) = 0$ in $[\alpha, \beta]$ will be

a. 0 **b.** 2 **c.** 3 **d.** 4

35. Suppose $f'(x)$ exists for each x and $h(x) = f(x) - f(x)^2 + (f(x))^3$ for every real number x, then incorrect statement is

a. $h(x)$ will be monotonic if $f(x)$ is monotonic

b. $h(x)$ will be periodic with same period if $f(x)$ is periodic

c. $f'(x) = h'(x)$ for maximum two value of x

d. Number and nature of local extremum points of $f(x)$ and $h(x)$ will be same.

36. If $f(x)=\begin{cases} e^{x+1}-e^{x} & x\le 0 \\ e^{1-x}-1 & 0<x<1 \\ x+\ln x & x\ge 1 \end{cases}$, then

a. $x = 0$ is point of local maxima, $x = 1$ is neither local maxima nor local minima.

b. $x = 1$ is point of local minima, $x = 0$ is point of local maxima

c. $x = 0$ and $x = 1$ both are points of local maxima

d. $x = 0$ and $x = 1$ both are points of local minima

37. If tangents to the curve $y=\frac{x^4}{4}+\frac{ax^3}{3}+\frac{ax^2}{2}+x+1$ $x\in R$ always lie below the curve, then range of a is

a. $[0, 3]$ **b.** $(-\infty, \infty)$

c. $(-\infty, 3]$ **d.** $(-\infty, 3)\cup[3, \infty)$

38. The number of points of intersection of curve $\sin x = \cos y$ and $x^2 + y^2 = 1$, is

a. 3 **b.** 2 **c.** 0 **d.** infinitely many

39. Equation of the normal to the curve $y = x^2$ which passes through (6, 3) is given by

a. $x + 4y = 18$ **b.** $x + 2y = 12$

c. $x + y = 9$ **d.** $x + 3y = 15$

40. If the curve $y^2 = ax^3 - 6x^2 + b$ passes through (0, 1) and has its tangent parallel to y-axis at $x = 2$, then

a. $a = 2, b = 1$ **b.** $a=\frac{23}{8}, b = 1$

c. $a=-\frac{8}{23}, b=1$ **d.** $a=-\frac{23}{8}, b=1$

41. The lines tangent to the curve $x^3 - y^3 + x^2y - yx^2 + 3x - 2y = 0$ and $x^5 - y^4 + 2x + 3y = 0$ at the origin intersect at an angle θ equal to

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

42. The equation of normal to the curve $x + y = x^y$, where it cuts x-axis is

a. $y = x + 1$ **b.** $y = -x + 1$

c. $y = x - 1$ **d.** $y = -x - 1$

43. If a spherical balloon has a variable diameter $3x+\frac{9}{2}$, then the rate of change of its volume with respect to x

a. $27\pi(2x + 3)^2$ **b.** $\frac{27\pi}{16}(2x+3)^2$

c. $\frac{27\pi}{8}(2x+3)^2$ **d.** None of these

44. Minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is

a. $\sqrt{21}$ **b.** $\sqrt{26}-\sqrt{5}$

c. $\sqrt{20}-\sqrt{5}$ **d.** $\sqrt{21}-\sqrt{5}$

45. Given $f'(x) > 0\ \forall\ x \in R$ and $f(a) = 0$, then which of the following statement must be "False" for the function $g(x)=\int_0^x f(t)dt$

a. $g(x)$ has a horizontal tangent at $x = a$

b. $g(x)$ is a differentiable function

c. $g(x)$ has a local maxima at $x = a$

d. $g(x)$ has a local minima at $x = a$

46. The least positive integral value of λ for which $f(x)=\frac{3x^3}{2}+\frac{\lambda x^2}{3}+x+7$ has a point of maxima

a. 6 **b.** 7 **c.** 8 **d.** 10

47. Given $f(x)=\begin{cases} |x-1|+a & \text{if } x\le 1 \\ 2x+3 & \text{if } x>1 \end{cases}$

If $f(x)$ has a local minimum at $x = 1$, then which of the following is true?

a. $a \le 5$ **b.** $a \ge 5$ **c.** $a \le 0$ **d.** $a \ge 0$

48. Let $P(x, y)$ be a variable point on $y=\frac{x^2}{2}-2x+5$. If the rate of decrease of y is thrice rate of decrease of x, then x is equal to

a. 4 **b.** 5 **c.** 7 **d.** 10

49. The number of integral values of a for which the function $f(x) = \begin{cases} x^3+1 & , x\le 0 \\ a^2-5a+7+x & , x>0 \end{cases}$ has maxima at $x = 0$ are

a. 0 **b.** 1 **c.** 2 **d.** Infinitely many

50. For all $x \in (0, 1)$

a. $e^x < 1 + x$ **b.** $\log_e(1 + x) < x$

c. $\sin x > x$ **d.** $\log_e x > x$

51. Consider the following statement S and R:

S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $(\pi/2, \pi)$

R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true?

a. both S and R are wrong

b. both S and R are correct, but R is not the correct explanation for S

c. S is correct and R is the correct explanation for S

d. S is correct and R is wrong

52. Let $f(x)=\int e^x(x-1)(x-2)dx$ then f decreases in the interval:

a. $(-\infty, 2)$ **b.** $(-2, -1)$ **c.** $(1, 2)$ **d.** $(2, +\infty)$

53. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is

a. increasing on $\left(-\frac{1}{2},1\right)$ **b.** decreasing on $\left[-\frac{1}{2},1\right]$

c. increasing on R **d.** decreasing on R

54. The length of a longest interval in which the function $f(x) = 3\sin x - 4\sin^3 x$ is increasing is

a. $\pi/3$ **b.** $\pi/2$

c. $3\pi/2$ **d.** π

55. Let $f(x)=\begin{cases} x^{\alpha}\ln x, & x>0 \\ 0, & x=0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$

a. -2 **b.** -1

c. 0 **d.** 1/2

56. If $f(x)$ is a twice differentiable function and given that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then

a. $f''(x) = 2$, for $x \in (1, 3)$

b. $f''(x) = f'(x) = 2$, for some $x \in (2, 3)$

c. $f''(x) = 3$, for $x \in (2, 3)$

d. $f''(x) = 2$, for some $x \in (1, 3)$

57. Let $f(x) = 2 + \cos x$ for all real x.

Statement-1: For each real t, there exists a point c is $[t, t + \pi]$ such that $f'(c) = 0$.

Statement-1: $f(t) = f(t + 2\pi)$ for each real t.

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

58. Let the function $g: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

a. even and is strictly increasing in $(0, \infty)$

b. odd and is strictly decreasing in $(-\infty, \infty)$

c. odd and is strictly increasing in $(-\infty, \infty)$

d. neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

59. Let $f: (0, 1) \to R$ be defined by $f(x) = \frac{b - x}{1 - bx}$, where b is a constant such that $0 < b < 1$. Then

a. f is not invertible on $(0, 1)$

b. $f \neq f^{-1}$ on $(0, 1)$ and $f'(b) = \frac{1}{f'(0)}$

c. $f = f^{-1}$ on $(0, 1)$ $f'(b)$ $\frac{1}{f'(0)}$

d. f^{-1} is differentiable on $(0, 1)$

60. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is

a. 6 b. 4 c. 2 d. 0

Chapter 4 Multiple Choice Questions

1. If $g(x) = 7x^2 e^{-x^2}\ \forall\ x \in R$, then $g(x)$ has

a. local maximum at $x = 0$

b. local minima $x = 0$

c. local maximum at $x = -1$

d. two local maxima and one local minima

2. The value of a for which the function $f(x) = \begin{cases} -x^3 + \cos^{-1} a, & 0 < x < 1 \\ x^2, & x \geq 1 \end{cases}$ has a local minimum at $x = 1$, is

a. -1 b. 1 c. 0 d. $-\frac{1}{2}$

3. The families of curves defined by the equations $y = bx$ and $x^2 + y^2 = a^2$ are orthogonal if

a. $a = 2, b = 4$ b. $a = -3, b = 5$

c. $a = -2, b = 3$ d. $a = 5, b = 2$

4. If $y = g(x)$ is a curve which is obtained by the reflection of $y = f(x) = \frac{e^x - e^{-x}}{2}$ about the line $y = x$, then which of the following is/are true.

a. $y = g(x)$ has exactly one tangent parallel to x-axis

b. $y = g(x)$ has no tangent parallel to x-axis

c. The tangent to $y = g(x)$ at $(0, 0)$ is $y = x$

d. $g(x)$ has no extremum

5. If $f(x)$ and $g(x)$ and two increasing and differentiable function such that $f(x), g(x) > 0\ x \in R$, then

a. $(f(x))^{g(x)}$ is always increasing

b. if $f(x) > 1$, then $(f(x))^{g(x)}$ is increasing

c. if$(f(x))^{g(x)}$ is decreasing then $f(x) < 1$

d. if $(f(x))^{g(x)}$ is decreasing then $f(x) > 1$

6. Consider the equation $\left|2 - \frac{1}{|x-1|}\right| = K$. Now which of the following is true?

a. The given equation has 4 solutions if $K \in (0, 2)$

b. There is no value of K for which given equation has exactly two solutions

c. There is no value of K for which given equation has exactly one solutions

d. The least positive value of K for which given equation has two solutions is 2.

7. If the curves $y = 2(x - a)^2$ and $y = e^{2x}$ touches each other, then a is less than

a. -1 b. 0 c. 1 d. 2

8. Let f be a twice differentiable function on $[0, 2]$ such that $f(0) = 0$, $f(1) = 2$, $f(2) = 4$, then

a. $f'(\alpha) = 2$ for some $\alpha \in (0, 1)$

b. $f''(x) = 0\ \forall\ x\ (0, 2)$

c. $f'(\beta) = 2$ for some $\beta \in (1, 2)$

d. $f''(\gamma) = 0$ for some $\gamma \in (0, 2)$

9. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha) \cdot f(\beta) < 0$, then which of the following can be true for the equation $f(x) = 0$?

a. three equal roots

b. three distinct real roots

c. one positive roots if $f(\alpha) < 0$ and $f(\beta) > 0$

d. one negative roots if $f(\alpha) > 0$ and $(\beta) < 0$

10. Let $f(x)$ be a cubic polynomial such that it has point of inflection at $x = 2$ and local minima at $x = 4$, then

a. $f(x)$ has local minima at $x = 0$

b. $f(x)$ has local maxima at $x = 0$

c. $\lim_{x\to\infty} f(x) \to \infty$

d. $\lim_{x\to\infty} f(x) \to -\infty$

11. Let $f(x)$ and $g(x)$ are polynomial of degree 3, where $g(\alpha) = 0$, $g'(\alpha) = 0$, $g''(\alpha) \neq 0$ and $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 0$. If $h(x) = f(x)\, g'(x) + f'(x) \cdot g(x)$, then which of the following is/are true

a. Number of solutions which are common to all $f(x) = 0$, $g(x) = 0$, $h(x) = 0$ is 1.

b. Number of solutions which are common to all $f(x) = 0$, $g(x) = 0$, $h(x) = 0$ is 2.

c. Number of distinct roots of equations $h(x) = 0$ is 2.

d. $h'''(\alpha) = 0$

12. A spherical rain drop evaporates at a rate proportional to its surface area at that instant. The radius of drop initial is 3 mm and after one hour it is found to be 2 mm. Which of the following statement(s) is/are correct?

a. The time it takes for drop to completely evaporate is 3 hours.

b. The time it takes for drop to completely evaporate is 2 hours

c. In 2 hours, approximately 96.3% of drop is evaporated.

d. In 2 hours, approximately 70% of drop is evaporated.

13. If α is only real root of the equation $x^3 + (\cos 1)\, x^2 + (\sin 1)\, x + 1 = 0$, then $\left(\tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha}\right)$ cannot be equal to

a. 0 **b.** $\frac{\pi}{2}$

c. $-\frac{\pi}{2}$ **d.** π

14. Given $f(x) = x^4 \cdot e^{-x^2}$, $x \in R$ then

a. minimum values of $f(x)$ is 0

b. maximum value of $f(x)$ is $4e^{-2}$

c. the line $y = 4$ cuts $y = x^4e^{-x^2}$ at 2 points

d. the graph $y = x^4e^{-x^2}$ is symmetrical about y-axis.

15. Water is flowing inside a cylinder (closed from bottom and open from top) at the rate of 20 cm^3/min whose radius = 3 cm and altitude = 4 cm. Identify the correct statement(s). (Assume at $t = 0$, cylinder is empty)

a. Cylinder will be full at $t = \frac{3\pi}{5}$ min

b. Cylinder will be full at $t = \frac{9\pi}{5}$ min

c. If it takes t_1 min for the cylinder to be half full then it will take $2t_1$ min for the cylinder to be full

d. As time increases, rate of depth of water in the cylinder decreases

16. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'(1/4) = 0$. Then

a. $f''(x)$ vanishes at least twice on [0, 1]

b. $f'(1/2) = 0$

c. $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x\,dx = 0$

d. $\int_{0}^{1/2} f(t)e^{\sin \pi t}dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t}dt$

17. For the function $f(x) = x \cos x\ \frac{1}{x} \geq 1$

a. for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$

b. $\lim_{x\to\infty} f'(x) = 1$

c. for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$

d. $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

18. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement is/are true?

a. $f''(x)$ exist for all $x \in (0, \infty)$

b. $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

c. there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

d. there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \in (0, \infty)$

19. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$

a. f has a local maximum at $x = 2$

b. f is decreasing on (2, 3)

c. there exists some $c \in (0, \infty)$ such that $f''(c) = 0$

d. f has a local minimum at $x = 3$

20. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at

a. a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

b. a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

c. a unique point in the interval $(n, n+1)$

d. two points in the interval $(n, n+1)$

21. If a function $f(x)$ is continuous for all $x \in R$ and has a relative maximum at $(-2, 2)$ and relative minimum at $(3, -2)$, then which of the following statement(s) is/are always **correct**?

a. The graph of $f(x)$ must change its concavity somewhere between $x = -2$ and $x = 3$.

b. The equation $f(x) = 0$ must have at least one real root.

c. The value of $f'(-2)$ must be zero.

d. The graph of $f(x)$ intersects both x-axis and y-axis.

22. For which of the following functions, **Rolle's Theorem** is applicable?

a. $f(x) = \frac{x}{2} + \frac{2}{x}$, $x \in [1, 4]$.

b. $f(x) = x + 1 - x^{\frac{3}{2}}$, $x \in [0, 1]$.

c. $f(x) = |x+1|^3$, $x \in [-2, 0]$.

d. $f(x) = \text{sgn}(x) + \text{sgn}(-x)$, $x \in \left[\frac{-5}{2}, \frac{1}{2}\right]$.

[**Note:** sgn (x) denotes signum function of x.]

23. If $y = mx + 5$ is a tangent to the curve $x^3y^3 = ax^3 + by^3$ at P (1, 2), then

a. $a + b = \frac{18}{5}$

b. $a > b$

c. $a < b$

d. $a + b = \frac{19}{5}$

24. Let $f(x) = \int_0^x e^{-t^2}(t-5)(t^2-7t+12)\,dt$ for all $x \in (0, \infty)$, then

a. f has a local maximum at $x = 4$ and a local minimum at $x = 3$.

b. f is decreasing on $(3, 4) \cup (5, \infty)$ and increasing on $(0, 3) \cup (4, 5)$.

c. There exists at least two $c_1, c_2 \in (0, \infty)$ such that $f''(c_1) = 0$ and $f''(c_2) = 0$.

d. There exists some $c \in (0, \infty)$ such that $f'''(c) = 0$.

25. Let $f(x) = x^3 + ax^2 + bx + c$, where $a, b, c \in R$, then which of the following statement(s) is(are) correct?

a. If the equation $f(x) = 0$ has exactly one real root then $f(x)$ must be strictly increasing on R.

b. If $f(x)$ has a negative point of local minimum, then both roots of equation $f'(x) = 0$ must be negative and distinct.

c. If $f(x_1) \cdot f(x_2) < 0$, $x_1 < x_2$, then the equation $f(x) = 0$ has at least one real root in (x_1, x_2).

d. $f(x)$ possesses exactly one point of inflexion.

Chapter 4 Passage Type Questions

Passage 1

Consider the curve C_1: $y = ax^2 + bx + c$, such that it touches the line $y = x$ at the point $x = 1$ and passes through the point $(-1, 0)$. Let the curve C_2: $y = \frac{e^{-x}}{4}$, intersect the given curve at P(m, n), where $m > -1$

Let α be the length of subtangent to C_2 at point P

β be the length of subnormal to C_1 at point P

γ be the length of subnormal to C_2 at point P

On the basis of above information, answer the following questions.

1. The quadratic equation $ax^2 + bx + c = 0$ has

a. imaginary roots

b. real distinct roots

c. coincident real roots

d. irrational roots

2. The angle of intersection between curve C_1 and C_2 at P(m, n) is

a. $\tan^{-1}\frac{6}{7}$ **b.** $\tan^{-1}\frac{7}{6}$ **c.** $\tan^{-1} 1$ **d.** $\frac{\pi}{2}$

3. $\frac{\alpha}{4}$, β, γ are in

a. A.P. **b.** G.P. **c.** H.P. **d.** None of these

Passage 2

Let $f(x)$ be a real-valued continuous function on R defined as $f(x) = x^2 e^{-|x|}$

1. The value of k for which the curve $y = kx^2$ ($k > 0$) intersect the curve $y = e^{|x|}$ at exactly two points, is

a. e^2 **b.** $\frac{e^2}{2}$ **c.** $\frac{e^2}{4}$ **d.** $\frac{e^2}{8}$

2. The area bounded by ordinates at points of local maximum, the curve $y = f(x)$ and x-axis is equal to

a. $\frac{4(e^2-5)}{e^2}$ **b.** $\frac{e^2+5}{e}$ **c.** $\frac{2(e^2+5)}{e^2}$ **d.** 4

3. Range of the function $f(x)$ is

a. $\left[0, \frac{2}{e^2}\right]$ **b.** $\left[0, \frac{4}{e^2}\right]$ **c.** $\left[0, \frac{4}{e}\right]$ **d.** $(0, e^2)$

Passage 3

Consider $f(x) = \frac{2|x|-1}{x-3}$

On the basis of above information answer the following:

1. Range of $f(x)$ is

a. $R - \{3\}$ **b.** $\left(-\infty, \frac{1}{3}\right] \cup (2, \infty)$

c. $\left(-2, \frac{1}{3}\right] \cup (2, \infty)$ **d.** R

2. Which of the following is false for $f(x)$?

a. It attains its local maximum at $x = 0$ only

b. It increases in $(-\infty, 0)$

c. Not differentiable at exactly one point of domain

d. None of these

3. The tangent at $(x_1 y_1)$ on the graph of $f(x)$ passes through origin, then x_1 is a/an

a. positive integer **b.** negative integer

c. rational number but not integer **d.** irrational number

Passage 4

Let $f(x) = x^3 - 3x + 1$, $x \in R$ and a, b, c are roots of $f(x) = 0$, P$(\alpha, f(\alpha))$ is point of local minima Q$(\beta, f(\beta))$ is point of local maxima and $R(\gamma, f(\gamma))$ is point of inflection in the graph of $y = f(x)$. On the basis of above information answer the following:

1. $\begin{vmatrix} f(a) & f''(a) & 1 \\ f(b) & f''(b) & 1 \\ f(c) & f''(c) & 1 \end{vmatrix}$ is equal to

a. $(a - b)(b - c)(c - a)$ **b.** $3(a - b)(b - c)(c - a)$
c. $\cos^{-1}(\cos 1)$ **d.** $\cos^{-1}1$

2. Triangle PQR is

a. equilateral **b.** isosceles
c. right angle **d.** none of these

3. Number of distinct real solution of $f(f(x)) = 0$ are

a. 3 **b.** 5 **c.** 7 **d.** 9

Passage 5

Consider a real-valued function $f: R \to R$ satisfying $f\left(\dfrac{2x+3y}{5}\right) = \dfrac{2f(x)+3f(y)}{5} \forall\ x, y \in R$ and $f'(0) = 2; f(0) = 1$.

1. Minimum distance of a point on graph of $y = f(x)$ from origin is less than or equal to

a. $\dfrac{1}{\sqrt{2}}$ units **b.** $\dfrac{1}{2}$ units
c. $\dfrac{1}{\sqrt{5}}$ units **d.** $\dfrac{1}{\sqrt{7}}$ units

2. Which of the following is true about f?

a. f is monotonic $\forall\ x \in R$ **b.** f is periodic function
c. f is an odd function **d.** f is a bijection

Passage 6

Consider $f(x) = x^3(x - 2)^2 (x - 1)$

On the basis of above information, answer the following questions:

1. If maxima of $f(x)$ occurs at x_i then $\sum x_i^2$ is equal to

a. $\dfrac{9}{4}$ **b.** $\dfrac{25}{4}$ **c.** $\dfrac{40}{9}$ **d.** $\dfrac{4}{9}$

2. If minima of $f(x)$ occurs at x_i then $\sum x_i^2$ is equal to

a. $\dfrac{40}{9}$ **b.** $\dfrac{25}{4}$ **c.** $\dfrac{9}{4}$ **d.** $\dfrac{4}{9}$

3. Least value of $f(x)$ is

a. $\dfrac{1}{9}$ **b.** $\dfrac{4}{27}$ **c.** $-\dfrac{128}{729}$ **d.** 0

Passage 7

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x, where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \le 0$ at

a. no point **b.** one point
c. two points **d.** more than two points

2. The positive value of k for which $ke^x - x = 0$ has only one roots is

a. $1/e$ **b.** 1 **c.** e **d.** $\log_e 2$

3. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is

a. $(0, 1/e)$ **b.** $(1/e, 1)$ **c.** $(1/e, \infty)$ **d.** $(0, 1)$

Chapter 4 Matrix Match Type Questions

1.

Column-I	Column-II
(a) The number of positive roots of the equation $(x - 1)(x - 2)(x - 3) + (x - 1)(x - 2)(x - 4) + (x - 2)(x - 3)(x - 4) + (x - 1)(x - 3)(x - 4) = 0$	(p) 1
(b) If the function $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f(6 - x^2)\ \forall\ x \in R$ increases in the interval (a, ∞), where $f''(x) > 0\ \forall\ x \in R$, then the value of a is	(q) 2
(c) If $f(x) = e^x\ \forall\ x \in [0, 1]$ and $f(1) - f(0) = f'(c)$, where $c \in (0, 1)$, then $\ln(e^c + 1)$ is equal to	(r) 3
(d) If for the function $f(x) = \begin{cases} mx + c, & x < 0 \\ e^x, & x \ge 0 \end{cases}$. Lagrange's mean value theorem is applicable then $(m + 3c)$ is	(s) 4

2. The function $f(x) = \sqrt{ax^3 + bx^2 + cx + d}$ has its non-zero local minimum and maximum value at $x = -2$ and $x = 2$, respectively. If a is one of the root of $x^2 - x - 6 = 0$, then match the following:

Column-I	Column-II
(a) The value of a is	(p) 0
(b) The value of b is	(q) 24
(c) The value of c is	(r) greater than 32
(d) The value of d is	(s) –2
	(t) 3

3. In the following $[x]$ denotes the greatest integer less than or equal to x.

Match the functions in column-I with the properties in Column-II

Column-I	Column-II
(a) $x\,\lvert x \rvert$	(p) continuous in (–1, 1)
(b) $\sqrt{\lvert x \rvert}$	(q) differentiable in (–1, 1)
(c) $x + \lvert x \rvert$	(r) strictly increasing in (–1, 1)
(d) $\lvert x - 1\rvert + \lvert x + 1\rvert$	(s) non differentiable at least at one point in (–1, 1)

4.

Column-I	Column-II
(a) The maximum value of the function $f(x) = \dfrac{64\sin^3 x \cdot \cos x}{1+\tan^2 x}$ is equal to	(p) 10
(b) Let f: $(0, 2\pi) \to R$ be defined as $f(x) = \sin x \cdot e^{\sin 2x} + \int_0^x e^{\sin 2t}(\sin t - \cos t)dt$. Then the number of points of local minima of $f(x)$ is equal to	(q) 8
(c) If the curves $C_1 : y = \dfrac{\ln x}{x}$ and $C_2 : y = \lambda x^2$ (where λ is constant) touch each other, then the value of λ is less than	(r) 6
(d) If P and Q are variable points on $C_1 : x^2 + y^2 = 4$ and $C_2 : x^2 + y^2 - 8x - 6y + 24 = 0$, respectively, then the maximum value of PQ, is equal to	(s) 3
	(t) 2

(*Continued*)

5. Four points A, B, C and D lie in order on the parabola $y = ax^2 + bx + c$ and the co-ordinates of A = (–2, 3); B = (–1,1) and D = (2,7), then match the following columns:

Column-I	Column-II
(a) The value of $(a + b + c)$ is equal to	(p) –1
(b) If the roots of the equation $ax^2 + bx + c = 0$ are α and β, then $(\alpha^{19} + \beta^7)$ is	(q) 8
(c) Least value of $(a+2)x^2 + \dfrac{(b+2)}{x^2} + c$ is	(r) 3
(d) If the area of the quadrilateral ABCD is greatest and and co-ordinates of C is (p, q), then $(2p + 4q)$ is equal to	(s) 7

Chapter 4 Integer Type Questions

1. Consider a continuous function $f(x) = \begin{cases} \dfrac{\sqrt[3]{8+x} - \sqrt[3]{8+x^2-x^3}}{\sqrt[3]{8+x} - \sqrt[3]{8+x^2+x^3}} & , \ x \neq 0 \\ k & , \ x = 0 \end{cases}$ and a circle $C \equiv (x - k)^2 + y^2 = k^2$.
Let the intercept of the line $y = x$ on circle C be AB. If the equation of circle with AB as diameter of the form $x^2 + y^2 + ax + by + c = 0$. Find the value of $6a - 7b + 8c$?

2. Given two curves: $y = f(x)$ passing through (0, 1) and $y = \int_{-\infty}^{x} f(t)dt$ passing through $\left(0, \dfrac{1}{3}\right)$. The tangents drawn to both the curves at the points with equal abscissas intersect on x-axis. Find the value of $\ln f(3)$.

3. The maximum value of a so that the function $f(x) = x^2 - 8ax + 2013$ is increasing in (2012, 2014) is

4. If it is enclosed by the tangent line to the curve $x^{2/3} + y^{2/3} = 9$ at the point $(1, 2^{9/2})$ and the coordinate axes is l, then $\dfrac{\lambda^2}{18}$ is

5. If a particle starts moving from (0, 0) and rate of change of its abscissa is 8 cm/s one the curve $y = \sqrt{x}$, then rate of change of area of triangle formed by coordinate axes and tangent at a point reached after 2 sec of start is

6. Let $f(x) = x^3 - 3x + 1$. Then the number of different real solutions of the equation $f(f(x)) = 0$ is equal to

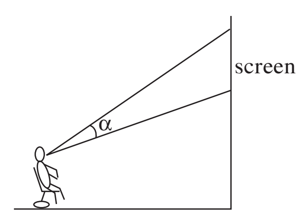
7. For $f(x) = \dfrac{x^{2015}}{2015} - \dfrac{x^{1015}}{1015} + \dfrac{x^{115}}{115} - \dfrac{x^{15}}{15} + x$. Number of integral value (s) of x at which function is strictly decreasing is equal to

8. Let f: $R \to R$ be a function such that $f'(x) = 2012(x - 2012)(x - 2013)^3(x - 2014)^2(x - 2015)\ \forall\ x \in$ R. If g is a function defined on R such that $g(x) = e^{f(x)}\ \forall\ x \in$ R, then number of points in R for which of has a local minimum is

9. On the interval [0, 2], the function $x^{15}(2 - x)^{45}$ takes its maximum value at $x = \dfrac{1}{k}$, where k is

10. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at origin intersect at an angle $\theta = \pi/k$, then k is

11. In a movie theatre with level floor, the bottom of the screen is 1 unit above the eye level and the top of the screen is 1 unit above that. if a person sits $\sqrt{N}$ distance back from the screen in order to maximize the viewing angle α (as shown in the figure), then N is equal to



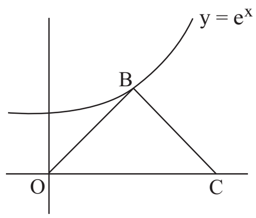
12. If (α, β) be a point on parabola $y^2 = 4x$ which is nearest to the circle $x^2 + (y - 12)^2 = 1$, then $(\alpha + \beta)$ is equal to

13. If $\dfrac{|x-1|}{|x|-1}|\ln x| = kx$ has one solution then number of non-negative values of k will be

14. If $f(x) = 2x^3 - x^2 + 15x + 1$ and $g(a) = \int_0^a f(x)dx + \int_0^{5-a} f(x)dx \forall$ $a \in (a, 5)$ then 'g' is increasing in $\left(\dfrac{\alpha}{\beta}, \gamma\right)$, where α and β are coprime numbers then $(\alpha + \beta) \cdot \gamma$ is

15. Let $f(x)$ and $g(x)$ be continuous and differentiable function. If $f(0) = f(2) = f(4); f(1) + f(3) = g(0) = g(2) = g(4) = 0$ and if $f(x) = 0$ and $g'(x) = 0$ do not have common root, then the minimum number of zeros of $f'(x)\, g'(x) + f(x) \cdot g''(x) = 0$ $[0, 4]$ is k, then $\left[\dfrac{500}{k}\right]$ is equal to (where $[\cdot]$ denotes greatest integer function).

16. Let point B is a variable point (α, e^{α}) lying on curve $y = e^x$, where $0 \le \alpha \le \ln 2$. If ΔOBC as shown in figure is such that OB = OC and C lies on x-axis. If maximum area of ΔOBC is $\lambda\sqrt{(\ln 2)^2 + 4}$, then (100λ) is



17. $f(x) = \int_0^x (t-2)^2 \log|t-1|\, dt$, if L and M denote number of points of local maxima and local minima of $f(x)$ respectively, then $(100\,L + M)$ is equal to

18. Let d be the minimum distance of $y^3 = x^2$ from point $\left(\dfrac{1}{3}, 2\right)$ measured in first quadrant, then $9d^2$ is equal to

19. If maximum area of trapezium ABCD formed by points A$(x, 0)$, $B\left(x, x + \dfrac{1}{x}\right)$, C$(-x, e^{-x})$, D$(-x, 0)$ is Δ, where $x \in (0, 1]$, then $[e\Delta]$ is equal to (where $[\cdot]$ denotes greatest integer function)

20. If the function $f(x) = \int_1^x t(e^{2t} - 1)(t - 100)^3\, dt$ has a local minimum at $x = k$, then the value of k is

21. Given $f'(x)$ is an increasing function and $h(x) = \dfrac{1}{6}f(3x^2 - 1) + \dfrac{1}{4}f(1 - 2x^2)$. If $h(x)$ is increasing in the interval $(a, 0) \cup (b, \infty)$, then $(100\,(a^4 + b^4))$ is equal to

22. If a is the number of point of extremum of the function $f(x) = \int_0^{x^2-1} t(t-1)(t-3)^3\, dt$ than a^2 is

23. A line is tangent to the curve $y = x^4 - 2x^2 - x$ at (α_1, β_1) as well as (α_2, β_2), then $|\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2|$ is equal to

24. The minimum value of k $(k \in I)$ for which the equation $e_x = kx^2$ has exactly three real solutions, is

25. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is

Chapter 4 Additional and Advanced Exercises

Functions and Derivatives

1. What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

2. Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

3. Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.

4. Local extrema Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x - 1)(x - 2)^2(x - 3)^3(x - 4)^4$$

to identify the points where f has local maximum and minimum values.

5. Local extrema

a. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x + 1)(x - 2)^2.$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

b. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6x(x + 1)(x - 2).$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \le 2$ for all x, what is the most the values of f can increase on $[0, 6]$? Give reasons for your answer.

7. Bounding a function Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. An inequality

a. Show that $-1/2 \le x/(1 + x^2) \le 1/2$ for every value of x.

b. Suppose that f is a function whose derivative is $f'(x) = x/(1 + x^2)$. Use the result in part (a) to show that

$$|f(b) - f(a)| \le \frac{1}{2}|b - a|$$

for any a and b.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. **Extrema and inflection points** Let $h = fg$ be the product of two differentiable functions of x.
    a. If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a, does h have a local maximum at a?
    b. If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. **Finding a function** Use the following information to find the values of a, b, and c in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.
    i) The values of a, b, and c are either 0 or 1.
    ii) The graph of f passes through the point $(-1, 0)$.
    iii) The line $y = 1$ is an asymptote of the graph of f.

12. **Horizontal tangent** For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

Optimization

13. **Largest inscribed triangle** Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the area of triangle ABC is largest when the triangle is isosceles? How do you know?

14. **Proving the second derivative test** The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:
    a. f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$
    b. f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

    To prove statement (a), let $\epsilon = (1/2)|f''(c)|$. Then use the fact that

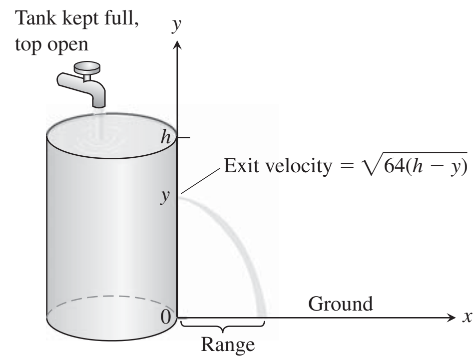
$$f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c + h)}{h}$$

to conclude that for some $\delta > 0$,

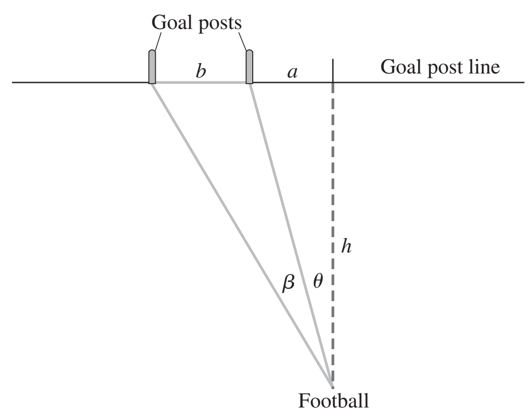
$$0 < |h| < \delta \quad \Rightarrow \quad \frac{f'(c + h)}{h} < f''(c) + \epsilon < 0.$$

Thus, $f'(c + h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

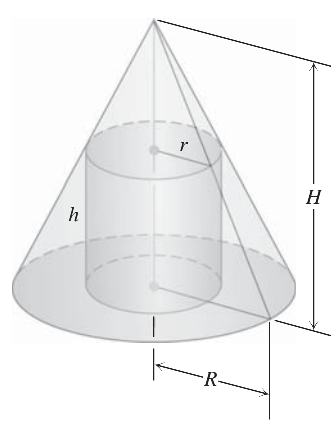
15. **Hole in a water tank** You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any, for the hole? (*Hint:* How long will it take an exiting droplet of water to fall from height y to the ground?)



16. **Kicking a field goal** An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b feet apart and that the hash mark line is a distance $a > 0$ feet from the right goal post. (See the accompanying figure.) Find the distance h from the goal post line that gives the kicker his largest angle β. Assume that the football field is flat.



17. **A max-min problem with a variable answer** Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H, as shown here. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.



18. Minimizing a parameter Find the smallest value of the positive constant m that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of x

Theory and Examples

19. Suppose that it costs a company $y = a + bx$ dollars to produce x units per week. It can sell x units per week at a price of $P = c - ex$ dollars per unit. Each of a, b, c, and e represents a positive constant. **(a)** What production level maximizes the profit? **(b)** What is the corresponding price? **(c)** What is the weekly profit at this level of production? **(d)** At what price should each item be sold to maximize profits if the government imposes a tax of t dollars per item sold? Comment on the difference between this price and the price before the tax.

20. Estimating reciprocals without division You can estimate the value of the reciprocal of a number a without ever dividing by a if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

a. Graph $y = (1/x) - 3$. Where does the graph cross the x-axis?

b. Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

21. To find $x = \sqrt[q]{a}$, we apply Newton's method to $f(x) = x^q - a$. Here we assume that a is a positive real number and q is a positive integer. Show that x_1 is a "weighted average" of x_0 and a/x_0^{q-1}, and find the coefficients m_0, m_1 such that

$$x_1 = m_0 x_0 + m_1\left(\frac{a}{x_0^{q-1}}\right), \qquad \begin{array}{l} m_0 > 0, m_1 > 0, \\ m_0 + m_1 = 1. \end{array}$$

What conclusion would you reach if x_0 and a/x_0^{q-1} were equal? What would be the value of x_1 in that case?

22. The family of straight lines $y = ax + b$ (a, b arbitrary constants) can be characterized by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2,$$

where h and r are arbitrary constants. (*Hint:* Eliminate h and r from the set of three equations including the given one and two obtained by successive differentiation.)

23. Assume that the brakes of an automobile produce a constant deceleration of k ft/sec^2. **(a)** Determine what k must be to bring an automobile traveling 60 mi/hr (88 ft/sec) to rest in a distance of 100 ft from the point where the brakes are applied. **(b)** With the same k, how far would a car traveling 30 mi/hr go before being brought to a stop?

24. Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.

25. Can there be a curve satisfying the following conditions? d^2y/dx^2 is everywhere equal to zero and, when $x = 0$, $y = 0$ and $dy/dx = 1$. Give a reason for your answer.

26. Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

27. A particle moves along the x-axis. Its acceleration is $a = -t^2$. At $t = 0$, the particle is at the origin. In the course of its motion, it reaches the point $x = b$, where $b > 0$, but no point beyond b. Determine its velocity at $t = 0$.

28. A particle moves with acceleration $a = \sqrt{t} - \left(1/\sqrt{t}\right)$. Assuming that the velocity $v = 4/3$ and the position $s = -4/15$ when $t = 0$, find

a. the velocity v in terms of t.

b. the position s in terms of t.

29. Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \ge 0$ for all real x if and only if $b^2 - ac \le 0$.

30. Schwarz's inequality

a. In Exercise 29, let

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

and deduce Schwarz's inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$
$$\le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

b. Show that equality holds in Schwarz's inequality only if there exists a real number x that makes a_ix equal $-b_i$ for every value of i from 1 to n.

31. Marginal revenue A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = \left[3 - (x/40)\right]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue.)

32. Industrial production

a. Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

b. Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

33. Motion of a particle The position at time $t \ge 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

a. What is the particle's starting position ($t = 0$)?

b. What are the points farthest to the left and right of the origin reached by the particle?

c. Find the particle's velocity and acceleration at the points in part (b).

d. When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

34. Shooting a paper clip On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In t sec after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

a. How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

b. On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

35. Velocities of two particles At time t sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

36. Velocity of a particle A particle of constant mass m moves along the x-axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m(v^2 - v_0{}^2) = \frac{1}{2}k(x_0{}^2 - x^2),$$

where k, v_0, and x_0 are constants. Show that whenever $v \neq 0$,

$$m\frac{dv}{dt} = -kx.$$

37. Average and instantaneous velocity

a. Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

b. What is the geometric significance of the result in part (a)?

38. The period of a clock pendulum The period T of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where T is measured in seconds, $g = 32.2$ ft/sec^2, and L, the length of the pendulum, is measured in feet. Find approximately

a. the length of a clock pendulum whose period is $T = 1$ sec.

b. the change dT in T if the pendulum in part (a) is lengthened 0.01 ft.

c. the amount the clock gains or loses in a day as a result of the period's changing by the amount dT found in part (b).

39. The melting ice cube Assume that an ice cube retains its cubical shape as it melts. If we call its edge length s, its volume is $V = s^3$ and its surface area is $6s^2$. We assume that V and s are differentiable functions of time t. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost $1/4$ of its volume during the first hour, and that the volume is V_0 when $t = 0$. How long will it take the ice cube to melt?

5

Integrals

OVERVIEW A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method to calculate the areas of very general shapes. This method, called *integration*, is a way to calculate much more than areas and volumes. The *definite integral* is the key tool in calculus for defining and calculating many important quantities, such as areas, volumes, lengths of curved paths, probabilities, averages, energy consumption, the weights of various objects, and the forces against a dam's floodgates, just to mention a few.

As with the derivative, the definite integral also arises as a limit, this time of increasingly fine approximations to the quantity of interest. The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces, and then summing the contributions from each piece. We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process. As the number of terms contributing to the sum approaches infinity and we take the limit of these sums in a way described in Section 5.3, the result is a definite integral. By considering the rate of change of the area under a graph, we prove that definite integrals are connected to antiderivatives, a connection that gives one of the most important relationships in calculus.

5.1 Antiderivatives

We have studied how to find the derivative of a function and how to use it to solve a wide range of problems. However, many other problems require that we recover a function from its known derivative (from its known rate of change). For instance, the laws of physics tell us the acceleration of an object falling from an initial height, and we can use this to compute its velocity and its height at any time. More generally, starting with a function f, we want to find a function F whose derivative is f. If such a function F exists, it is called an *antiderivative* of f.

Finding Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \sec^2 x + \dfrac{1}{2\sqrt{x}}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = \tan x + \sqrt{x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \tan x + \sqrt{x}$ is $\sec^2 x + \left(1/2\sqrt{x}\right)$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.5 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

THEOREM 1 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Thus the most general antiderivative of f on I is a *family* of functions $F(x) + C$ whose graphs are vertical translations of one another. We can select a particular antiderivative from this family by assigning a specific value to C. Here is an example showing how such an assignment might be made.

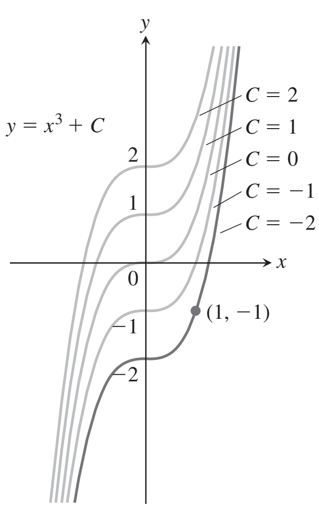


FIGURE 5.1 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for C selects the particular curve from the family of curves $y = x^3 + C$ that passes through the point $(1, -1)$ in the plane (Figure 5.1). ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant C in the general expression representing all antiderivatives of a given function. Table 5.1 gives antiderivative formulas for a number of important functions.

TABLE 5.1 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

The rules in Table 5.1 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $(\tan kx)/k + C$ is $\sec^2 kx$, whatever the value of the constants C or $k \neq 0$, and this establishes Formula 4 for the most general antiderivative of $\sec^2 kx$.

EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \sqrt{x}$

(c) $h(x) = \sin 2x$ **(d)** $i(x) = \cos \frac{x}{2}$

Solution In each case, we can use one of the formulas listed in Table 5.1.

(a) $F(x) = \frac{x^6}{6} + C$ Formula 1 with $n = 5$

(b) $g(x) = x^{1/2}$, so

$$G(x) = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$$ Formula 1 with $n = 1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin\frac{x}{2} + C$ Formula 3 with $k = 1/2$ ■

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives and multiply them by constants.

TABLE 5.2 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule*:	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule*:	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule*:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

The formulas in Table 5.2 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function. Formula 2 is the special case $k = -1$ in Formula 1.

EXAMPLE 4 Find the general antiderivative of

$$f(x) = 3\sqrt{x} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2x^{3/2}/3$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3\left(2x^{3/2}/3\right) = 2x^{3/2}$ is an antiderivative of $3g(x) = 3\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 2x^{3/2} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Initial Value Problems and Differential Equations

Antiderivatives play several important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus. Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science.

The most general antiderivative $F(x) + C$ (such as $x^3 + C$ in Example 2) of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$. In Example 2, the function $y = x^3 - 2$ is the particular solution of the differential equation $dy/dx = 3x^2$ satisfying the initial condition $y(1) = -1$.

Antiderivatives and Motion

We have seen that the derivative of the position function of an object gives its velocity, and the derivative of its velocity function gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an

antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.5. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

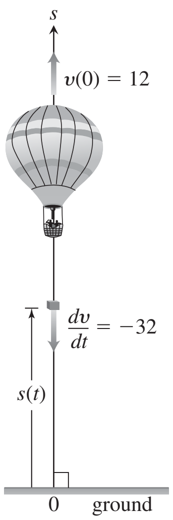


FIGURE 5.2 A package dropped from a rising hot-air balloon (Example 5).

EXAMPLE 5 A hot-air balloon ascending at the rate of 12 ft/sec is at a height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution Let $v(t)$ denote the velocity of the package at time t, and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 32 ft/sec^2. Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -32. \qquad \text{Negative because gravity acts in the direction of decreasing } s$$

This leads to the following initial value problem (Figure 5.2):

$$\begin{aligned} &\textit{Differential equation:} && \frac{dv}{dt} = -32 \\ &\textit{Initial condition:} && v(0) = 12. && \text{Balloon initially rising} \end{aligned}$$

This is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation*: The general formula for an antiderivative of -32 is

$$v = -32t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C*:

$$\begin{aligned} 12 &= -32(0) + C && \text{Initial condition } v(0) = 12 \\ C &= 12. \end{aligned}$$

The solution of the initial value problem is

$$v = -32t + 12.$$

Since velocity is the derivative of height, and the height of the package is 80 ft at time $t = 0$ when it is dropped, we now have a second initial value problem:

$$\begin{aligned} &\textit{Differential equation:} && \frac{ds}{dt} = -32t + 12 && \text{Set } v = ds/dt \text{ in the previous equation.} \\ &\textit{Initial condition:} && s(0) = 80. \end{aligned}$$

We solve this initial value problem to find the height as a function of t.

1. *Solve the differential equation*: Finding the general antiderivative of $-32t + 12$ gives

$$s = -16t^2 + 12t + C.$$

2. *Evaluate C*:

$$\begin{aligned} 80 &= -16(0)^2 + 12(0) + C && \text{Initial condition } s(0) = 80 \\ C &= 80. \end{aligned}$$

The package's height above ground at time t is

$$s = -16t^2 + 12t + 80.$$

Use the solution: To find how long it takes the package to reach the ground, we set s equal to 0 and solve for t:

$$\begin{aligned} -16t^2 + 12t + 80 &= 0 \\ -4t^2 + 3t + 20 &= 0 \\ t &= \frac{-3 \pm \sqrt{329}}{-8} \qquad \text{Quadratic formula} \\ t \approx -1.89, \qquad t &\approx 2.64. \end{aligned}$$

The package hits the ground about 2.64 sec after it is dropped from the balloon. (The negative root has no physical meaning.) ■

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int \left(\sec^2 x + \frac{1}{2\sqrt{x}}\right) dx = \tan x + \sqrt{x} + C$$

This notation is related to the main application of antiderivatives. Antiderivatives play a key role in computing limits of certain infinite sums, an unexpected and wonderfully useful role that is described in a central result called the Fundamental Theorem of Calculus.

EXAMPLE 6 Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine $C_1, -2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the possible antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end. ■

Exercises 5.1

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$

2. a. $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$

3. a. $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$

4. a. $2x^{-3}$ **b.** $\frac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$

5. a. $\frac{1}{x^2}$ **b.** $\frac{5}{x^2}$ **c.** $2 - \frac{5}{x^2}$

6. a. $-\frac{2}{x^3}$ **b.** $\frac{1}{2x^3}$ **c.** $x^3 - \frac{1}{x^3}$

7. a. $\frac{3}{2}\sqrt{x}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \frac{1}{\sqrt{x}}$

8. a. $\frac{4}{3}\sqrt[3]{x}$ **b.** $\frac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

9. a. $\frac{2}{3}x^{-1/3}$ **b.** $\frac{1}{3}x^{-2/3}$ **c.** $-\frac{1}{3}x^{-4/3}$

10. a. $x^{\sqrt{3}}$ **b.** x^{π} **c.** $x^{\sqrt{2}-1}$

11. a. $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$

12. a. $\pi \cos \pi x$ **b.** $\frac{\pi}{2}\cos\frac{\pi x}{2}$ **c.** $\cos\frac{\pi x}{2} + \pi \cos x$

13. a. $\frac{1}{2}\sec^2 x$ **b.** $\frac{2}{3}\sec^2\frac{x}{3}$ **c.** $-\sec^2\frac{3x}{2}$

14. a. $\csc^2 x$ **b.** $-\frac{3}{2}\csc^2\frac{3x}{2}$ **c.** $1 - 8\csc^2 2x$

15. a. $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc\frac{\pi x}{2}\cot\frac{\pi x}{2}$

16. a. $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec\frac{\pi x}{2}\tan\frac{\pi x}{2}$

Finding Indefinite Integrals

In Exercises 17–56, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

17. $\int (x + 1)\,dx$ **18.** $\int (5 - 6x)\,dx$

19. $\int \left(3t^2 + \frac{t}{2}\right)dt$ **20.** $\int \left(\frac{t^2}{2} + 4t^3\right)dt$

21. $\int (2x^3 - 5x + 7)\,dx$ **22.** $\int (1 - x^2 - 3x^5)\,dx$

23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right)dx$ **24.** $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right)dx$

25. $\int x^{-1/3}\,dx$ **26.** $\int x^{-5/4}\,dx$

27. $\int (\sqrt{x} + \sqrt[3]{x})\, dx$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

29. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

30. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

31. $\int 2x(1 - x^{-3})\, dx$

32. $\int x^{-3}(x + 1)\, dx$

33. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$

34. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$

35. $\int (-2 \cos t)\, dt$

36. $\int (-5 \sin t)\, dt$

37. $\int 7 \sin \frac{\theta}{3}\, d\theta$

38. $\int 3 \cos 5\theta\, d\theta$

39. $\int (-3 \csc^2 x)\, dx$

40. $\int \left(-\frac{\sec^2 x}{3}\right) dx$

41. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$

42. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$

43. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

44. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$

45. $\int (\sin 2x - \csc^2 x)\, dx$

46. $\int (2 \cos 2x - 3 \sin 3x)\, dx$

47. $\int \frac{1 + \cos 4t}{2}\, dt$

48. $\int \frac{1 - \cos 6t}{2}\, dt$

49. $\int 3x^{\sqrt{3}}\, dx$

50. $\int x^{\sqrt{2}-1}\, dx$

51. $\int (1 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

52. $\int (2 + \tan^2 \theta)\, d\theta$

53. $\int \cot^2 x\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

54. $\int (1 - \cot^2 x)\, dx$

55. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

56. $\int \frac{\csc \theta}{\csc \theta - \sin \theta}\, d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 57–62 by differentiation.

57. $\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

58. $\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

59. $\int \sec^2 (5x - 1)\, dx = \frac{1}{5} \tan (5x - 1) + C$

60. $\int \csc^2\left(\frac{x - 1}{3}\right) dx = -3 \cot\left(\frac{x - 1}{3}\right) + C$

61. $\int \frac{1}{(x + 1)^2}\, dx = -\frac{1}{x + 1} + C$

62. $\int \frac{1}{(x + 1)^2}\, dx = \frac{x}{x + 1} + C$

63. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int x \sin x\, dx = \frac{x^2}{2} \sin x + C$

b. $\int x \sin x\, dx = -x \cos x + C$

c. $\int x \sin x\, dx = -x \cos x + \sin x + C$

64. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{\sec^3 \theta}{3} + C$

b. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \tan^2 \theta + C$

c. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \sec^2 \theta + C$

65. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int (2x + 1)^2\, dx = \frac{(2x + 1)^3}{3} + C$

b. $\int 3(2x + 1)^2\, dx = (2x + 1)^3 + C$

c. $\int 6(2x + 1)^2\, dx = (2x + 1)^3 + C$

66. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x + C}$

b. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x} + C$

c. $\int \sqrt{2x + 1}\, dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

67. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4}\, dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

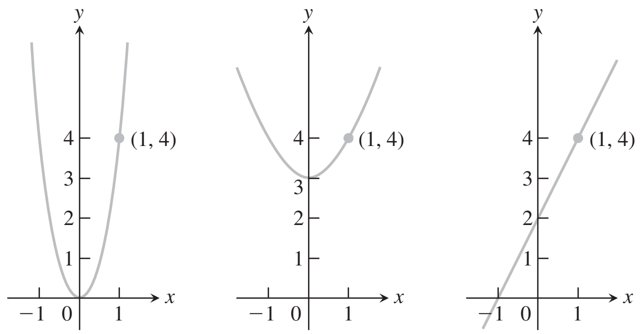
68. Right, or wrong? Give a brief reason why.

$$\int \frac{x \cos (x^2) - \sin (x^2)}{x^2}\, dx = \frac{\sin (x^2)}{x} + C$$

Initial Value Problems

69. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

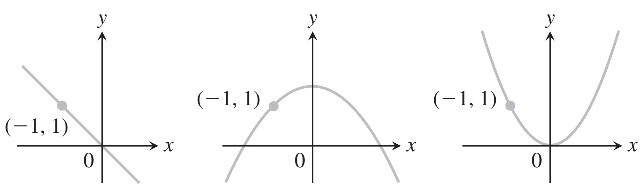


(a) (b) (c)

Give reasons for your answer.

70. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 71–90.

71. $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$ **72.** $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

73. $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

74. $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

75. $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

76. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

77. $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

78. $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

79. $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

80. $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$ **81.** $\dfrac{dv}{dt} = \dfrac{1}{2}\sec t \tan t, \quad v(0) = 1$

82. $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

83. $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

84. $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

85. $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

86. $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

87. $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

88. $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

89. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

90. $y^{(4)} = -\cos x + 8\sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

91. Find the curve $y = f(x)$ in the xy-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

92. a. Find a curve $y = f(x)$ with the following properties:

i) $\dfrac{d^2y}{dx^2} = 6x$

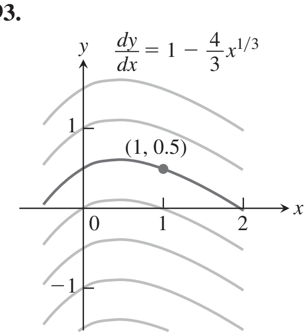
ii) Its graph passes through the point (0, 1) and has a horizontal tangent there.

b. How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 93–96 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

93.



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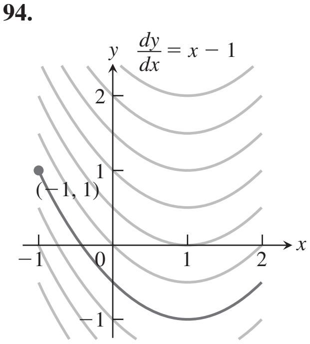

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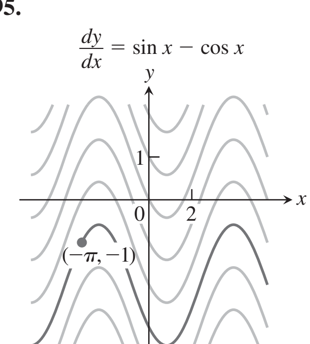

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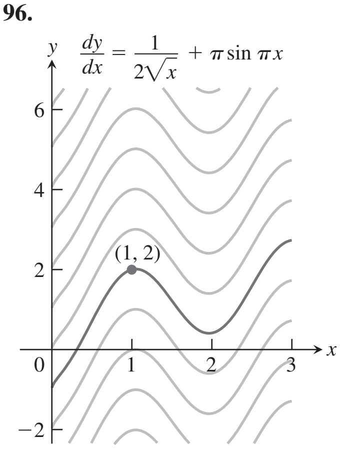


Applications

97. Finding displacement from an antiderivative of velocity

a. Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

98. Liftoff from Earth A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

99. Stopping a car in time You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

1. Solve the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = -k \qquad (k \text{ constant})$

Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

3. Find the value of k that makes $s = 242$ for the value of t you found in Step 2.

100. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

101. Motion along a coordinate line A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

a. the velocity $v = ds/dt$ in terms of t

b. the position s in terms of t.

T **102. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\dfrac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$

Initial conditions: $\dfrac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

103. Motion with constant acceleration The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \tag{1}$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = a$

Initial conditions: $\dfrac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

104. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/sec^2, Equation (1) in Exercise 103 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \tag{2}$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 103, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

105. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\displaystyle\int f(x)\,dx$ **b.** $\displaystyle\int g(x)\,dx$

c. $\displaystyle\int [-f(x)]\,dx$ **d.** $\displaystyle\int [-g(x)]\,dx$

e. $\displaystyle\int [f(x) + g(x)]\,dx$ **f.** $\displaystyle\int [f(x) - g(x)]\,dx$

106. Uniqueness of solutions If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

5.2 Area and Estimating with Finite Sums

The basis for formulating definite integrals is the construction of appropriate approximations by finite sums. In this section we consider three examples of this construction process: finding the area under a graph, the distance traveled by a moving object, and the average value of a function. Although we need to define precisely what we mean by the area of a general region in the plane, or the average value of a function over a closed interval, we do have intuitive ideas of what these notions mean. So in this section we begin our approach to integration by *approximating* these quantities with finite sums. We also consider what happens when we take more and more terms in the summation process. In subsequent sections we look at taking the limit of these sums as the number of terms goes to infinity, which then leads to precise definitions of the quantities being approximated here.

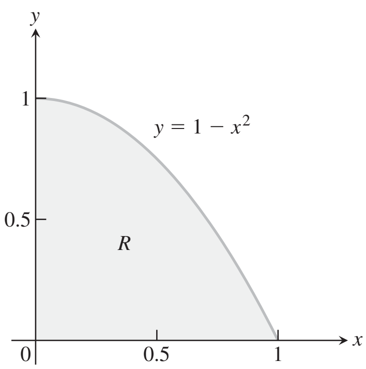


FIGURE 5.3 The area of the region R cannot be found by a simple formula.

Area

Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (Figure 5.3). Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region R. How, then, can we find the area of R?

While we do not yet have a method for determining the exact area of R, we can approximate it in a simple way. Figure 5.4a shows two rectangles that together contain the region R. Each rectangle has width $1/2$ and they have heights 1 and $3/4$, moving from left to right. The height of each rectangle is the maximum value of the function f in each subinterval. Because the function f is decreasing, the height is its value at the left endpoint of the subinterval of $[0, 1]$ forming the base of the rectangle. The total area of the two rectangles approximates the area A of the region R,

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

This estimate is larger than the true area A since the two rectangles contain R. We say that 0.875 is an **upper sum** because it is obtained by taking the height of each rectangle as the maximum (uppermost) value of $f(x)$ for a point x in the base interval of the rectangle. In Figure 5.4b, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region R. These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

which is still greater than A since the four rectangles contain R.

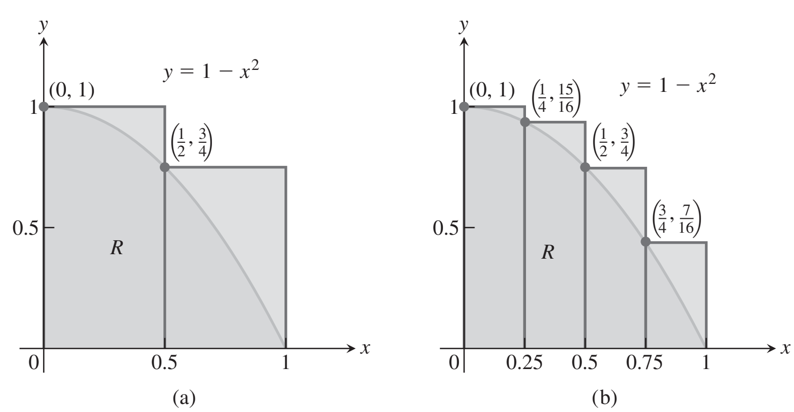


FIGURE 5.4 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

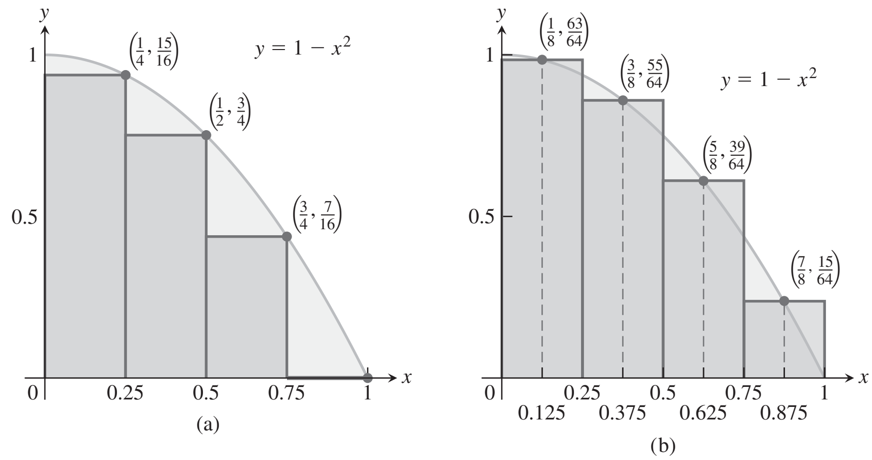


FIGURE 5.5 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

Suppose instead we use four rectangles contained *inside* the region R to estimate the area, as in Figure 5.5a. Each rectangle has width $1/4$ as before, but the rectangles are shorter and lie entirely beneath the graph of f. The function $f(x) = 1 - x^2$ is decreasing on $[0, 1]$, so the height of each of these rectangles is given by the value of f at the right endpoint of the subinterval forming its base. The fourth rectangle has zero height and therefore contributes no area. Summing these rectangles with heights equal to the minimum value of $f(x)$ for a point x in each base subinterval gives a **lower sum** approximation to the area,

$$A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125.$$

This estimate is smaller than the area A since the rectangles all lie inside of the region R. The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

By considering both lower and upper sum approximations, we get not only estimates for the area, but also a bound on the size of the possible error in these estimates, since the true value of the area lies somewhere between them. Here the error cannot be greater than the difference $0.78125 - 0.53125 = 0.25$.

Yet another estimate can be obtained by using rectangles whose heights are the values of f at the midpoints of their bases (Figure 5.5b). This method of estimation is called the **midpoint rule** for approximating the area. The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it is not quite so clear whether it overestimates or underestimates the true area. With four rectangles of width $1/4$ as before, the midpoint rule estimates the area of R to be

$$A \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = \frac{172}{64} \cdot \frac{1}{4} = 0.671875.$$

In each of our computed sums, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (also called length) $\Delta x = (b - a)/n$, and f was evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on. The finite sums then all take the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

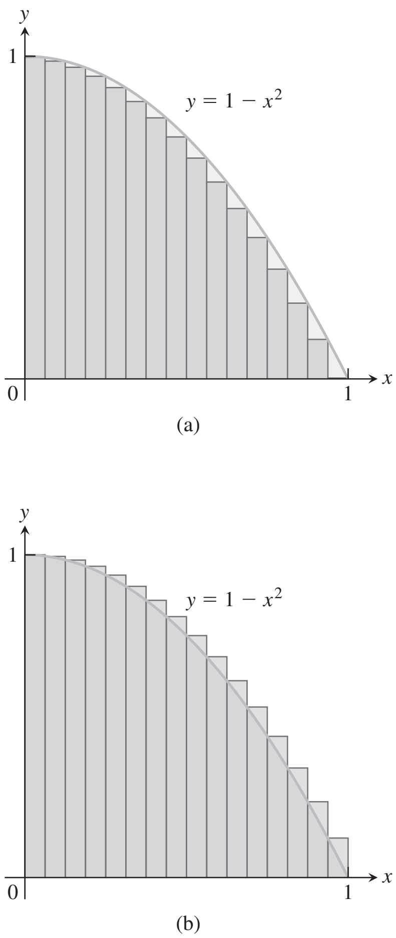


FIGURE 5.6 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

By taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region R.

Figure 5.6a shows a lower sum approximation for the area of R using 16 rectangles of equal width. The sum of their areas is 0.634765625, which appears close to the true area, but is still smaller since the rectangles lie inside R.

Figure 5.6b shows an upper sum approximation using 16 rectangles of equal width. The sum of their areas is 0.697265625, which is somewhat larger than the true area because the rectangles taken together contain R. The midpoint rule for 16 rectangles gives a total area approximation of 0.6669921875, but it is not immediately clear whether this estimate is larger or smaller than the true area.

EXAMPLE 1 Table 5.3 shows the values of upper and lower sum approximations to the area of R, using up to 1000 rectangles. In Section 5.3 we will see how to get an exact value of the areas of regions such as R by taking a limit as the base width of each rectangle goes to zero and the number of rectangles goes to infinity. With the techniques developed there, we will be able to show that the area of R is exactly $2/3$. ■

TABLE 5.3 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Distance Traveled

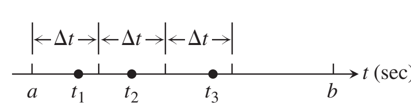
Suppose we know the velocity function $v(t)$ of a car moving down a highway, without changing direction, and want to know how far it traveled between times $t = a$ and $t = b$. The position function $s(t)$ of the car has derivative $v(t)$. If we can find an antiderivative $F(t)$ of $v(t)$ then we can find the car's position function $s(t)$ by setting $s(t) = F(t) + C$. The distance traveled can then be found by calculating the change in position, $s(b) - s(a) = F(b) - F(a)$. If the velocity function is known only by the readings at various times of a speedometer on the car, then we have no formula from which to obtain an antiderivative function for velocity. So what do we do in this situation?

When we don't know an antiderivative for the velocity function $v(t)$, we can approximate the distance traveled with finite sums in a way similar to our estimates for area discussed before. We subdivide the interval $[a, b]$ into short time intervals on each of which the velocity is considered to be fairly constant. Then we approximate the distance traveled on each time subinterval with the usual distance formula

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the results across $[a, b]$.

Suppose the subdivided interval looks like



with the subintervals all of equal length Δt. Pick a number t_1 in the first interval. If Δt is so small that the velocity barely changes over a short time interval of duration Δt, then the distance traveled in the first time interval is about $v(t_1)\,\Delta t$. If t_2 is a number in the second interval, the distance traveled in the second time interval is about $v(t_2)\,\Delta t$. The sum of the distances traveled over all the time intervals is

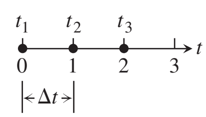
$$D \approx v(t_1)\,\Delta t + v(t_2)\,\Delta t + \cdots + v(t_n)\,\Delta t,$$

where n is the total number of subintervals.

EXAMPLE 2 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m? (You will learn how to compute the exact value easily in Section 5.14.)

Solution We explore the results for different numbers of intervals and different choices of evaluation points. Notice that $f(t)$ is decreasing, so choosing left endpoints gives an upper sum estimate; choosing right endpoints gives a lower sum estimate.

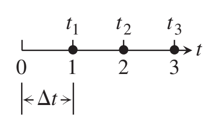
(a) *Three subintervals of length* 1, *with* f *evaluated at left endpoints giving an upper sum*:



With f evaluated at $t = 0$, 1, and 2, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &= 450.6. \end{aligned}$$

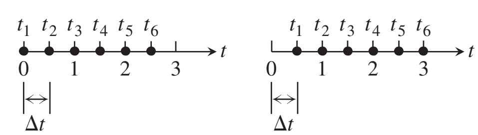
(b) *Three subintervals of length* 1, *with* f *evaluated at right endpoints giving a lower sum*:



With f evaluated at $t = 1$, 2, and 3, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &= 421.2. \end{aligned}$$

(c) *With six subintervals of length* $1/2$, *we get*



These estimates give an upper sum using left endpoints: $D \approx 443.25$; and a lower sum using right endpoints: $D \approx 428.55$. These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

As we can see in Table 5.4, the left-endpoint upper sums approach the true value 435.9 from above, whereas the right-endpoint lower sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entries is 0.23, a small percentage of the true value.

TABLE 5.4 Travel-distance estimates

Number of subintervals	Length of each subinterval	Upper sum	Lower sum
3	1	450.6	421.2
6	1/2	443.25	428.55
12	1/4	439.58	432.23
24	1/8	437.74	434.06
48	1/16	436.82	434.98
96	1/32	436.36	435.44
192	1/64	436.13	435.67

$$\begin{aligned}\text{Error magnitude} &= \left|\text{true value} - \text{calculated value}\right| \\ &= \left|435.9 - 435.67\right| = 0.23.\end{aligned}$$

$$\text{Error percentage} = \frac{0.23}{435.9} \approx 0.05\%.$$

It would be reasonable to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ■

Displacement Versus Distance Traveled

If an object with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by summing the distance traveled over small intervals, as in Example 2. If the object reverses direction one or more times during the trip, then we need to use the object's *speed* $|v(t)|$, which is the absolute value of its velocity function, $v(t)$, to find the total distance traveled. Using the velocity itself, as in Example 2, gives instead an estimate to the object's **displacement**, $s(b) - s(a)$, the difference between its initial and final positions.

To see why using the velocity function in the summation process gives an estimate to the displacement, partition the time interval $[a, b]$ into small enough equal subintervals Δt so that the object's velocity does not change very much from time t_{k-1} to t_k. Then $v(t_k)$ gives a good approximation of the velocity throughout the interval. Accordingly, the change in the object's position coordinate, which is its displacement during the time interval, is about

$$v(t_k)\,\Delta t.$$

The change is positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled by the object during the subinterval is about

$$|v(t_k)|\,\Delta t.$$

The **total distance traveled** over the time interval is approximately the sum

$$|v(t_1)|\Delta t + |v(t_2)|\Delta t + \cdots + |v(t_n)|\,\Delta t.$$

We revisit these ideas in Section 5.14.

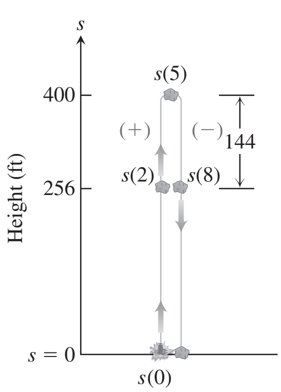


FIGURE 5.7 The rock in Example 3. The height $s = 256$ ft is reached at $t = 2$ and $t = 8$ sec. The rock falls 144 ft from its maximum height when $t = 8$.

EXAMPLE 3 In Example 4 in Section 3.4, we analyzed the motion of a heavy rock blown straight up by a dynamite blast. In that example, we found the velocity of the rock at any time during its motion to be $v(t) = 160 - 32t$ ft/sec. The rock was 256 ft above the ground 2 sec after the explosion, continued upward to reach a maximum height of 400 ft at 5 sec after the explosion, and then fell back down to reach the height of 256 ft again at $t = 8$ sec after the explosion. (See Figure 5.7.)

TABLE 5.5 Velocity function

t	$v(t)$	t	$v(t)$
0	160	4.5	16
0.5	144	5.0	0
1.0	128	5.5	-16
1.5	112	6.0	-32
2.0	96	6.5	-48
2.5	80	7.0	-64
3.0	64	7.5	-80
3.5	48	8.0	-96
4.0	32		

If we follow a procedure like that presented in Example 2, and use the velocity function $v(t)$ in the summation process over the time interval $[0, 8]$, we will obtain an estimate to the rock's 256 ft *height* above the ground at $t = 8$. The positive upward motion (which yields a positive distance change of 144 ft from the height of 256 ft to the maximum height) is canceled by the negative downward motion (giving a negative change of 144 ft from the maximum height down to 256 ft again), so the displacement or height above the ground is estimated from the velocity function.

On the other hand, if the absolute value $|v(t)|$ is used in the summation process, we will obtain an estimate to the *total distance* the rock has traveled: the maximum height reached of 400 ft plus the additional distance of 144 ft it has fallen back down from that maximum when it again reaches the height of 256 ft at $t = 8$ sec. That is, using the absolute value of the velocity function in the summation process over the time interval $[0, 8]$, we obtain an estimate to 544 ft, the total distance up and down that the rock has traveled in 8 sec. There is no cancelation of distance changes due to sign changes in the velocity function, so we estimate distance traveled rather than displacement when we use the absolute value of the velocity function (that is, the speed of the rock).

As an illustration of our discussion, we subdivide the interval $[0, 8]$ into sixteen subintervals of length $\Delta t = 1/2$ and take the right endpoint of each subinterval in our calculations. Table 5.5 shows the values of the velocity function at these endpoints.

Using $v(t)$ in the summation process, we estimate the displacement at $t = 8$:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16$$
$$+ 0 - 16 - 32 - 48 - 64 - 80 - 96) \cdot \frac{1}{2} = 192$$
$$\text{Error magnitude} = 256 - 192 = 64$$

Using $|v(t)|$ in the summation process, we estimate the total distance traveled over the time interval $[0, 8]$:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16$$
$$+ 0 + 16 + 32 + 48 + 64 + 80 + 96) \cdot \frac{1}{2} = 528$$
$$\text{Error magnitude} = 544 - 528 = 16$$

If we take more and more subintervals of $[0, 8]$ in our calculations, the estimates to these values 256 ft and 544 ft improve, approaching them as shown in Table 5.6. ■

TABLE 5.6 Travel estimates for a rock blown straight up during the time interval $[0, 8]$

Number of subintervals	Length of each subinterval	Displacement	Total distance
16	1/2	192.0	528.0
32	1/4	224.0	536.0
64	1/8	240.0	540.0
128	1/16	248.0	542.0
256	1/32	252.0	543.0
512	1/64	254.0	543.5

Average Value of a Nonnegative Continuous Function

The average value of a collection of n numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by n. But what is the average value of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the

temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

When a function is constant, this question is easy to answer. A function with constant value c on an interval $[a, b]$ has average value c. When c is positive, its graph over $[a, b]$ gives a rectangle of height c. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (Figure 5.8a).

What if we want to find the average value of a nonconstant function, such as the function g in Figure 5.8b? We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height c equals the area under the graph of g divided by $b - a$. We are led to *define* the average value of a nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.4, but for now we look at an example.

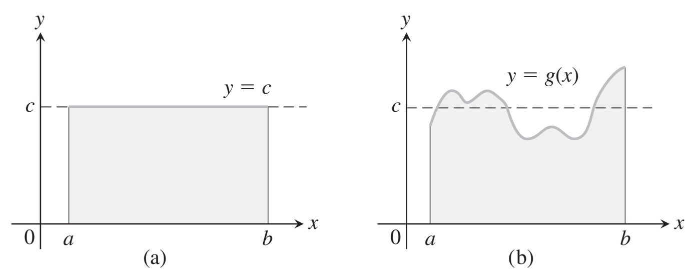


FIGURE 5.8 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

EXAMPLE 4 Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

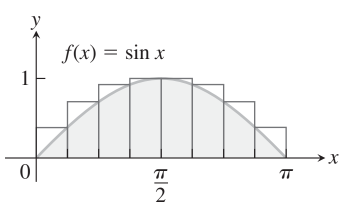


FIGURE 5.9 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 4).

Solution Looking at the graph of $\sin x$ between 0 and π in Figure 5.9, we can see that its average height is somewhere between 0 and 1. To find the average, we need to calculate the area A under the graph and then divide this area by the length of the interval, $\pi - 0 = \pi$.

We do not have a simple way to determine the area, so we approximate it with finite sums. To get an upper sum approximation, we add the areas of eight rectangles of equal width $\pi/8$ that together contain the region beneath the graph of $y = \sin x$ and above the x-axis on $[0, \pi]$. We choose the heights of the rectangles to be the largest value of $\sin x$ on each subinterval. Over a particular subinterval, this largest value may occur at the left endpoint, the right endpoint, or somewhere between them. We evaluate $\sin x$ at this point to get the height of the rectangle for an upper sum. The sum of the rectangular areas then estimates the total area (Figure 5.9):

$$\begin{aligned} A &\approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8} \\ &\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38)\cdot\frac{\pi}{8} = (6.02)\cdot\frac{\pi}{8} \approx 2.364. \end{aligned}$$

To estimate the average value of $\sin x$ on $[0, \pi]$ we divide the estimated area by the length π of the interval and obtain the approximation $2.364/\pi \approx 0.753$.

TABLE 5.7 Average value of $\sin x$ on $0 \le x \le \pi$

Number of subintervals	Upper sum estimate
8	0.75342
16	0.69707
32	0.65212
50	0.64657
100	0.64161
1000	0.63712

Since we used an upper sum to approximate the area, this estimate is greater than the actual average value of $\sin x$ over $[0, \pi]$. If we use more and more rectangles, with each rectangle getting thinner and thinner, we get closer and closer to the true average value as shown in Table 5.7. Using the techniques covered in Section 5.4, we will show that the true average value is $2/\pi \approx 0.63662$.

As before, we could just as well have used rectangles lying under the graph of $y = \sin x$ and calculated a lower sum approximation, or we could have used the midpoint rule. In Section 5.4 we will see that in each case, the approximations are close to the true area if all the rectangles are sufficiently thin. ■

Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function over an interval can all be approximated by finite sums constructed in a certain way. First we subdivide the interval into subintervals, treating some function f as if it were constant over each particular subinterval. Then we multiply the width of each subinterval by the value of f at some point within it, and add these products together. If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the kth subinterval, this process gives a finite sum of the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

The choices for the c_k could maximize or minimize the value of f in the kth subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. The finite sum approximations we looked at improved as we took more subintervals of thinner width.

Exercises 5.2

Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.

b. a lower sum with four rectangles of equal width.

c. an upper sum with two rectangles of equal width.

d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

6. $f(x) = x^3$ between $x = 0$ and $x = 1$.

7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. Distance traveled The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

10. Distance traveled upstream You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

a. left-endpoint values.

b. right-endpoint values.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

11. Length of a road You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

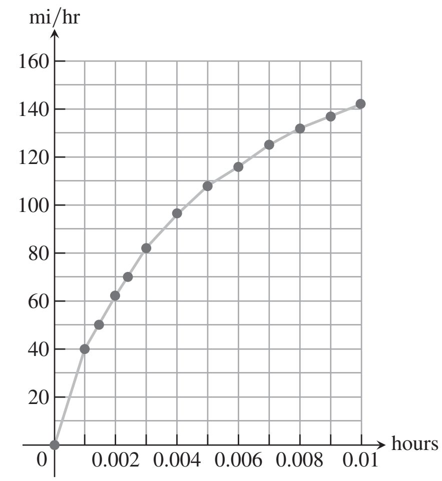
a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

12. Distance from velocity data The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

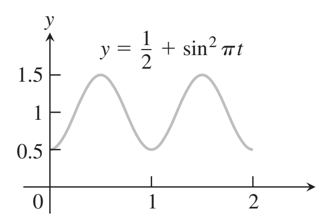
Average Value of a Function

In Exercises 13–16, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

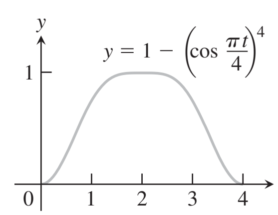
13. $f(x) = x^3$ on $[0, 2]$

14. $f(x) = 1/x$ on $[1, 9]$

15. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



16. $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$



Examples of Estimations

17. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4	5	6	7	8
Leakage (gal/h)	50	70	97	136	190	265	369	516	720

a. Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b. Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c. The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

18. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) **b.** 8 (octagon) **c.** 16

d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

19. (*Continuation of Exercise 18.*)

a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

c. Repeat the computations in parts (a) and (b) for a circle of radius r.

5.3 Sigma Notation and Limits of Finite Sums

In estimating with finite sums in Section 5.2, we encountered sums with many terms (up to 1000 in Table 5.3, for instance). In this section we introduce a more convenient notation for sums with a large number of terms. After describing the notation and stating several of its properties, we look at what happens to a finite sum approximation as the number of terms approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

■

EXAMPLE 2 Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k = 0$ or $k = 1$, but we can start with any integer.

$$\textit{Starting with } k = 0: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{4} (2k + 1)$$

$$\textit{Starting with } k = 1: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5} (2k - 1)$$

$$\textit{Starting with } k = 2: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6} (2k - 3)$$

$$\textit{Starting with } k = -3: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1} (2k + 7)$$

■

When we have a sum such as

$$\sum_{k=1}^{3} (k + k^2)$$

we can rearrange its terms,

$$\begin{aligned} \sum_{k=1}^{3} (k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms.} \\ &= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2. \end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Four such rules are given below.

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. *Difference Rule:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule:* $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)

4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$ Difference Rule and Constant Multiple Rule

(b) $\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1 \cdot \sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$ Constant Multiple Rule

(c) $\sum_{k=1}^{3}(k + 4) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4$ Sum Rule

$= (1 + 2 + 3) + (3 \cdot 4)$ Constant Value Rule

$= 6 + 12 = 18$

(d) $\sum_{k=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$ Constant Value Rule ($1/n$ is constant) ■

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first n integers.

HISTORICAL BIOGRAPHY
Carl Friedrich Gauss
(1777–1855)

EXAMPLE 4 Show that the sum of the first n integers is

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}.$$

Solution The formula tells us that the sum of the first 4 integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n - 1) & + & (n - 2) & + & \cdots & + & 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n - 1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the n columns together we get n terms, each equal to $n + 1$, for a total of $n(n + 1)$. Since this is twice the desired quantity, the sum of the first n integers is $(n)(n + 1)/2$. ■

We state them here.

$$\text{The first } n \text{ squares:} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{The first } n \text{ cubes:} \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Limits of Finite Sums

The finite sum approximations we considered in Section 5.2 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and their number grows to infinity.

EXAMPLE 5 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.6a.)

Solution We compute a lower sum approximation using n rectangles of equal width $\Delta x = (1 - 0)/n$, and then we see what happens as $n \to \infty$. We start by subdividing $[0, 1]$ into n equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$\left[f\left(\frac{1}{n}\right)\right]\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{k}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\left(\frac{1}{n}\right).$$

We write this in sigma notation and simplify,

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) &= \sum_{k=1}^{n}\left(1 - \left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3} && \text{Difference Rule} \\
&= n \cdot \frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 && \text{Constant Value and Constant Multiple Rules} \\
&= 1 - \left(\frac{1}{n^3}\right)\frac{(n)(n+1)(2n+1)}{6} && \text{Sum of the First } n \text{ Squares} \\
&= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}. && \text{Numerator expanded}
\end{aligned}$$

We have obtained an expression for the lower sum that holds for any n. Taking the limit of this expression as $n \to \infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n \to \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^{n} f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region R as this limiting value. In Section 5.4 we study the limits of such finite approximations in a general setting. ■

Riemann Sums

HISTORICAL BIOGRAPHY
Georg Friedrich Bernhard Riemann
(1826–1866)

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral studied in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.10, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between a and b satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

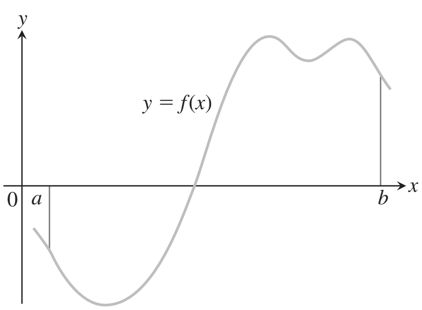


FIGURE 5.10 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

To make the notation consistent, we denote a by x_0 and b by x_n, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set

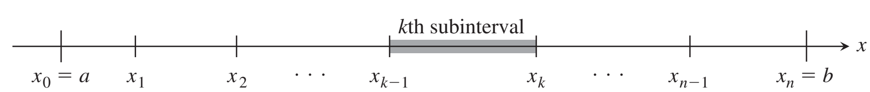
$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

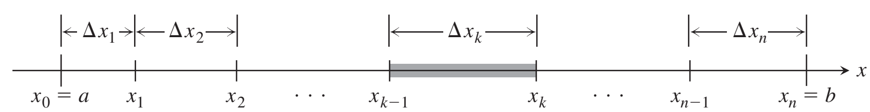
$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the ***k*th subinterval** of P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



HISTORICAL BIOGRAPHY
Richard Dedekind
(1831–1916)

The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (Figure 5.11).

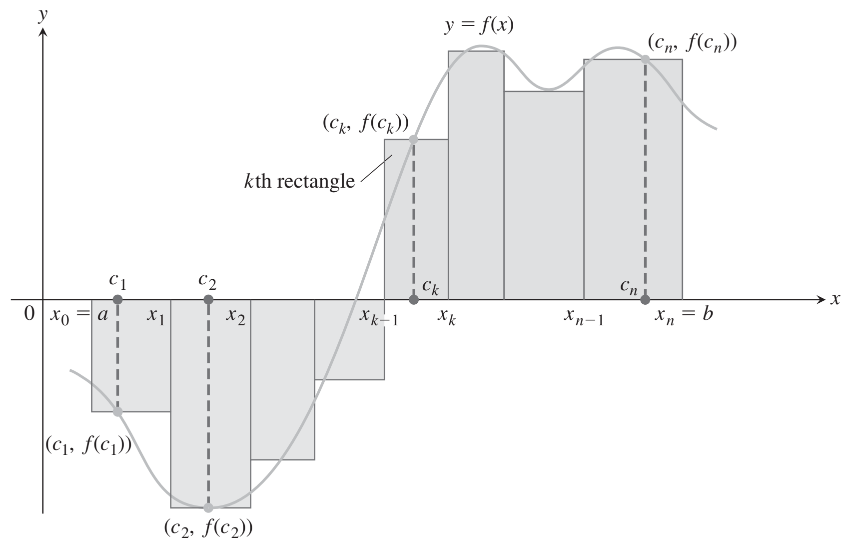


FIGURE 5.11 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.10 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\, \Delta x_k .$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**. There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b - a)}{n}\right) \cdot \left(\frac{b - a}{n}\right).$$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n. When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width. Let's look at an example of these ideas.

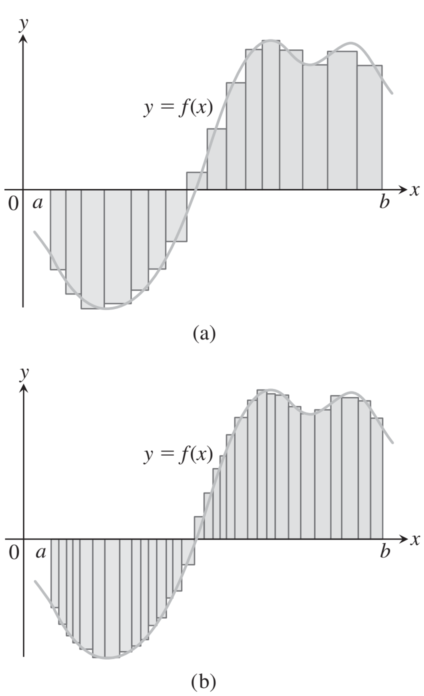
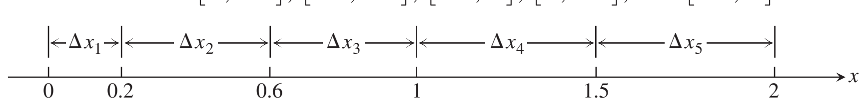


FIGURE 5.12 The curve of Figure 5.11 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.12. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.

Exercises 5.3

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$ **4.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$

5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

a. $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

a. $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a. $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

10. Which formula is not equivalent to the other two?

a. $\displaystyle\sum_{k=1}^{4} (k-1)^2$ **b.** $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ **c.** $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

11. $1 + 2 + 3 + 4 + 5 + 6$ **12.** $1 + 4 + 9 + 16$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ **14.** $2 + 4 + 6 + 8 + 10$

15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ **16.** $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

a. $\displaystyle\sum_{k=1}^{n} 3a_k$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\displaystyle\sum_{k=1}^{n} (a_k + b_k)$

d. $\displaystyle\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\displaystyle\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

a. $\displaystyle\sum_{k=1}^{n} 8a_k$ **b.** $\displaystyle\sum_{k=1}^{n} 250b_k$

c. $\displaystyle\sum_{k=1}^{n} (a_k + 1)$ **d.** $\displaystyle\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. a. $\displaystyle\sum_{k=1}^{10} k$ **b.** $\displaystyle\sum_{k=1}^{10} k^2$ **c.** $\displaystyle\sum_{k=1}^{10} k^3$

20. a. $\displaystyle\sum_{k=1}^{13} k$ **b.** $\displaystyle\sum_{k=1}^{13} k^2$ **c.** $\displaystyle\sum_{k=1}^{13} k^3$

21. $\displaystyle\sum_{k=1}^{7} (-2k)$ **22.** $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\displaystyle\sum_{k=1}^{6} (3 - k^2)$ **24.** $\displaystyle\sum_{k=1}^{6} (k^2 - 5)$

25. $\displaystyle\sum_{k=1}^{5} k(3k + 5)$ **26.** $\displaystyle\sum_{k=1}^{7} k(2k + 1)$

27. $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\displaystyle\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

29. a. $\displaystyle\sum_{k=1}^{7} 3$ **b.** $\displaystyle\sum_{k=1}^{500} 7$ **c.** $\displaystyle\sum_{k=3}^{264} 10$

30. a. $\displaystyle\sum_{k=9}^{36} k$ **b.** $\displaystyle\sum_{k=3}^{17} k^2$ **c.** $\displaystyle\sum_{k=18}^{71} k(k-1)$

31. a. $\displaystyle\sum_{k=1}^{n} 4$ **b.** $\displaystyle\sum_{k=1}^{n} c$ **c.** $\displaystyle\sum_{k=1}^{n} (k-1)$

32. a. $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\displaystyle\sum_{k=1}^{n} \frac{k}{n^2}$

5.4 The Definite Integral

In Section 5.3 we investigated the limit of a finite sum for a function defined over a closed interval $[a, b]$ using n subintervals of equal width (or length), $(b - a)/n$. In this section we consider the limit of more general Riemann sums as the norm of the partitions of $[a, b]$ approaches zero. For general Riemann sums, the subintervals of the partitions need not have equal widths. The limiting process then leads to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Definition of the Definite Integral

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann sums approach a limiting value J. What we mean by this limit is that a Riemann sum will be close to the number J provided that the norm of its partition is sufficiently small (so that all of its subintervals have thin enough widths). We introduce the symbol ϵ as a small positive number that specifies how close to J the Riemann sum must be, and the symbol δ as a second small positive number that specifies how small the norm of a partition must be in order for convergence to happen. We now define this limit precisely.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero.

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J, no matter what choices are made. When the limit exists we write it as the definite integral

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

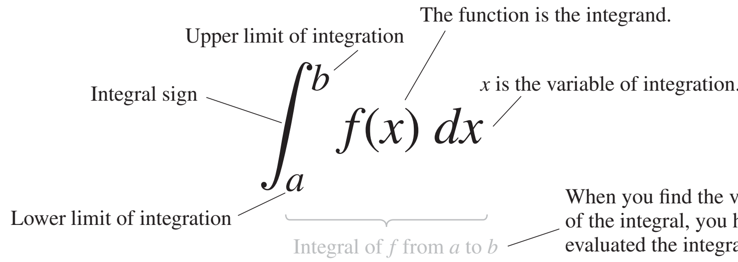
The limit of any Riemann sum is always taken as the norm of the partitions approaches zero and the number of subintervals goes to infinity.

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums. He envisioned the finite sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ becoming an infinite sum of function values $f(x)$ multiplied by "infinitesimal" subinterval widths dx. The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter "S." The function values $f(c_k)$ are replaced by a continuous selection of function values $f(x)$. The subinterval widths Δx_k become the differential dx. It is as if we are summing all products of the form $f(x) \cdot dx$ as x goes from a to b. While this notation captures the process of constructing an integral, it is Riemann's definition that gives a precise meaning to the definite integral.

The symbol for the number J in the definition of the **definite integral** is

$$\int_a^b f(x)\,dx,$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



When the condition in the definition is satisfied, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x)\,dx$ and that f is **integrable** over $[a, b]$.

In the cases where the subintervals all have equal width $\Delta x = (b - a)/n$, we can form each Riemann sum as

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right), \qquad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

where c_k is chosen in the kth subinterval. When the limit of these Riemann sums as $n \to \infty$ exists and is equal to J, then J is the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) \qquad \|P\| \to 0 \text{ means } n \to \infty.$$

If we pick the point c_k at the right endpoint of the kth subinterval, so $c_k = a + k\,\Delta x = a + k(b - a)/n$, then the formula for the definite integral becomes

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \qquad (1)}$$

Equation (1) gives one explicit formula that can be used to compute definite integrals. Other choices of partitions and locations of points c_k result in the same value for the definite integral when we take the limit as $n \to \infty$ provided that the norm of the partition approaches zero.

The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number that is defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a **dummy variable** representing the real numbers in the closed interval $[a, b]$.

Integrable and Nonintegrable Functions

Not every function defined over the closed interval $[a, b]$ is integrable there, even if the function is bounded. That is, the Riemann sums for some functions may not converge to the same limiting value, or to any value at all. A full development of exactly which

functions defined over $[a, b]$ are integrable requires advanced mathematical analysis, but fortunately most functions that commonly occur in applications are integrable. In particular, every *continuous* function over $[a, b]$ is integrable over this interval, and so is every function having no more than a finite number of jump discontinuities on $[a, b]$. The following theorem, which is proved in more advanced courses, establishes these results.

> **THEOREM 2—Integrability of Continuous Functions** If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

The idea behind Theorem 1 for continuous functions is given in Exercises 86 and 87. Briefly, when f is continuous we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles. Our first example shows a function that is not integrable over a closed interval.

EXAMPLE 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there is both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x-axis to be approximated by rectangles, no matter how thin they are. We show, in fact, that upper sum approximations and lower sum approximations converge to different limiting values.

If we pick a partition P of $[0, 1]$ and choose c_k to be the point giving the maximum value for f on $[x_{k-1}, x_k]$ then the corresponding Riemann sum is

$$U = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (1)\,\Delta x_k = 1,$$

since each subinterval $[x_{k-1}, x_k]$ contains a rational number where $f(c_k) = 1$. Note that the lengths of the intervals in the partition sum to 1, $\sum_{k=1}^{n} \Delta x_k = 1$. So each such Riemann sum equals 1, and a limit of Riemann sums using these choices equals 1.

On the other hand, if we pick c_k to be the point giving the minimum value for f on $[x_{k-1}, x_k]$, then the Riemann sum is

$$L = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (0)\,\Delta x_k = 0,$$

since each subinterval $[x_{k-1}, x_k]$ contains an irrational number c_k where $f(c_k) = 0$. The limit of Riemann sums using these choices equals zero. Since the limit depends on the choices of c_k, the function f is not integrable. ■

5.5 Indefinite Integrals and the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. We defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by $\int f(x)\,dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as a derivative.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C. \tag{1}$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$$

which suggests that the simpler expression du can be substituted for $(du/dx)\,dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx}\,dx.$$

EXAMPLE 1 Find the integral $\displaystyle\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)\, dx = \int u^5\, du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\, dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u. \qquad \blacksquare$$

EXAMPLE 2 Find $\int \sqrt{2x + 1}\, dx$.

Solution The integral does not fit the formula

$$\int u^n\, du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx}\, dx = 2\, dx$$

is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by a factor of $1/2$ in front of the integral sign. So we write

$$\int \sqrt{2x + 1}\, dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u} \cdot \underbrace{2\, dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}\, du \qquad \text{Let } u = 2x + 1,\ du = 2\, dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u. \qquad \blacksquare$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 3—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) \qquad \text{Chain Rule}$$

$$= f(g(x)) \cdot g'(x). \qquad F' = f$$

If we make the substitution $u = g(x)$, then

$$\int f(g(x))g'(x)\, dx = \int \frac{d}{dx}F(g(x))\, dx$$

$$
\begin{aligned}
&= F(g(x)) + C \\
&= F(u) + C && u = g(x) \\
&= \int F'(u)\,du \\
&= \int f(u)\,du. && F' = f \qquad \blacksquare
\end{aligned}
$$

The use of the variable u in the Substitution Rule is traditional (sometimes it is referred to as u-substitution), but any letter can be used, such as v, t, θ and so forth. The rule provides a method for evaluating an integral of the form $\int f(g(x))g'(x)\,dx$ given that the conditions of Theorem 3 are satisfied. The primary challenge is deciding what expression involving x you want to substitute for in the integrand. Our examples to follow give helpful ideas.

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

EXAMPLE 3 Find $\int \sec^2(5x + 1) \cdot 5\,dx$

Solution We substitute $u = 5x + 1$ and $du = 5\,dx$. Then,

$$
\begin{aligned}
\int \sec^2(5x + 1) \cdot 5\,dx &= \int \sec^2 u\,du && \text{Let } u = 5x + 1,\ du = 5\,dx. \\
&= \tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \tan(5x + 1) + C. && \text{Substitute } 5x + 1 \text{ for } u. \qquad \blacksquare
\end{aligned}
$$

EXAMPLE 4 Find $\int \cos(7\theta + 3)\,d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$
\begin{aligned}
\int \cos(7\theta + 3)\,d\theta &= \frac{1}{7}\int \cos(7\theta + 3) \cdot 7\,d\theta && \text{Place factor } 1/7 \text{ in front of integral.} \\
&= \frac{1}{7}\int \cos u\,du && \text{Let } u = 7\theta + 3,\ du = 7\,d\theta. \\
&= \frac{1}{7}\sin u + C && \text{Integrate.} \\
&= \frac{1}{7}\sin(7\theta + 3) + C. && \text{Substitute } 7\theta + 3 \text{ for } u.
\end{aligned}
$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7\,d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7)\,du$. Then the integral becomes

$$\int \cos(7\theta + 3)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du \qquad \text{Let } u = 7\theta + 3,\ du = 7\, d\theta, \text{ and } d\theta = (1/7)\, du.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C. \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$. ■

EXAMPLE 5 Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of x. This situation occurs in the following integration.

$$\int x^2 \cos x^3\, dx = \int \cos x^3 \cdot x^2\, dx$$

$$= \int \cos u \cdot \frac{1}{3}\, du \qquad \text{Let } u = x^3,\ du = 3x^2\, dx,\ (1/3)\, du = x^2\, dx.$$

$$= \frac{1}{3}\int \cos u\, du$$

$$= \frac{1}{3}\sin u + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}\sin x^3 + C \qquad \text{Replace } u \text{ by } x^3.$$ ■

HISTORICAL BIOGRAPHY
George David Birkhoff
(1884–1944)

It may happen that an extra factor of x appears in the integrand when we try a substitution $u = g(x)$. In that case, it may be possible to solve the equation $u = g(x)$ for x in terms of u. Replacing the extra factor of x with that expression may then allow for an integral we can evaluate. Here's an example of this situation.

EXAMPLE 6 Evaluate $\displaystyle\int x\sqrt{2x + 1}\, dx$.

Solution Our previous integration in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2\, dx$. Then,

$$\sqrt{2x + 1}\, dx = \frac{1}{2}\sqrt{u}\, du.$$

However, in this case the integrand contains an extra factor of x multiplying the term $\sqrt{2x + 1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ to obtain $x = (u - 1)/2$, and find that

$$x\sqrt{2x + 1}\, dx = \frac{1}{2}(u - 1) \cdot \frac{1}{2}\sqrt{u}\, du.$$

The integration now becomes

$$\int x\sqrt{2x + 1}\, dx = \frac{1}{4}\int (u - 1)\sqrt{u}\, du = \frac{1}{4}\int (u - 1)u^{1/2}\, du \qquad \text{Substitute.}$$

$$= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du \qquad \text{Multiply terms.}$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \qquad \text{Integrate.}$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C. \qquad \text{Replace } u \text{ by } 2x+1. \quad \blacksquare$$

EXAMPLE 7 Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can evaluate using the Substitution Rule.

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

(c) $$\int (1-2\sin^2 x)\sin 2x\,dx = \int (\cos^2 x - \sin^2 x)\sin 2x\,dx$$

$$= \int \cos 2x \sin 2x\,dx \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \int \frac{1}{2}\sin 4x\,dx = \int \frac{1}{8}\sin u\,du \qquad u = 4x,\ du = 4\,dx$$

$$= -\cos 4x + C. \quad \blacksquare$$

EXAMPLE 8 $I = \int (\sin x + \cos x)^5 (\cos x - \sin x)\mathrm{d}x$

Based upon above, we can clearly see put $(\sin x + \cos x) = t$

$\Rightarrow \qquad (\cos x - \sin x)\,\mathrm{d}x = \mathrm{d}t$

$\therefore \qquad I = \int t^5 \mathrm{d}t = \frac{t^6}{6} + C = \frac{(\sin x + \cos x)^6}{6} + C$

We have some special integrals which must be learnt by heart. $\blacksquare$

EXAMPLE 9 $I = \int \tan x\mathrm{d}x$

Now, $\qquad I = \int \frac{\sin x}{\cos x}\mathrm{d}x$

Put $\qquad \cos x = t$

$\Rightarrow \qquad \sin x\,\mathrm{d}x = -\mathrm{d}t$

$\therefore \qquad I = \int \frac{-\mathrm{d}t}{t} = -\ln|t| + \mathrm{C} \Rightarrow \int \tan x\mathrm{d}x = \ln|\sec x| + \mathrm{C}$

Also (2) $I = \int \cot x\mathrm{d}x = \ln|\sin x| + \mathrm{C}$

This integral can be done as above.

(3) $I = \int \sec x \, dx$

In this case, we can multiply the integrand by (sec x + tan x)

$$\therefore \quad I = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

Put $\quad \sec x + \tan x = t$

$$\Rightarrow \quad (\sec x \tan x + \sec^2 x)\, dx = dt$$

$$\therefore \quad I = \int \frac{dt}{t} = \ln|\sec x + \tan x| + C$$

$$\therefore \quad \boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C}$$

Similarly

$$\boxed{\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C}$$ ■

EXAMPLE 10 $\quad I = \int \frac{\sin x}{\sin(x - \alpha)} dx$

Here, put $x - \alpha = t$

$$\Rightarrow \quad dx = dt$$

$$\int \frac{\sin t \cos\alpha + \cos t \sin\alpha}{\sin t} dt = \cos\alpha \int dt + \sin\alpha \int \cot t dt$$

$$= \cos\alpha \times (x - \alpha) + \sin a \ln|\sin(x - \alpha)| + C$$ ■

EXAMPLE 11 $\quad \int \frac{dx}{\sin(x - a)\sin(x - b)}$

Solution $\quad I = \frac{1}{\sin(b - a)} \int \frac{\sin((x - a)(x - b))}{\sin(x - a)\sin(x - b)} dx$

$$= \frac{1}{\sin(b - a)} \int \frac{\sin(x - a)\cos(x - b) - \cos(x - a)\sin(x - b)}{\sin(x - a)\sin(x - b)} dx$$

$$= \frac{1}{\sin(b - a)} \left[\int \cot(x - b)dx - \int \cot(x - a)dx\right]$$

$$= \frac{1}{\sin(b - a)} [\ln|\sin(x - b)| - \ln|\sin(x - a)|] + C$$ ■

EXAMPLE 12 $\quad I = \int \frac{\sec x}{\sqrt{\cos(2x + \alpha) + \cos\alpha}} dx$

Solution $\quad I = \int \frac{\sec x}{\sqrt{2\cos(x + \alpha)\cos x}} dx = \int \frac{\sec x}{\sqrt{2\cos x(\cos x \cos\alpha - \sin x \sin\alpha)}} dx$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^2 x}{\sqrt{(\cos\alpha - \tan x \cdot \sin\alpha)}} dx$$

Put $\quad \cos\alpha - \tan x.\sin\alpha = t^2$

$\Rightarrow \quad -\sec^2 x \sin\alpha \, dx = 2t\, dt$

$\therefore$
$$I = \frac{1}{\sqrt{2}}\int \frac{2t\,dt}{(-\sin\alpha)\cdot t}$$
$$= -\sqrt{2}\operatorname{cosec}\alpha\sqrt{\cos\alpha - \tan x\cdot\sin\alpha} + C$$

EXAMPLE 13 $\quad I = \int \dfrac{dx}{\cos x\cdot\sin^2 x}$

Solution $\quad I = \int \dfrac{\cos^2 x + \sin^2 x}{\cos x\cdot\sin^2 x}dx = \int \dfrac{\cos x}{\sin^2 x}dx + \int \sec x\,dx$

$$= \int \cot x \operatorname{cosec} x\,dx + \int \sec x\,dx$$

$$= -\operatorname{cosec} x + \ln|\sec x \tan x| + C$$

EXAMPLE 14 $\quad I = \int \dfrac{dx}{\sec x - \operatorname{cosec} x}$

Solution $\quad I = \int \dfrac{\sin x\cdot\cos x}{\sin x - \cos x}dx$

$$= \frac{1}{2}\int \frac{1-(\sin x - \cos x)^2}{(\sin x - \cos x)}dx$$

$$= \frac{1}{2}\left[\int \frac{dx}{\sin x - \cos x} - \int \sin x\,dx + \int \cos x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(x - \frac{\pi}{4}\right)} + \cos x + \sin x\right]$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + \frac{1}{2}\ (\cos x + \sin x) + C$$

Trying Different Substitutions

The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. Finding the right substitution gets easier with practice and experience. If the first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand.

EXAMPLE 15 $\quad$ Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case, and both substitutions are successful.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}. \quad \blacksquare$$

Exercises 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx, \quad u = 2x + 4$

2. $\int 7\sqrt{7x-1}\,dx, \quad u = 7x - 1$

3. $\int 2x(x^2+5)^{-4}\,dx, \quad u = x^2 + 5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4 + 1$

5. $\int (3x+2)(3x^2+4x)^4\,dx, \quad u = 3x^2 + 4x$

6. $\int \frac{\left(1+\sqrt{x}\right)^{1/3}}{\sqrt{x}}\,dx, \quad u = 1 + \sqrt{x}$

7. $\int \sin 3x\,dx, \quad u = 3x$

8. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$

9. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$

10. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

16. $\int \frac{dx}{\sqrt{5x+8}}$

a. Using $u = 5x + 8$ **b.** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–50.

17. $\int \sqrt{3-2s}\,ds$

18. $\int \frac{1}{\sqrt{5s+4}}\,ds$

19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

20. $\int 3y\sqrt{7-3y^2}\,dy$

21. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x + 2)\,dx$

24. $\int \tan^2 x \sec^2 x\,dx$

25. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\,dx$

26. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18} - 1\right)^5 dr$

28. $\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2} \sin(x^{3/2} + 1)\,dx$

30. $\int \csc\left(\frac{v - \pi}{2}\right)\cot\left(\frac{v - \pi}{2}\right) dv$

31. $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)}\,dt$

32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2} \cos\left(\frac{1}{t} - 1\right) dt$

34. $\int \frac{1}{\sqrt{t}} \cos\left(\sqrt{t} + 3\right) dt$

35. $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1 + x}}\,dx$

38. $\int \sqrt{\frac{x - 1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2 - \frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2 - 1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3 - 3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3 - 1}}\,dx$

43. $\int x(x - 1)^{10}\,dx$

44. $\int x\sqrt{4 - x}\,dx$

45. $\int (x + 1)^2(1 - x)^5\,dx$

46. $\int (x + 5)(x - 5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2 + 1}\,dx$

48. $\int 3x^5\sqrt{x^3 + 1}\,dx$

49. $\int \frac{x}{(x^2 - 4)^3}\,dx$

50. $\int \frac{x}{(2x - 1)^{2/3}}\,dx$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 51 and 52.

51. $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2}\,dx$
   - **a.** $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$
   - **b.** $u = \tan^3 x$, followed by $v = 2 + u$
   - **c.** $u = 2 + \tan^3 x$

52. $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1)\,dx$
   - **a.** $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$
   - **b.** $u = \sin(x - 1)$, followed by $v = 1 + u^2$
   - **c.** $u = 1 + \sin^2(x - 1)$

Evaluate the integrals in Exercises 53 and 54.

53. $\int \frac{(2r - 1)\cos\sqrt{3(2r - 1)^2 + 6}}{\sqrt{3(2r - 1)^2 + 6}}\,dr$

54. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta \cos^3 \sqrt{\theta}}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 55–60.

55. $\frac{ds}{dt} = 12t(3t^2 - 1)^3, \quad s(1) = 3$

56. $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, \quad y(0) = 0$

57. $\frac{ds}{dt} = 8 \sin^2\left(t + \frac{\pi}{12}\right), \quad s(0) = 8$

58. $\frac{dr}{d\theta} = 3 \cos^2\left(\frac{\pi}{4} - \theta\right), \quad r(0) = \frac{\pi}{8}$

59. $\frac{d^2s}{dt^2} = -4 \sin\left(2t - \frac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

60. $\frac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

61. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6 \sin 2t$ m/sec for all t. If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ sec.

62. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/sec^2 for all t. If $s = 0$ and $v = 8$ m/sec when $t = 0$, find s when $t = 1$ sec.

5.6 Techniques of Integration

Finding antiderivatives (or indefinite integrals) is not as straightforward as finding derivatives. We need to develop some techniques to help us. Nevertheless, we note that it is not always possible to find an antiderivative expressed in terms of elementary functions.

In this chapter we study a number of important techniques which apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we also develop some numerical methods for calculating definite integrals.

Using Basic Integration Formulas

Table 5.8 summarizes the forms of indefinite integrals for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules with algebraic methods and trigonometric identities to help us use Table 5.8.

Sometimes we have to rewrite an integral to match it to a standard form in Table 5.8. We have used this procedure before, but here is another example.

TABLE 5.8 Basic integration formulas

1. $\int k\,dx = kx + C \qquad$ (any number k)
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x > a > 0)$

EXAMPLE 1 Evaluate the integral

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}}\,dx.$$

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.15, to find

$$\begin{aligned}
\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}}\,dx &= \int_1^{11} \frac{du}{\sqrt{u}} && \begin{array}{l} u = x^2 - 3x + 1,\ du = (2x - 3)\,dx; \\ u = 1 \text{ when } x = 3,\ u = 11 \text{ when } x = 5 \end{array} \\
&= \int_1^{11} u^{-1/2}\,du \\
&= 2\sqrt{u}\Big]_1^{11} = 2\left(\sqrt{11} - 1\right) \approx 4.63. && \text{Table 5.8, Formula 2}
\end{aligned}$$

■

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned}
8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\
&= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\
&= \int \frac{du}{\sqrt{a^2 - u^2}} && \begin{array}{l} a = 4,\ u = (x - 4), \\ du = dx \end{array} \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 5.8, Formula 18} \\
&= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.
\end{aligned}$$

■

EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\,dx.$$

Solution Here we can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned}
\int (\cos x \sin 2x + \sin x \cos 2x)\,dx &= \int (\sin (x + 2x))\,dx \\
&= \int \sin 3x\,dx \\
&= \int \frac{1}{3} \sin u\,du && u = 3x,\ du = 3\,dx \\
&= -\frac{1}{3} \cos 3x + C. && \text{Table 5.8, Formula 6}
\end{aligned}$$

■

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$, which is simply a multiplication by a form of the number one. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned}
\int_0^{\pi/4} \frac{dx}{1-\sin x} &= \int_0^{\pi/4} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx \\
&= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\, dx \\
&= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\, dx \\
&= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx \qquad \text{Use Table 5.8, Formulas 8 and 10} \\
&= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \big(1 + \sqrt{2} - (0+1)\big) = \sqrt{2}. \qquad \blacksquare
\end{aligned}$$

EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x+2}\, dx.$$

$$\begin{array}{r}
x \;-\; 3 \\
3x + 2 \,\big)\, \overline{3x^2 - 7x} \\
\underline{3x^2 + 2x} \\
-9x \\
\underline{-9x - 6} \\
+\,6
\end{array}$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x+2} = x - 3 + \frac{6}{3x+2}.$$

Therefore,

$$\int \frac{3x^2-7x}{3x+2}\, dx = \int \left(x - 3 + \frac{6}{3x+2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x+2| + C. \qquad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 5.10.

EXAMPLE 6 Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then we obtain

$$\begin{aligned}3\int \frac{x\,dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du \\ &= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1.\end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1} x + C_2. \qquad \text{Table 5.8, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = -3\sqrt{1 - x^2} + 2\sin^{-1} x + C.$$

■

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the next example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{\left(1 + \sqrt{x}\right)^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but we quickly realize the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $\left(1 + \sqrt{x}\right)$, and it turns out this works:

$$\int \frac{dx}{\left(1 + \sqrt{x}\right)^3} = \int \frac{2(u - 1)\,du}{u^3} \qquad \begin{aligned}&u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;\\ &dx = 2\sqrt{x}\,du = 2(u - 1)\,du\end{aligned}$$

$$= \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$= \frac{1 - 2u}{u^2} + C$$

$$= \frac{1 - 2\left(1 + \sqrt{x}\right)}{\left(1 + \sqrt{x}\right)^2} + C$$

$$= C - \frac{1 + 2\sqrt{x}}{\left(1 + \sqrt{x}\right)^2}.$$

■

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx$.

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx = 0. \qquad \text{Theorem 2, Section 5.15}$$

∎

Exercises 5.6

Assorted Integrations

The integrals in Exercises 1–40 are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate, and then use a substitution to reduce it to a standard form.

1. $\displaystyle\int_0^1 \frac{16x}{8x^2 + 2}\, dx$

2. $\displaystyle\int \frac{x^2}{x^2 + 1}\, dx$

3. $\displaystyle\int (\sec x - \tan x)^2\, dx$

4. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$

5. $\displaystyle\int \frac{1 - x}{\sqrt{1 - x^2}}\, dx$

6. $\displaystyle\int \frac{dx}{x - \sqrt{x}}$

7. $\displaystyle\int \frac{e^{-\cot z}}{\sin^2 z}\, dz$

8. $\displaystyle\int \frac{2^{\ln z^3}}{16z}\, dz$

9. $\displaystyle\int \frac{dz}{e^z + e^{-z}}$

10. $\displaystyle\int_1^2 \frac{8\, dx}{x^2 - 2x + 2}$

11. $\displaystyle\int_{-1}^0 \frac{4\, dx}{1 + (2x + 1)^2}$

12. $\displaystyle\int_{-1}^3 \frac{4x^2 - 7}{2x + 3}\, dx$

13. $\displaystyle\int \frac{dt}{1 - \sec t}$

14. $\displaystyle\int \csc t \sin 3t\, dt$

15. $\displaystyle\int_0^{\pi/4} \frac{1 + \sin\theta}{\cos^2\theta}\, d\theta$

16. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$

17. $\displaystyle\int \frac{\ln y}{y + 4y \ln^2 y}\, dy$

18. $\displaystyle\int \frac{2^{\sqrt{y}}\, dy}{2\sqrt{y}}$

19. $\displaystyle\int \frac{d\theta}{\sec\theta + \tan\theta}$

20. $\displaystyle\int \frac{dt}{t\sqrt{3 + t^2}}$

21. $\displaystyle\int \frac{4t^3 - t^2 + 16t}{t^2 + 4}\, dt$

22. $\displaystyle\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}}\, dx$

23. $\displaystyle\int_0^{\pi/2} \sqrt{1 - \cos\theta}\, d\theta$

24. $\displaystyle\int (\sec t + \cot t)^2\, dt$

25. $\displaystyle\int \frac{dy}{\sqrt{e^{2y} - 1}}$

26. $\displaystyle\int \frac{6\, dy}{\sqrt{y}\,(1 + y)}$

27. $\displaystyle\int \frac{2\, dx}{x\sqrt{1 - 4\ln^2 x}}$

28. $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

29. $\displaystyle\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

30. $\displaystyle\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$

31. $\displaystyle\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1}\, dx$

32. $\displaystyle\int_{-1}^1 \sqrt{1 + x^2}\, \sin x\, dx$

33. $\displaystyle\int_{-1}^0 \sqrt{\frac{1 + y}{1 - y}}\, dy$

34. $\displaystyle\int e^{z + e^z}\, dz$

35. $\displaystyle\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$

36. $\displaystyle\int \frac{dx}{(2x + 1)\sqrt{4x + 4x^2}}$

37. $\displaystyle\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\, d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta - 1}$

39. $\displaystyle\int \frac{dx}{1 + e^x}$

Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1 + x^3}\, dx$

Hint: Let $u = x^{3/2}$.

41. The functions $y = e^{x^3}$ and $y = x^3 e^{x^3}$ do not have elementary antiderivatives, but $y = (1 + 3x^3)e^{x^3}$ does. Evaluate

$$\int (1 + 3x^3)e^{x^3} dx.$$

42. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1 + \sin^2 x}.$$

43. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7\sqrt{x^4 + 1}\, dx.$$

44. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3} dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3,$ and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1} \sqrt{x}$

e. $u = \tan^{-1} ((x - 1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

5.7 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\, dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x \cos x\, dx \qquad \text{and} \qquad \int x^2 e^x\, dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\, dx \qquad \text{and} \qquad \int e^x \cos x\, dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int [f'(x)g(x) + f(x)g'(x)]\, dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\, dx = \int \frac{d}{dx}[f(x)g(x)]\, dx - \int f'(x)g(x)\, dx,$$

leading to the **integration by parts** formula

$$\boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx \qquad (1)}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du and v, and finally we apply the formula in Equation (2).

EXAMPLE 1 Find

$$\int x \cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C. \qquad \blacksquare$$

There are four apparent choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x \cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $du = (\cos x - x \sin x)\,dx$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\,dx.$$

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. When finding v from dv, any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in v because it would simply cancel out of the right-hand side of Equation (2).

If we put $u = f(x)$ and $v = g(x)$, then the formulae for integration by parts become:

$$\int f(x)g'(x)\mathrm{d}x = f(x)\,g(x) - \int f'(x)g(x)\mathrm{d}x$$

We are assuming $f(x)$ to be first function and $g'(x)$ to be second function. Hence, the above can be written as

$$\int(\text{first function})\times(\text{second function})\mathrm{d}x = (\text{first function})\times(\text{integral of second function}) - \int(\text{derivative of first function})\times(\text{integral of second function})\,\mathrm{d}x$$

While applying above formula, we need to select first and second functions appropriately using following guidelines:

(1) The rule of ILATE can be applied. Here, "I" stands for inverse trigonometric function, "L" stands for logarithmic function, "A" stands for algebraic function, "T" stands for trigonometric function and "E" stands for exponential function. In the above rule which ever letter comes first will be taken as first function and remaining as second function. ■

EXAMPLE 2 $I = \int x \tan^{-1} x \, dx$

Solution Here, the integrand consists of product of two functions. No techniques of substitution can help here. We can apply the rule of ILATE which suggest x must be taken as second function and $\tan^{-1}x$ must be taken as first function.

$$\therefore \quad \int x \tan^{-1} x \, dx = (\tan^{-1}x) \times \frac{x^2}{2} - \int \frac{1}{1+x^2} \times \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\left[\int 1.dx - \int \frac{dx}{x^2+1}\right]$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$ ■

(2) If integrand consists of product of two trigonometric functions, then we can take that function as second function whose integral is simpler.

EXAMPLE 3 $I = \int \sec^3 x dx$

Solution $\sec^3 x$ can be written as $(\sec x) \cdot (\sec^2 x)$

$$\therefore \quad I = \int \underbrace{(\sec x)}_{\text{1st function}} \times \underbrace{(\sec^2 x)}_{\text{2nd function}} dx = \sec x \cdot \tan x - \int \tan x \cdot \sec x \cdot \tan x \, dx$$

$$\therefore \quad I = \sec \tan x - \int \sec x.(\sec^2 x - 1)dx$$

$$= \sec x \times \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\Rightarrow \quad I = \sec x \cdot \tan x - I + \ln |\sec x + \tan x| + C$$

$$\Rightarrow \quad I = \frac{1}{2} (\sec x \cdot \tan x + \ln |\sec x + \tan x|) + C$$

(3) If the integrand consists of a single function which is not directly integrable, then by parts can be applied by assuming '1' as second function. ■

EXAMPLE 4 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $dv = dx$ Easy to integrate

$$du = \frac{1}{x}\,dx, \qquad\qquad v = x. \qquad \text{Simplest antiderivative}$$

Then from Equation (2),

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C. \qquad \blacksquare$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 5 Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\ &= x^2 e^x - 2x e^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. $\blacksquare$

The technique of Example 3 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 6 Evaluate

$$\int e^x \cos x\,dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int(-\cos x)(e^x\,dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

EXAMPLE 7 Obtain a formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx. \qquad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

Two Important Integrals

(1) $\int e^x(f(x)+f'(x))\,\mathrm{d}x = e^x f(x)+C$

(2) $\int (xf'(x)+f(x))\mathrm{d}x = xf(x)+C$

Proof

(1) $\int e^x(f(x)+f'(x))\mathrm{d}x \;=\; \int e^x f(x)dx + \int e^x f'(x)\mathrm{d}x$

In first integral, we are applying by parts and converting it into second integral. Hence, e^x is taken as second function and $f(x)$ is taken as first function.

$\therefore \qquad \int e^x f(x)\mathrm{d}x + \int e^x f'(x)\mathrm{d}x \;= e^x\cdot f(x) - \int e^x f'(x)\mathrm{d}x + \int e^x f'(x)\mathrm{d}x \;= e^x f(x) + C$

Similarly, the second integral can be executed.

EXAMPLE 8 $\quad I = \int e^{\cos^{-1}x}\left(\dfrac{(x+1)+\sqrt{1-x^2}}{(x+1)^2\sqrt{1-x^2}}\right)\mathrm{d}x$

Solution Put $\qquad \cos^{-1}x = t$

$$\Rightarrow \qquad \frac{\mathrm{d}x}{\sqrt{1-x^2}} = -\mathrm{d}t$$

$$\therefore \qquad I = -\int e^t\left(\frac{\cos t+1+\sin t}{(1+\cos t)^2}\right)\mathrm{d}t$$

$$= -\int e^t\left(\underbrace{\frac{1}{(1+\cos t)}}_{f(t)} + \underbrace{\frac{\sin t}{(1+\cos t)^2}}_{f'(x)}\right)dx$$

$$= -e^t\,\frac{1}{1+\cos t}+C = \frac{-e^{\cos^{-1}x}}{1+x}+C$$

EXAMPLE 9 $\quad I = \int \dfrac{e^x(x^4+2)}{(1+x^2)^{5/2}}\mathrm{d}x$

Solution $\quad I = \int e^x\left(\dfrac{(x^2+1)^2+1-2x^2}{(1+x^2)^{5/2}}\right)\mathrm{d}x \;=\; \int e^x\left(\dfrac{1}{\sqrt{1+x^2}}+\dfrac{1-2x^2}{(1+x^2)^{5/2}}\right)\mathrm{d}x$

In this example, we can see the integrand is not of the form $e^x(f(x) + f'(x))$ but $e^x(f(x) - f''(x))$. So, we can add and substract $f'(x)$ to get the desirable form.

$$\therefore \qquad \int e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{(1+x^2)^{3/2}} + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right)\mathrm{d}x$$

$$= e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{(1+x^2)^{3/2}}\right)+C$$

EXAMPLE 10 $\quad \mathrm{I} = \int\left(\ln(\ln x)+\dfrac{1}{\ln^2 x}\right)\mathrm{d}x$

Solution Put $\qquad \ln x = t$

$$\Rightarrow \qquad x = e^t$$

$$\Rightarrow \qquad dx = e^t \, dt$$

$$\therefore \qquad I = \int \left(\ln t + \frac{1}{t^2} \right) e^t dt$$

$$= \int e^t \left(\ln t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$= e^t \left(\ln t - \frac{1}{t} \right) + C = x \left(\ln(\ln x) - \frac{1}{\ln x} \right) + C$$

EXAMPLE 11 $\quad I = \int \sin 4x \cdot e^{\tan^2 x} dx$

Solution $\quad \int 2\cos 2x \cdot 2 \cdot \dfrac{\sin x}{\cos x} \cdot \cos^2 x \cdot e^{\tan^2 x} dx$

Put $\qquad \tan^2 x = t \Rightarrow \cos 2x = \dfrac{1-t}{1+t}$ and $\sec^2 x = 1+t$

$$\Rightarrow \qquad 2\tan x \sec^2 x \, dx = dt$$

$$= 2\int \frac{e^t(1-t)}{(1+t)^3} dt = -2\int \frac{e^t(t+1-2)}{(1+t)^3} dt$$

$$= -2e^{\tan^2 x} \cdot \cos^4 x + C$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

Tabular Integration Can Simplify Repeated Integrations

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the notation and calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a nice way to organize the calculations that prevents these pitfalls and simplifies the work. It is called **tabular integration** and is illustrated in the next examples.

EXAMPLE 12 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ **and its derivatives**		$g(x)$ **and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3. ■

EXAMPLE 13 Find the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution The integral is

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx &= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx \\ &= \frac{1}{n\pi} \sin nx \Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx.\end{aligned}$$

Using tabular integration to find an antiderivative, we have

$f(x)$ **and its derivatives**		$g(x)$ **and its integrals**
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n}\sin nx$
$6x$	$(+)$	$-\frac{1}{n^2}\cos nx$
6	$(-)$	$-\frac{1}{n^3}\sin nx$
0		$\frac{1}{n^4}\cos nx$

$$\frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{3x^3}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{3\pi^2 \cos n\pi}{n^2} - \frac{6 \cos n\pi}{n^4} + \frac{6}{n^4}\right)$$

$$= \frac{3}{\pi}\left(\frac{\pi^2 n^2(-1)^n + 2(-1)^{n+1} + 2}{n^4}\right). \qquad \cos n\pi = (-1)^n$$

■

Integrals like those in Example 8 occur frequently in electrical engineering.

Exercises 5.7

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int xe^x\, dx$
8. $\int xe^{3x}\, dx$
9. $\int x^2 e^{-x}\, dx$
10. $\int (x^2 - 2x + 1)e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3 e^x\, dx$
16. $\int p^4 e^{-p}\, dp$
17. $\int (x^2 - 5x)e^x\, dx$
18. $\int (r^2 + r + 1)e^r\, dr$
19. $\int x^5 e^x\, dx$
20. $\int t^2 e^{4t}\, dt$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1 - x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln (x + x^2)\, dx$
29. $\int \sin (\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x \sec x^2\, dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$
33. $\int x (\ln x)^2\, dx$
34. $\int \frac{1}{x (\ln x)^2}\, dx$
35. $\int \frac{\ln x}{x^2}\, dx$
36. $\int \frac{(\ln x)^3}{x}\, dx$
37. $\int x^3 e^{x^4}\, dx$
38. $\int x^5 e^{x^3}\, dx$
39. $\int x^3\sqrt{x^2 + 1}\, dx$
40. $\int x^2 \sin x^3\, dx$
41. $\int \sin 3x \cos 2x\, dx$
42. $\int \sin 2x \cos 4x\, dx$
43. $\int \sqrt{x} \ln x\, dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
45. $\int \cos \sqrt{x}\, \mathrm{dx}$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\, dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$
51. $\int x \tan^{-1} x\, dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\, dx$

Reduction Formulas

In Exercises 53–57, use integration by parts to establish the reduction formula.

53. $\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$
54. $\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$
55. $\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx, \quad a \neq 0$

56. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

57. $\int x^m(\ln x)^n\,dx = \dfrac{x^{m+1}}{m+1}(\ln x)^n - \dfrac{n}{m+1}\cdot$

$$\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$$

58. Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \left(\dfrac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

59. Show that

$$\int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx.$$

60. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x), \quad x = f(y) \\ &&& dx = f'(y)\,dy \\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with} \\ &&& u = y,\ dv = f'(y)\,dy \\ &= xf^{-1}(x) - \int f(y)\,dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x\,dx &= \int ye^y\,dy && y = \ln x, \quad x = e^y \\ &&& dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C. \end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned} \int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x \\ &= x\cos^{-1} x - \sin y + C \\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 61–64. Express your answers in terms of x.

61. $\int \sin^{-1} x\,dx$

62. $\int \tan^{-1} x\,dx$

63. $\int \sec^{-1} x\,dx$

64. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 65 and 66 compare the results of using Equations (4) and (5).

65. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

66. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

5.8 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If m **is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m **is even and** n **is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both** m **and** n **are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Case 4 If $(m + n)$ is a negative even integer, then convert the integral into $\tan x$ and $\sec^2 x$ or $\cot x$ and $\operatorname{cosec}^2 x$

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d\,(\cos x)) && \sin x\,dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2)\, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

■

EXAMPLE 2 Evaluate

$$\int \cos^5 x\, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned}
\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx &= \int (1 - \sin^2 x)^2\, d(\sin x) && \cos x\, dx = d(\sin x) \\
&= \int (1 - u^2)^2\, du && u = \sin x \\
&= \int (1 - 2u^2 + u^4)\, du && \text{Square } 1 - u^2. \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

■

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x\, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x\, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx && m \text{ and } n \text{ both even} \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]
\end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned}
\int \cos^2 2x\, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx \\
&= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). && \text{Omitting the constant of integration until the final result}
\end{aligned}$$

For the $\cos^3 2x$ term, we have

$$\begin{aligned}
\int \cos^3 2x\, dx &= \int (1 - \sin^2 2x)\cos 2x\, dx && u = \sin 2x,\ du = 2\cos 2x\, dx \\
&= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). && \text{Again omitting } C
\end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

EXAMPLE 4 $I = \int \dfrac{dx}{\sqrt{\operatorname{cosec} x \sec^7 x}}$

Solution $I = \int (\sin x)^{\frac{-1}{2}} (\cos x)^{\frac{-7}{2}} dx$

Clearly, $(m + n) = -4$, i.e., negative even integer

$$\therefore \qquad I = \int \frac{(\sin x)^{\frac{-1}{2}}}{(\cos x)^{\frac{-1}{2}}} (\cos x)^{-4}\, dx = \int \frac{1}{\sqrt{\tan x}} \sec^2 x \cdot \sec^2 x dx$$

Put $\tan x = t^2$

$\Rightarrow \quad \sec^2 x\, dx = 2t\, dt$

$$\therefore \qquad I = \int \frac{1}{t}(1+t^4)\, 2t\, dt = 2\left(t + \frac{t^5}{5}\right) + C = 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C \qquad \blacksquare$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 5 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \begin{array}{l}\cos 2x \geq 0 \text{ on} \\ [0, \pi/4]\end{array} \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}. \qquad \blacksquare
\end{aligned}$$

Integrals of Powers of tan *x* and sec *x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 6 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\int \tan^4 x\,dx = \int \tan^2 x \cdot \tan^2 x\,dx = \int \tan^2 x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\,dx$$

and have

$$\int u^2\,du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C. \qquad \blacksquare$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x\,dx.$$

Solution

$$\int (\tan^4 x)(\sec^4 x)\,dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\,dx \qquad \sec^2 x = 1 + \tan^2 x$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\,dx$$

$$= \int (\tan^4 x)(\sec^2 x)\,dx + \int (\tan^6 x)(\sec^2 x)\,dx$$

$$= \int u^4\,du + \int u^6\,du = \frac{u^5}{5} + \frac{u^7}{7} + C \qquad u = \tan x,\ du = \sec^2 x\,dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C \qquad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \text{and} \qquad \int \cos mx \cos nx\,dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]. \tag{5}$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x\, dx &= \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Exercises 5.8

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x\, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx$
3. $\int \cos^3 x \sin x\, dx$
4. $\int \sin^4 2x \cos 2x\, dx$
5. $\int \sin^3 x\, dx$
6. $\int \cos^3 4x\, dx$
7. $\int \sin^5 x\, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$
9. $\int \cos^3 x\, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x\, dx$

11. $\int \sin^3 x \cos^3 x\, dx$

12. $\int \cos^3 2x \sin^5 2x\, dx$

13. $\int \cos^2 x\, dx$

14. $\int_0^{\pi/2} \sin^2 x\, dx$

15. $\int_0^{\pi/2} \sin^7 y\, dy$

16. $\int 7 \cos^7 t\, dt$

17. $\int_0^{\pi} 8 \sin^4 x\, dx$

18. $\int 8 \cos^4 2\pi x\, dx$

19. $\int 16 \sin^2 x \cos^2 x\, dx$

20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y\, dy$

21. $\int 8 \cos^3 2\theta \sin 2\theta\, d\theta$

22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\dfrac{1 - \cos x}{2}}\, dx$

24. $\int_0^{\pi} \sqrt{1 - \cos 2x}\, dx$

25. $\int_0^{\pi} \sqrt{1 - \sin^2 t}\, dt$

26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta}\, d\theta$

27. $\int_{\pi/3}^{\pi/2} \dfrac{\sin^2 x}{\sqrt{1 - \cos x}}\, dx$

28. $\int_0^{\pi/6} \sqrt{1 + \sin x}\, dx$

$\left(\textit{Hint:} \text{ Multiply by } \sqrt{\dfrac{1 - \sin x}{1 - \sin x}}.\right)$

29. $\int_{5\pi/6}^{\pi} \dfrac{\cos^4 x}{\sqrt{1 - \sin x}}\, dx$

30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x}\, dx$

31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta}\, d\theta$

32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2}\, dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x\, dx$

34. $\int \sec x \tan^2 x\, dx$

35. $\int \sec^3 x \tan x\, dx$

36. $\int \sec^3 x \tan^3 x\, dx$

37. $\int \sec^2 x \tan^2 x\, dx$

38. $\int \sec^4 x \tan^2 x\, dx$

39. $\int_{-\pi/3}^{0} 2 \sec^3 x\, dx$

40. $\int e^x \sec^3 e^x\, dx$

41. $\int \sec^4 \theta\, d\theta$

42. $\int 3 \sec^4 3x\, dx$

43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta\, d\theta$

44. $\int \sec^6 x\, dx$

45. $\int 4 \tan^3 x\, dx$

46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x\, dx$

47. $\int \tan^5 x\, dx$

48. $\int \cot^6 2x\, dx$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx$

50. $\int 8 \cot^4 t\, dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 51–56.

51. $\int \sin 3x \cos 2x\, dx$

52. $\int \sin 2x \cos 3x\, dx$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx$

54. $\int_0^{\pi/2} \sin x \cos x\, dx$

55. $\int \cos 3x \cos 4x\, dx$

56. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\, dx$

Exercises 57–62 require the use of various trigonometric identities before you evaluate the integrals.

57. $\int \sin^2 \theta \cos 3\theta\, d\theta$

58. $\int \cos^2 2\theta \sin \theta\, d\theta$

59. $\int \cos^3 \theta \sin 2\theta\, d\theta$

60. $\int \sin^3 \theta \cos 2\theta\, d\theta$

61. $\int \sin \theta \cos \theta \cos 3\theta\, d\theta$

62. $\int \sin \theta \sin 2\theta \sin 3\theta\, d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 63–68.

63. $\int \dfrac{\sec^3 x}{\tan x}\, dx$

64. $\int \dfrac{\sin^3 x}{\cos^4 x}\, dx$

65. $\int \dfrac{\tan^2 x}{\csc x}\, dx$

66. $\int \dfrac{\cot x}{\cos^2 x}\, dx$

67. $\int x \sin^2 x\, dx$

68. $\int x \cos^3 x\, dx$

5.9 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 5.13.

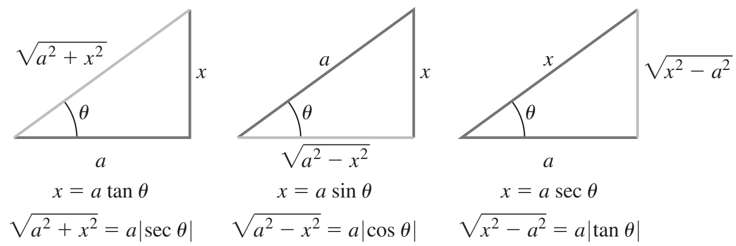


FIGURE 5.13 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

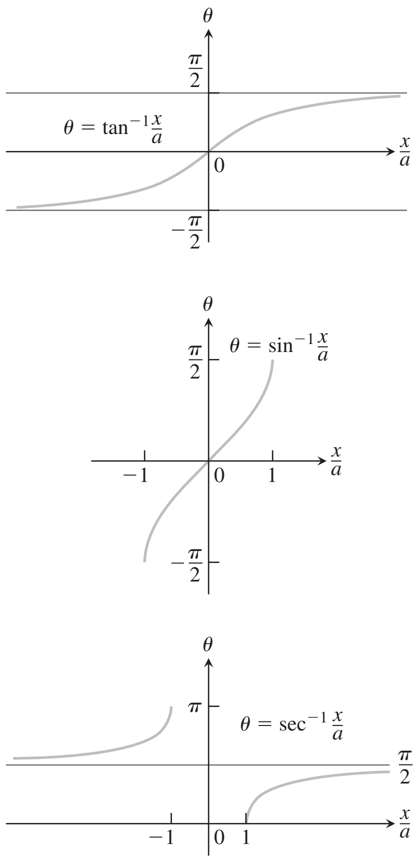


FIGURE 5.14 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

The functions in these substitutions have inverses only for selected values of θ (Figure 5.14). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\[2ex] \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4 + x^2}} &= \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|} && \sqrt{\sec^2 \theta} = |\sec \theta| \\
&= \int \sec \theta \, d\theta && \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln |\sec \theta + \tan \theta| + C \\
&= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C. && \text{From Fig. 5.15}
\end{aligned}$$

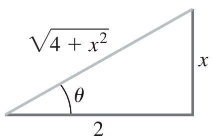


FIGURE 5.15 Reference triangle for $x = 2 \tan \theta$ (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4 + x^2}}{2}.$$

Notice how we expressed $\ln|\sec \theta + \tan \theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2 \tan \theta$ (Figure 5.15) and read the ratios from the triangle. ■

EXAMPLE 2 Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in Example 1, we find that

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \sec \theta \, d\theta && x = a \tan \theta, \ dx = a \sec^2 \theta \, d\theta \\
&= \ln |\sec \theta + \tan \theta| + C \\
&= \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C && \text{Fig. 5.13}
\end{aligned}$$

From Table 7.11, $\sinh^{-1}(x/a)$ is also an antiderivative of $1/\sqrt{a^2 + x^2}$, so the two antiderivatives differ by a constant, giving

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C.$$

Setting $x = 0$ in this last equation, we find $0 = \ln|1| + C$, so $C = 0$. Since $\sqrt{a^2 + x^2} > |x|$, we conclude that

$$\boxed{\sinh^{-1} \frac{x}{a} = \ln \left(\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right)}$$

■

EXAMPLE 3 Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{From Fig. 5.16}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

■

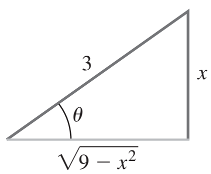


FIGURE 5.16 Reference triangle for $x = 3\sin\theta$ (Example 3):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 4 Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

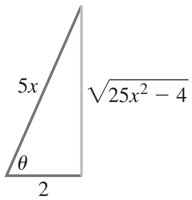


FIGURE 5.17 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 4).

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \begin{array}{l}\tan\theta > 0 \text{ for} \\ 0 < \theta < \pi/2\end{array}$$

With these substitutions, we have

$$\int\frac{dx}{\sqrt{25x^2 - 4}} = \int\frac{dx}{5\sqrt{x^2 - (4/25)}} = \int\frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int\sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{From Fig. 5.17}$$

■

EXAMPLE 5 $I = \int\frac{\sqrt{(25-x^2)^3}}{x^6}\mathrm{d}x$

Solution Clearly we can see $(25 - x^2)$ in square roots

$\therefore$ Put $\qquad x = 5\sin\theta$

$\Rightarrow \qquad \mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$

$$\therefore \qquad I = \int\frac{\sqrt{(25-25\sin^2\theta)^3}}{5^6\sin^6\theta}\cdot 5\cos\theta\,\mathrm{d}\theta = \int\frac{5^3\cdot\cos^3\theta\cdot 5\cos\theta}{5^6\sin^6\theta}\mathrm{d}\theta$$

$$I = \frac{1}{25}\int\frac{\cos^4\theta}{\sin^4\theta}\operatorname{cosec}^2\theta\,\mathrm{d}\theta$$

$$I = \frac{-1}{25}\int\cot^4\theta\,\mathrm{d}(\cot\theta)$$

$$I = \frac{-1}{25}\frac{\cot^5\theta}{5} + C = -\frac{1}{125}\left(\frac{\sqrt{25-x^2}}{x}\right)^5 + C$$

Alternatively:

We can take x^2 common

$$I = \int\frac{x^3\sqrt{\left(\frac{25}{x^2}-1\right)^3}}{x^6}\mathrm{d}x = \int\frac{\left(\frac{25}{x^2}-1\right)^{3/2}}{x^3}\mathrm{d}x$$

Put $\qquad \frac{25}{x^2} - 1 = z^2$

$\Rightarrow \qquad -\frac{25}{x^3}dx = z\mathrm{d}z$

$$\Rightarrow \qquad I = \frac{-1}{25}\int z^3\cdot z\mathrm{d}z = \frac{-1}{125}z^5 + C = -\frac{1}{125}\left(\frac{25}{x^2}-1\right)^{5/2} + C$$

■

Exercises 5.9

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}},\quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}},\quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy,\quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy,\quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}},\quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}},\quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

16. $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

17. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

19. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

20. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

21. $\displaystyle\int \sqrt{\frac{x+1}{1-x}}\,dx$

22. $\displaystyle\int x\sqrt{x^2-4}\,dx$

23. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

24. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

25. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}},\quad x > 1$

26. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}},\quad x > 1$

27. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

28. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

29. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

30. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

31. $\displaystyle\int \frac{x^3\,dx}{x^2-1}$

32. $\displaystyle\int \frac{x\,dx}{25+4x^2}$

33. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

34. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

38. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

39. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

40. $\displaystyle\int \frac{dx}{1+x^2}$

41. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

42. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

43. $\displaystyle\int \frac{x\,dx}{\sqrt{1+x^4}}$

44. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

45. $\displaystyle\int \sqrt{\frac{4-x}{x}}\,dx$

(*Hint:* Let $x = u^2$.)

46. $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\,dx$

(*Hint:* Let $u = x^{3/2}$.)

47. $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\,dx$

48. $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 49–52 for y as a function of x.

49. $x\dfrac{dy}{dx} = \sqrt{x^2-4},\quad x \ge 2,\quad y(2) = 0$

50. $\sqrt{x^2-9}\,\dfrac{dy}{dx} = 1,\quad x > 3,\quad y(5) = \ln 3$

51. $(x^2+4)\dfrac{dy}{dx} = 3,\quad y(2) = 0$

52. $(x^2+1)^2\dfrac{dy}{dx} = \sqrt{x^2+1},\quad y(0) = 1$

53. Evaluate $\int x^3\sqrt{1-x^2}\,dx$ using

a. integration by parts.

b. a u-substitution.

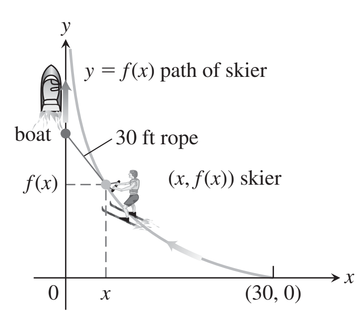
c. a trigonometric substitution.

54. Path of a water skier Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.

a. Show that $f'(x) = \dfrac{-\sqrt{900-x^2}}{x}$.

(*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)

b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



NOT TO SCALE

5.10 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\begin{aligned}\int \frac{5x - 3}{(x + 1)(x - 3)}\,dx &= \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx \\ &= 2\ln|x + 1| + 3\ln|x - 3| + C.\end{aligned}$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \tag{1}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$*. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- *We must know the factors of $g(x)$*. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate
$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx.$$

Solution The partial fraction decomposition has the form
$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$
To find the values of the undetermined coefficients A, B, and C, we clear fractions and get
$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A\left(x^2 + 4x + 3\right) + B\left(x^2 + 2x - 3\right) + C\left(x^2 - 1\right) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$
The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining
$$\begin{aligned} &\text{Coefficient of } x^2\text{:} && A + B + C = 1 \\ &\text{Coefficient of } x^1\text{:} && 4A + 2B = 4 \\ &\text{Coefficient of } x^0\text{:} && 3A - 3B - C = 1 \end{aligned}$$
There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4, B = 1/2$, and $C = -1/4$. Hence we have
$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x - 1} + \frac{1}{2}\frac{1}{x + 1} - \frac{1}{4}\frac{1}{x + 3}\right] dx \\ &= \frac{3}{4}\ln|x - 1| + \frac{1}{2}\ln|x + 1| - \frac{1}{4}\ln|x + 3| + K, \end{aligned}$$
where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\, dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned}\frac{6x + 7}{(x + 2)^2} &= \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \\ 6x + 7 &= A(x + 2) + B && \text{Multiply both sides by } (x + 2)^2. \\ &= Ax + (2A + B)\end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned}\int \frac{6x + 7}{(x + 2)^2}\, dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2}\, dx \\ &= 6 \ln|x + 2| + 5(x + 2)^{-1} + C.\end{aligned}$$

■

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g.

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{\smash{)}\, 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} - 3 \\ 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned}\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx &= \int 2x\, dx + \int \frac{5x - 3}{x^2 - 2x - 3}\, dx \\ &= \int 2x\, dx + \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx \\ &= x^2 + 2 \ln|x + 1| + 3 \ln|x - 3| + C.\end{aligned}$$

■

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx.$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)\\ &= (A+C)x^3 + (-2A+B-C+D)x^2\\ &\quad + (A-2B+C)x + (B-C+D).\end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned}&\text{Coefficients of } x^3: & 0 &= A+C\\ &\text{Coefficients of } x^2: & 0 &= -2A+B-C+D\\ &\text{Coefficients of } x^1: & -2 &= A-2B+C\\ &\text{Coefficients of } x^0: & 4 &= B-C+D\end{aligned}$$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$\begin{aligned}&-4 = -2A, \quad A = 2 && \text{Subtract fourth equation from second.}\\ &C = -A = -2 && \text{From the first equation}\\ &B = (A+C+2)/2 = 1 && \text{From the third equation and } C = -A\\ &D = 4 - B + C = 1. && \text{From the fourth equation.}\end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int\left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx\\ &= \int\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx\\ &= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C. \qquad \blacksquare\end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x\\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex\\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A = 1,\ B = -1,\ C = 0,\ D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} && \begin{array}{l} u = x^2+1, \\ du = 2x\,dx \end{array} \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

■

HISTORICAL BIOGRAPHY
Oliver Heaviside
(1850–1925)

The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\begin{aligned}
\frac{(1)^2+1}{(1-2)(1-3)} &= A + 0 + 0, \\
A &= 1.
\end{aligned}$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} \tag{4}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2+1}{\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x-1)}}\,(1-2)(1-3)} = \frac{2}{(-1)(-2)} = 1.$$

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Expression (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2+1}{(2-1)\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x-2)}}(2-3)} = \frac{5}{(1)(-1)} = -5.$$

Finally, C is found by covering the $(x - 3)$ in Expression (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3-1)(3-2)\underbrace{\boxed{(x-3)}}_{\Uparrow \text{ Cover}}} = \frac{10}{(2)(1)} = 5. \qquad \blacksquare$$

> **Heaviside Method**
>
> **1.** *Write the quotient with $g(x)$ factored:*
>
> $$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$
>
> **2.** *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:
>
> $$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$
>
> $$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$
>
> $$\vdots$$
>
> $$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$
>
> **3.** *Write the partial fraction expansion of $f(x)/g(x)$* as
>
> $$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 7 Use the Heaviside Method to evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of the cubic polynomial $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0 + 4}{\underbrace{\boxed{x}}_{\Uparrow \text{ Cover}}(0 - 2)(0 + 5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$$A_2 = \frac{2 + 4}{2\underbrace{\boxed{(x - 2)}}_{\Uparrow \text{ Cover}}(2 + 5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$$A_3 = \frac{-5 + 4}{(-5)(-5 - 2)\underbrace{\boxed{(x + 5)}}_{\Uparrow \text{ Cover}}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C. \qquad \blacksquare$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}. \qquad \blacksquare$$

In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \qquad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0)\\
2 &= 2A\\
A &= 1\\
x = 2: \qquad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0)\\
5 &= -B\\
B &= -5
\end{aligned}$$

$$x = 3: \qquad (3)^2 + 1 = A(0) + B(0) + C(2)(1)$$
$$10 = 2C$$
$$C = 5.$$

Conclusion:

$$\frac{x^2 + 1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}. \qquad \blacksquare$$

EXAMPLE 10 $\quad I = \int \frac{dx}{x^3 + 1}$

Solution Here, the integrand contains $x^3 + 1$ in denominator which can be factorized as $(x + 1)$ and $(x^2 - x + 1)$ and then partial fraction technique can be applied but in this case we can make the calculation a bit simpler by creating derivative of $x^3 + 1$ in numerator.

$$\therefore \qquad I = \int \frac{\mathrm{d}x}{x^3+1} = \int \frac{x^2 - (x^2 - 1)}{x^3 + 1}\mathrm{d}x \underbrace{\int \frac{x^2}{x^3+1}\mathrm{d}x}_{I_1} - \underbrace{\int \frac{x-1}{x^2 - x + 1}\mathrm{d}x}_{I_2}$$

$$\therefore \qquad I_1 = \frac{1}{3}\int \frac{\mathrm{d}(x^3)}{x^3+1} = \frac{1}{3}\ln|x^3 + 1|$$

$$I_2 = \int \frac{x-1}{x^2 - x + 1}\mathrm{d}x$$

Here, the numerator, i.e., $(x - 1)$ can be written as a derivative of $x^2 - x + 1$.

$$\text{i.e.,} \qquad (x-1) = l\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x + 1)\right) + m$$

$$x - 1 = \ell(2x - 1) + m$$

Clearly $\qquad 2\ell = 1$ and $m - \ell = -1$

$$\Rightarrow \qquad \ell = \frac{1}{2} \quad m = \frac{-1}{2}$$

$$\therefore \qquad I_2 = \int \frac{\frac{1}{2}(2x-1) - \frac{1}{2}}{x^2 - x + 1}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{2x-1}{x^2 - x + 1} - \frac{1}{2}\int \frac{\mathrm{d}x}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\ln(x^2 - x + 1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\therefore \qquad I = \frac{1}{3}\ln|x^3 + 1| + \frac{1}{2}\ln(x^2 - x + 1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C \qquad \blacksquare$$

Exercises 5.10

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{x^2 - x + 2}{x^3 - 1}\,dx$

28. $\displaystyle\int \frac{1}{x^4 + x}\,dx$

29. $\displaystyle\int \frac{x^2}{x^4 - 1}\,dx$

30. $\displaystyle\int \frac{x^2 + x}{x^4 - 3x^2 - 4}\,dx$

31. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

32. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

34. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

35. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

36. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

37. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

38. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–50.

39. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

40. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

41. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

42. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

43. $\displaystyle\int \frac{(x - 2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x - 2)^2}\,dx$

44. $\displaystyle\int \frac{(x + 1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x + 1)^2}\,dx$

45. $\displaystyle\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

46. $\displaystyle\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$
(*Hint:* Let $x = u^6$.)

47. $\displaystyle\int \frac{\sqrt{x + 1}}{x}\,dx$
(*Hint:* Let $x + 1 = u^2$.)

48. $\displaystyle\int \frac{1}{x\sqrt{x + 9}}\,dx$

49. $\displaystyle\int \frac{1}{x(x^4 + 1)}\,dx$
$\left(\textit{Hint:} \text{ Multiply by } \dfrac{x^3}{x^3}.\right)$

50. $\displaystyle\int \frac{1}{x^6(x^5 + 4)}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 51–54 for x as a function of t.

51. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

52. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

53. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

54. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

5.11 Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx \tag{1}$$

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx \tag{2}$$

$$\int \sin^n x \cos^m x\, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\, dx \qquad (n \neq -m). \tag{3}$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 1 Find

$$\int \tan^5 x\, dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \int \tan x\, dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'. \qquad \blacksquare$$

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 5.7.)

Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. These integrals can sometimes be expressed with infinite series or approximated using numerical methods for their evaluation. Examples of nonelementary integrals include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

and integrals such as

$$\int \sin x^2\, dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\, dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1 - k^2 \sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives, as a consequence of the Fundamental Theorem of Calculus, Part 1, because they are continuous. However, none of the antiderivatives are elementary. The integrals you are asked to evaluate in the present chapter have elementary antiderivatives, but you may encounter nonelementary integrals in your other work.

Exercises 5.11

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–26.

1. $\int \frac{dx}{x\sqrt{x-3}}$

2. $\int \frac{dx}{x\sqrt{x+4}}$

3. $\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\int x\sqrt{2x-3}\,dx$

6. $\int x(7x+5)^{3/2}\,dx$

7. $\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\int x\sqrt{4x-x^2}\,dx$

10. $\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\int e^{2t}\cos 3t\,dt$

16. $\int e^{-3t}\sin 4t\,dt$

17. $\int x\cos^{-1}x\,dx$

18. $\int x\tan^{-1}x\,dx$

19. $\int x^2\tan^{-1}x\,dx$

20. $\int \frac{\tan^{-1}x}{x^2}\,dx$

21. $\int \sin 3x\cos 2x\,dx$

22. $\int \sin 2x\cos 3x\,dx$

23. $\int 8\sin 4t\sin\frac{t}{2}\,dt$

24. $\int \sin\frac{t}{3}\sin\frac{t}{6}\,dt$

25. $\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta$

26. $\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta$

Substitution and Integral Tables

In Exercises 27–40, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

27. $\int \frac{x^3+x+1}{(x^2+1)^2}\,dx$

28. $\int \frac{x^2+6x}{(x^2+3)^2}\,dx$

29. $\int \sin^{-1}\sqrt{x}\,dx$

30. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

31. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

32. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx$

33. $\int \cot t\sqrt{1-\sin^2 t}\,dt, \quad 0 < t < \pi/2$

34. $\int \frac{dt}{\tan t\sqrt{4-\sin^2 t}}$

35. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}}$

36. $\int \tan^{-1}\sqrt{y}\,dy$

37. $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$
(*Hint:* Complete the square.)

38. $\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx$

39. $\int \sqrt{5-4x-x^2}\,dx$

40. $\int x^2\sqrt{2x-x^2}\,dx$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 41–50.

41. $\int \sin^5 2x\,dx$

42. $\int 8\cos^4 2\pi t\,dt$

43. $\int \sin^2 2\theta\cos^3 2\theta\,d\theta$

44. $\int 2\sin^2 t\sec^4 t\,dt$

45. $\int 4\tan^3 2x\,dx$

46. $\int 8\cot^4 t\,dt$

47. $\int 2 \sec^3 \pi x \, dx$

48. $\int 3 \sec^4 3x \, dx$

49. $\int \csc^5 x \, dx$

50. $\int 16x^3(\ln x)^2 \, dx$

Evaluate the integrals in Exercises 51–56 by making a substitution (possibly trigonometric) and then applying a reduction formula.

51. $\int e^t \sec^3 (e^t - 1) \, dt$

52. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}} \, d\theta$

53. $\int_0^1 2\sqrt{x^2 + 1} \, dx$

54. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

55. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r} \, dr$

56. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

57. What is the largest value

$$\int_a^b \sqrt{x - x^2} \, dx$$

can have for any a and b? Give reasons for your answer.

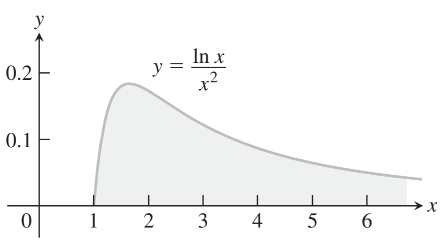
58. What is the largest value

$$\int_a^b x\sqrt{2x - x^2} \, dx$$

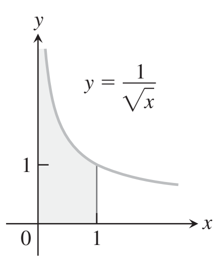
can have for any a and b? Give reasons for your answer.

5.12 Improper Integrals

Up to now, we have required definite integrals to have two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 5.18a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 5.18b). In either case, the integrals are said to be *improper* and are calculated as limits.



(a)



(b)

FIGURE 5.18 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 5.19a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 5.19b).

$$A(b) = \int_0^b e^{-x/2} \, dx = -2e^{-x/2} \Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b \to \infty} A(b) = \lim_{b \to \infty} \left(-2e^{-b/2} + 2\right) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2} \, dx = \lim_{b \to \infty} \int_0^b e^{-x/2} \, dx = 2.$$

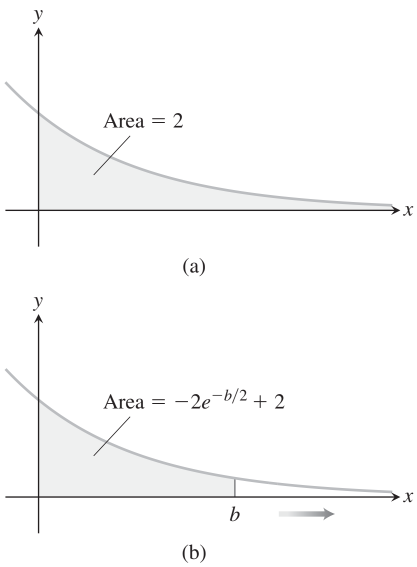


FIGURE 5.19 (a) The area in the first quadrant under the curve $y = e^{-x/2}$. (b) The area is an improper integral of the first type.

> **DEFINITION** Integrals with infinite limits of integration are **improper integrals of Type I**.
>
> **1.** If $f(x)$ is continuous on $[a, \infty)$, then
>
> $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$
>
> **2.** If $f(x)$ is continuous on $(-\infty, b]$, then
>
> $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$
>
> **3.** If $f(x)$ is continuous on $(-\infty, \infty)$, then
>
> $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$
>
> where c is any real number.
>
> In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

It can be shown that the choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^\infty f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 5.19 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 5.20). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

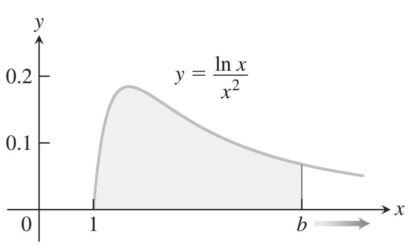


FIGURE 5.20 The area under this curve is an improper integral (Example 1).

The limit of the area as $b \to \infty$ is

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] \\ &= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1 \\ &= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1. \qquad \text{l'Hôpital's Rule}\end{aligned}$$

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} \\ &= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0 \\ &= \lim_{a\to-\infty}(\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \tan^{-1} x\Big]_0^b \\ &= \lim_{b\to\infty}(\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

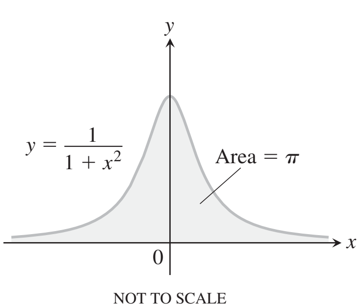


FIGURE 5.21 The area under this curve is finite (Example 2).

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 5.21). ■

The Integral $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}\left(b^{-p+1} - 1\right) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases}\dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1\end{cases}\end{aligned}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1.\end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \int_1^{\infty} \frac{dx}{x} \\ &= \lim_{b\to\infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \ln x\bigg]_1^b \\ &= \lim_{b\to\infty} (\ln b - \ln 1) = \infty.\end{aligned}$$

■

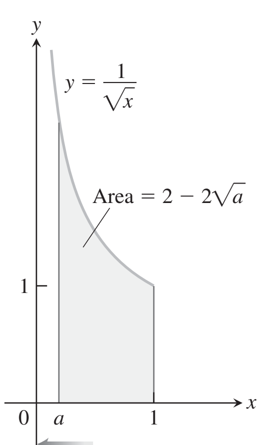


FIGURE 5.22 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 5.18b). First we find the area of the portion from a to 1 (Figure 5.22):

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\bigg]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

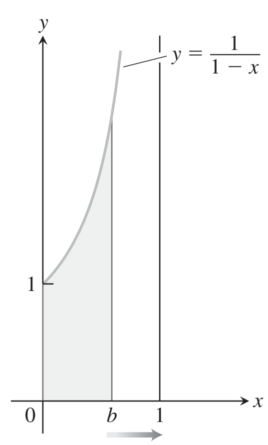


FIGURE 5.23 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 5.23). We evaluate the integral as

$$\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b$$
$$= \lim_{b\to 1^-} \left[-\ln(1-b) + 0\right] = \infty.$$

The limit is infinite, so the integral diverges. ■

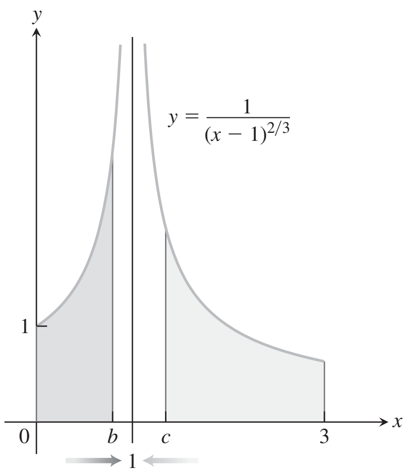


FIGURE 5.24 Example 5 shows that the area under the curve exists (so it is a real number).

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 5.24). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}}\\ &= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b\\ &= \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right] = 3\end{aligned}$$

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}}\\ &= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3\\ &= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

5.13 Properties of Definite Integrals

In defining $\int_a^b f(x)\,dx$ as a limit of sums $\sum_{k=1}^n f(c_k)\,\Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$? Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive. With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral $\int_b^a f(x)\,dx$. Since we have not previously given a meaning to integrating backward, we are led to define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Although we have only defined the integral over an interval $[a, b]$ when $a < b$, it is convenient to have a definition for the integral over $[a, b]$ when $a = b$, that is, for the integral over an interval of zero width. Since $a = b$ gives $\Delta x = 0$, whenever $f(a)$ exists we define

$$\int_a^a f(x)\,dx = 0.$$

Theorem 4 states basic properties of integrals, given as rules that they satisfy, including the two just discussed. These rules, listed in Table 5.9, become very useful in the process of computing integrals. We will refer to them repeatedly to simplify our calculations. Rules 2 through 7 have geometric interpretations, shown in Figure 5.25. The graphs in these figures are of positive functions, but the rules apply to general integrable functions.

THEOREM 4 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.9.

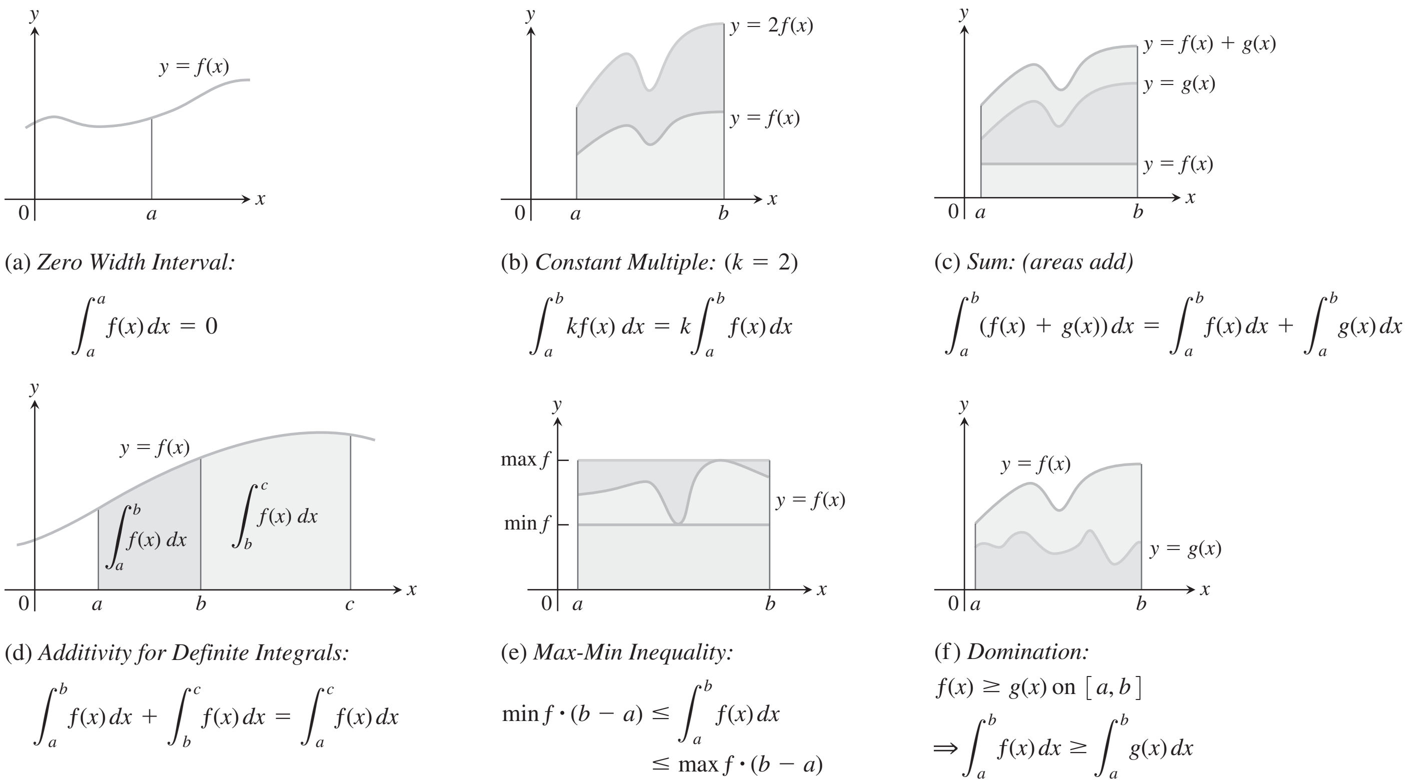
While Rules 1 and 2 are definitions, Rules 3 to 7 of Table 5.9 must be proved. The following is a proof of Rule 6. Similar proofs can be given to verify the other properties in Table 5.9.

TABLE 5.9 Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A definition when $f(a)$ exists
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$	
7. *Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
	$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special case)

Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k,

$$\begin{aligned} \min f \cdot (b - a) &= \min f \cdot \sum_{k=1}^{n} \Delta x_k && \textstyle\sum_{k=1}^{n} \Delta x_k = b - a \\ &= \sum_{k=1}^{n} \min f \cdot \Delta x_k && \text{Constant Multiple Rule} \end{aligned}$$



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$

(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$

(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.25 Geometric interpretations of Rules 2–7 in Table 5.9.

$$
\begin{aligned}
&\le \sum_{k=1}^{n} f(c_k)\,\Delta x_k && \min f \le f(c_k)\\
&\le \sum_{k=1}^{n} \max f \cdot \Delta x_k && f(c_k) \le \max f\\
&= \max f \cdot \sum_{k=1}^{n} \Delta x_k && \text{Constant Multiple Rule}\\
&= \max f \cdot (b - a).
\end{aligned}
$$

In short, all Riemann sums for f on $[a, b]$ satisfy the inequality

$$\min f \cdot (b - a) \le \sum_{k=1}^{n} f(c_k)\,\Delta x_k \le \max f \cdot (b - a).$$

Hence their limit, the integral, does too. ■

EXAMPLE 1 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\displaystyle\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$ Rule 1

2. $\displaystyle\int_{-1}^{1}[2f(x)+3h(x)]\,dx = 2\int_{-1}^{1}f(x)\,dx + 3\int_{-1}^{1}h(x)\,dx$ Rules 3 and 4

$\displaystyle = 2(5)+3(7) = 31$

3. $\displaystyle\int_{-1}^{4}f(x)\,dx = \int_{-1}^{1}f(x)\,dx + \int_{1}^{4}f(x)\,dx = 5+(-2) = 3$ Rule 5 ■

EXAMPLE 2 Show that the value of $\int_0^1\sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f\cdot(b-a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $\max f\cdot(b-a)$ is an *upper bound*. The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1}=\sqrt{2}$, so

$$\int_0^1\sqrt{1+\cos x}\,dx \le \sqrt{2}\cdot(1-0) = \sqrt{2}.$$ ■

Exercises 5.13

Using the Definite Integral Rules

1. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4,\quad \int_1^5 f(x)\,dx = 6,\quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.9 to find

a. $\displaystyle\int_2^2 g(x)\,dx$ **b.** $\displaystyle\int_5^1 g(x)\,dx$

c. $\displaystyle\int_1^2 3f(x)\,dx$ **d.** $\displaystyle\int_2^5 f(x)\,dx$

e. $\displaystyle\int_1^5 [f(x)-g(x)]\,dx$ **f.** $\displaystyle\int_1^5 [4f(x)-g(x)]\,dx$

2. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1,\quad \int_7^9 f(x)\,dx = 5,\quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.9 to find

a. $\displaystyle\int_1^9 -2f(x)\,dx$ **b.** $\displaystyle\int_7^9 [f(x)+h(x)]\,dx$

c. $\displaystyle\int_7^9 [2f(x)-3h(x)]\,dx$ **d.** $\displaystyle\int_9^1 f(x)\,dx$

e. $\displaystyle\int_1^7 f(x)\,dx$ **f.** $\displaystyle\int_9^7 [h(x)-f(x)]\,dx$

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\displaystyle\int_1^2 f(u)\,du$ **b.** $\displaystyle\int_1^2 \sqrt{3}f(z)\,dz$

c. $\displaystyle\int_2^1 f(t)\,dt$ **d.** $\displaystyle\int_1^2 [-f(x)]\,dx$

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\displaystyle\int_0^{-3} g(t)\,dt$ **b.** $\displaystyle\int_{-3}^0 g(u)\,du$

c. $\displaystyle\int_{-3}^0 [-g(x)]\,dx$ **d.** $\displaystyle\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

5. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\displaystyle\int_3^4 f(z)\,dz$ **b.** $\displaystyle\int_4^3 f(t)\,dt$

6. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\displaystyle\int_1^3 h(r)\,dr$ **b.** $\displaystyle -\int_3^1 h(u)\,du$

Using Known Areas to Find Integrals

In Exercises 7–14, graph the integrands and use known area formulas to evaluate the integrals.

7. $\displaystyle\int_{-2}^4\left(\frac{x}{2}+3\right)dx$ **8.** $\displaystyle\int_{1/2}^{3/2}(-2x+4)\,dx$

9. $\displaystyle\int_{-3}^3\sqrt{9-x^2}\,dx$ **10.** $\displaystyle\int_{-4}^0\sqrt{16-x^2}\,dx$

11. $\displaystyle\int_{-2}^1|x|\,dx$ **12.** $\displaystyle\int_{-1}^1(1-|x|)\,dx$

13. $\displaystyle\int_{-1}^1(2-|x|)\,dx$ **14.** $\displaystyle\int_{-1}^1\left(1+\sqrt{1-x^2}\right)dx$

Use known area formulas to evaluate the integrals in Exercises 15–20.

15. $\int_0^b \frac{x}{2}dx, \quad b > 0$

16. $\int_0^b 4x\,dx, \quad b > 0$

17. $\int_a^b 2s\,ds, \quad 0 < a < b$

18. $\int_a^b 3t\,dt, \quad 0 < a < b$

19. $f(x) = \sqrt{4 - x^2}$ on **a.** $[-2, 2]$, **b.** $[0, 2]$

20. $f(x) = 3x + \sqrt{1 - x^2}$ on **a.** $[-1, 0]$, **b.** $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 21–32.

21. $\int_1^{\sqrt{2}} x\,dx$

22. $\int_{0.5}^{2.5} x\,dx$

23. $\int_\pi^{2\pi} \theta\,d\theta$

24. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$

25. $\int_0^{\sqrt[3]{7}} x^2\,dx$

26. $\int_0^{0.3} s^2\,ds$

27. $\int_0^{1/2} t^2\,dt$

28. $\int_0^{\pi/2} \theta^2\,d\theta$

29. $\int_a^{2a} x\,dx$

30. $\int_a^{\sqrt{3}a} x\,dx$

31. $\int_0^{\sqrt[3]{b}} x^2\,dx$

32. $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.9 and Equations (2)–(4) to evaluate the integrals in Exercises 33–42.

33. $\int_3^1 7\,dx$

34. $\int_0^2 5x\,dx$

35. $\int_0^2 (2t - 3)\,dt$

36. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right)dt$

37. $\int_2^1 \left(1 + \frac{z}{2}\right)dz$

38. $\int_3^0 (2z - 3)\,dz$

39. $\int_1^2 3u^2\,du$

40. $\int_{1/2}^1 24u^2\,du$

41. $\int_0^2 (3x^2 + x - 5)\,dx$

42. $\int_1^0 (3x^2 + x - 5)\,dx$

Definite Integrals as Limits of Sums

Use the method of Example 4a or Equation (1) to evaluate the definite integrals in Exercises 43–50.

43. $\int_a^b c\,dx$

44. $\int_0^2 (2x + 1)\,dx$

45. $\int_a^b x^2\,dx, \quad a < b$

46. $\int_{-1}^0 (x - x^2)\,dx$

47. $\int_{-1}^2 (3x^2 - 2x + 1)\,dx$

48. $\int_{-1}^1 x^3\,dx$

49. $\int_a^b x^3\,dx, \quad a < b$

50. $\int_0^1 (3x - x^3)\,dx$

Theory and Examples

51. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

52. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

53. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}dx.$$

54. (*Continuation of Exercise 53.*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}dx.$$

55. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

56. Show that the value of $\int_0^1 \sqrt{x + 8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

57. Integrals of nonnegative functions Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \geq 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \geq 0.$$

58. Integrals of nonpositive functions Show that if f is integrable then

$$f(x) \leq 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \leq 0.$$

59. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

60. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

61. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\text{av}(f)$ should have the same integral over $[a, b]$ as f. Does it? That is, does

$$\int_a^b \text{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

62. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$

b. $\text{av}(kf) = k\,\text{av}(f)$ (any number k)

c. $\text{av}(f) \leq \text{av}(g)$ if $f(x) \leq g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

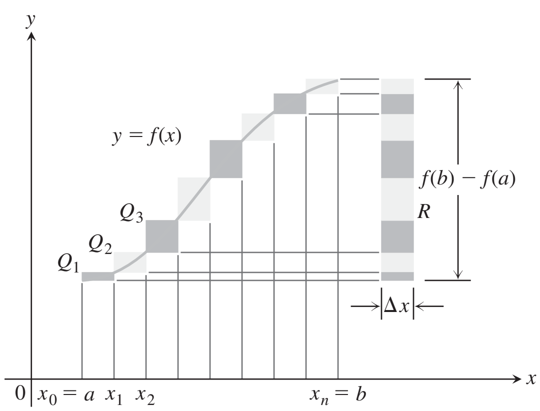
63. Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \leq |f(b) - f(a)|\, \Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.



64. Upper and lower sums for decreasing functions (*Continuation of Exercise 63.*)

a. Draw a figure like the one in Exercise 63 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 63a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)|\, \Delta x_{\max}$$

of Exercise 83b still holds and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.

65. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2 \sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

66. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1\, \Delta x_1 + m_2\, \Delta x_2 + \cdots + m_n\, \Delta x_n$$

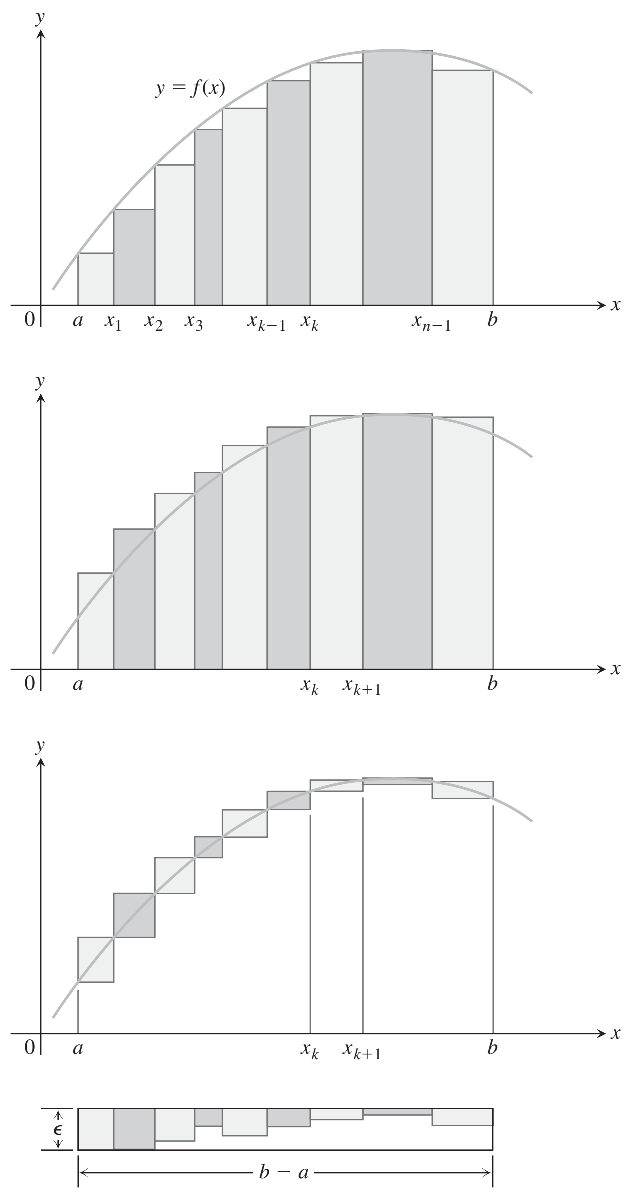
and the shaded regions in the first part of the figure.

b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1\, \Delta x_1 + M_2\, \Delta x_2 + \cdots + M_n\, \Delta x_n$$

and the shaded regions in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



67. We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \leq \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.

68. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer.

5.14 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY
Sir Isaac Newton
(1642–1727)

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.4. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem. We also find that the net change of a function over an interval is the integral of its rate of change, as suggested by Example 3 in Section 5.2.

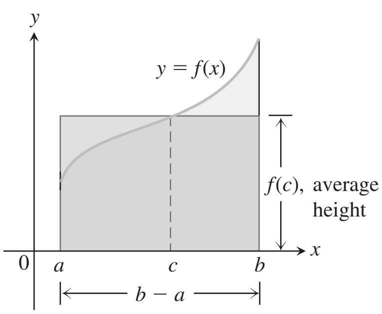


FIGURE 5.26 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\, dx.$$

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\, dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.26 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

THEOREM 5—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\, dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.7, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a}\int_a^b f(x)\, dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.5) says that f must assume every value between min f and max f. It must therefore assume the value $(1/(b - a))\int_a^b f(x)\, dx$ at some point c in $[a, b]$. ■

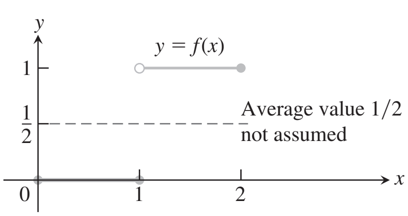


FIGURE 5.27 A discontinuous function need not assume its average value.

The continuity of f is important here. It is possible that a discontinuous function never equals its average value (Figure 5.27).

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$. ■

Fundamental Theorem, Part 1

It can be very difficult to compute definite integrals by taking the limit of Riemann sums. We now develop a powerful new method for evaluating definite integrals, based on using antiderivatives. This method combines the two strands of calculus. One strand involves the idea of taking the limits of finite sums to obtain a definite integral, and the other strand contains derivatives and antiderivatives. They come together in the Fundamental Theorem of Calculus. We begin by considering how to differentiate a certain type of function that is described as an integral.

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

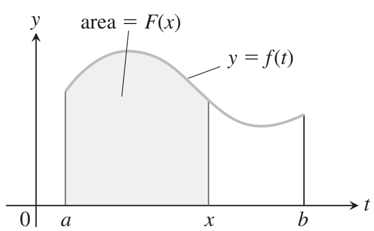


FIGURE 5.28 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x (Figure 5.28). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

Equation (1) gives a way to define new functions, but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x, it asserts that

$$\frac{d}{dx}F(x) = f(x).$$

To gain some insight into why this result holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of the derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x+h) - F(x)}{h}.$$

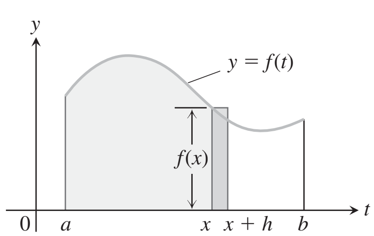


FIGURE 5.29 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x+h)$ is the area to the left of $x+h$. The difference quotient $[F(x+h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$ (Figure 5.29). If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h, which can be seen from Figure 5.29. That is,

$$F(x+h) - F(x) \approx hf(x).$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x).$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 6—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Before proving Theorem 6, we look at several examples to gain a good understanding of what it says. In each example, notice that the independent variable appears in a limit of integration, possibly in a formula.

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$ **(b)** $y = \int_x^5 3t \sin t\, dt$

(c) $y = \int_1^{x^2} \cos t\, dt$ **(d)** $y = \int_{1+3x^2}^4 \frac{1}{2 + t}\, dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\, dt = x^3 + 1 \qquad \text{Eq. (2) with } f(t) = t^3 + 1$$

(b) $$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\, dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right) && \text{Table 5.9, Rule 1} \\ &= -\frac{d}{dx}\int_5^x 3t \sin t\, dt \\ &= -3x \sin x && \text{Eq. (2) with } f(t) = 3t \sin t \end{aligned}$$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du}\int_1^u \cos t\, dt\right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ &= \cos(x^2) \cdot 2x \\ &= 2x \cos x^2 \end{aligned}$$

(d) $$\begin{aligned} \frac{d}{dx}\int_{1+3x^2}^4 \frac{1}{2 + t}\, dt &= \frac{d}{dx}\left(-\int_4^{1+3x^2} \frac{1}{2 + t}\, dt\right) && \text{Rule 1} \\ &= -\frac{d}{dx}\int_4^{1+3x^2} \frac{1}{2 + t}\, dt \\ &= -\frac{1}{2 + (1 + 3x^2)} \cdot \frac{d}{dx}(1 + 3x^2) && \text{Eq. (2) and the Chain Rule} \\ &= -\frac{2x}{1 + x^2} \end{aligned}$$

■

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x + h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b). Doing so, we find

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. && \text{Table 5.9, Rule 5} \end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c). \tag{4}$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x). \tag{5}$$

In conclusion, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= \lim_{h\to 0} f(c) && \text{Eq. (4)} \\ &= f(x). && \text{Eq. (5)} \end{aligned}$$

If $x = a$ or b, then the limit of Equation (3) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. Then Theorem 1 in Section 3.2 shows that F is continuous over $[a, b]$. This concludes the proof. ■

Fundamental Theorem, Part 2 (The Evaluation Theorem)

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

> **THEOREM 6 (Continued)—The Fundamental Theorem of Calculus, Part 2**
> If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.5). Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

■

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

1. Find an antiderivative F of f, and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x)\,dx$.

This process is much easier than using a Riemann sum computation. The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of *any* antiderivative F at only the two endpoints a and b. The usual notation for the difference $F(b) - F(a)$ is

$$F(x)\Big]_a^b \qquad \text{or} \qquad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $$\begin{aligned} \int_0^{\pi} \cos x\,dx &= \sin x\Big]_0^{\pi} \\ &= \sin \pi - \sin 0 = 0 - 0 = 0 \end{aligned}$$

$\frac{d}{dx}\sin x = \cos x$

(b) $$\begin{aligned} \int_{-\pi/4}^{0} \sec x \tan x\,dx &= \sec x\Big]_{-\pi/4}^{0} \\ &= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2} \end{aligned}$$

$\frac{d}{dx}\sec x = \sec x \tan x$

(c) $$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\ &= [8 + 1] - [5] = 4 \end{aligned}$$

$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$

■

Exercise 72 offers another proof of the Evaluation Theorem, bringing together the ideas of Riemann sums, the Mean Value Theorem, and the definition of the definite integral.

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the last equation asserts that the integral of F' is just the *net change* in F as x changes from a to b. Formally, we have the following result.

THEOREM 7—The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{6}$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 7,

$$\int_{x_1}^{x_2} c'(x)\,dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 7 says that

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \le t \le t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.2. ■

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x)\,dx,$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval. So if $v(t)$ represents the velocity function of an object moving along a coordinate line, this means that the object's final position $s(t_2)$ over a time interval $t_1 \le t \le t_2$ is its initial position $s(t_1)$ plus its net change in position along the line (see Example 4b).

EXAMPLE 5 Consider again our analysis of a heavy rock blown straight up from the ground by a dynamite blast (Example 3, Section 5.2). The velocity of the rock at any time t during its motion was given as $v(t) = 160 - 32t$ ft/sec.

(a) Find the displacement of the rock during the time period $0 \le t \le 8$.

(b) Find the total distance traveled during this time period.

Solution

(a) From Example 4b, the displacement is the integral

$$\int_0^8 v(t)\,dt = \int_0^8 (160 - 32t)\,dt = \left[160t - 16t^2\right]_0^8$$
$$= (160)(8) - (16)(64) = 256.$$

This means that the height of the rock is 256 ft above the ground 8 sec after the explosion, which agrees with our conclusion in Example 3, Section 5.2.

(b) As we noted in Table 5.5, the velocity function $v(t)$ is positive over the time interval $[0, 5]$ and negative over the interval $[5, 8]$. Therefore, from Example 4b, the total distance traveled is the integral

$$\begin{aligned}\int_0^8 |v(t)|\,dt &= \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt \\ &= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt \\ &= \left[160t - 16t^2\right]_0^5 - \left[160t - 16t^2\right]_5^8 \\ &= [(160)(5) - (16)(25)] - [(160)(8) - (16)(64) - ((160)(5) - (16)(25))] \\ &= 400 - (-144) = 544.\end{aligned}$$

Again, this calculation agrees with our conclusion in Example 3, Section 5.1. That is, the total distance of 544 ft traveled by the rock during the time period $0 \le t \le 8$ is (i) the maximum height of 400 ft it reached over the time interval $[0, 5]$ plus (ii) the additional distance of 144 ft the rock fell over the time interval $[5, 8]$. ■

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t)\,dt = F(x) - F(a),$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) for every continuous function f.

Exercises 5.14

Evaluating Integrals

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int_0^2 x(x - 3)\,dx$

2. $\displaystyle\int_{-1}^1 (x^2 - 2x + 3)\,dx$

3. $\displaystyle\int_{-2}^2 \frac{3}{(x + 3)^4}\,dx$

4. $\displaystyle\int_{-1}^1 x^{299}\,dx$

5. $\displaystyle\int_1^4 \left(3x^2 - \frac{x^3}{4}\right)dx$

6. $\displaystyle\int_{-2}^3 (x^3 - 2x + 3)\,dx$

7. $\int_0^1 (x^2 + \sqrt{x})\,dx$

8. $\int_1^{32} x^{-6/5}\,dx$

9. $\int_0^{\pi/3} 2\sec^2 x\,dx$

10. $\int_0^{\pi} (1 + \cos x)\,dx$

11. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\,d\theta$

12. $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$

13. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\,dt$

14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$

15. $\int_0^{\pi/4} \tan^2 x\,dx$

16. $\int_0^{\pi/6} (\sec x + \tan x)^2\,dx$

17. $\int_0^{\pi/8} \sin 2x\,dx$

18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)dt$

19. $\int_1^{-1} (r + 1)^2\,dr$

20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4)\,dt$

21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right)du$

22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\,dy$

23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\,ds$

24. $\int_1^{8} \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}}\,dx$

25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

26. $\int_0^{\pi/3} (\cos x + \sec x)^2\,dx$

27. $\int_{-4}^{4} |x|\,dx$

28. $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$

In Exercises 29–32, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep in mind the Chain Rule in guessing an antiderivative. You will learn how to find such antiderivatives in the next section.)

29. $\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$

30. $\int_1^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

31. $\int_2^{5} \frac{x\,dx}{\sqrt{1 + x^2}}$

32. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 33–38.

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

33. $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$

34. $\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt$

35. $\frac{d}{dt}\int_0^{t^4} \sqrt{u}\,du$

36. $\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy$

37. $\frac{d}{dx}\int_0^{x^3} t^{-2/3}\,dt$

38. $\frac{d}{dt}\int_1^{\sqrt{t}} \left(x^4 + \frac{3}{x^3}\right)dx$

Find dy/dx in Exercises 39–46.

39. $y = \int_0^x \sqrt{1 + t^2}\,dt$

40. $y = \int_1^x \frac{1}{t}\,dt, \quad x > 0$

41. $y = \int_{\sqrt{x}}^{0} \sin(t^2)\,dt$

42. $y = x\int_2^{x^2} \sin(t^3)\,dt$

43. $y = \int_{-1}^{x} \frac{t^2}{t^2 + 4}\,dt - \int_3^x \frac{t^2}{t^2 + 4}\,dt$

44. $y = \left(\int_0^x (t^3 + 1)^{10}\,dt\right)^3$

45. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1 - t^2}}, \quad |x| < \frac{\pi}{2}$

46. $y = \int_{\tan x}^{0} \frac{dt}{1 + t^2}$

Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 47–50. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t}\,dt - 3$

b. $y = \int_0^x \sec t\,dt + 4$

c. $y = \int_{-1}^x \sec t\,dt + 4$

d. $y = \int_{\pi}^x \frac{1}{t}\,dt - 3$

47. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$

48. $y' = \sec x, \quad y(-1) = 4$

49. $y' = \sec x, \quad y(0) = 4$

50. $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 51 and 52 in terms of integrals.

51. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$

52. $\frac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

53. The temperature T(°F) of a room at time t minutes is given by

$$T = 85 - 3\sqrt{25 - t} \quad \text{for} \quad 0 \le t \le 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \le t \le 25$.

54. The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \le t \le 8$.

55. Suppose that $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

56. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos\pi x$.

57. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1 + t}\,dt$$

at $x = 1$.

58. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$$

at $x = -1$.

59. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

60. Another proof of the Evaluation Theorem

a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i). Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.

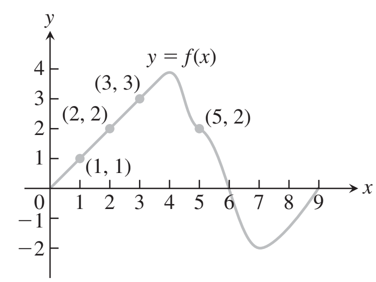
c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

61. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

f. When is the particle moving toward the origin? Away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

62. The marginal cost of manufacturing x units of an electronic device is $0.001x^2 - 0.5x + 115$. If 600 units are produced, what is the production cost per unit?

5.15 Definite Integral Substitutions

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We apply the new formula introduced here to the problem of computing the area between two curves.

The Substitution Formula

The following formula shows how the limits of integration change when the variable of integration is changed by substitution.

THEOREM 8—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof Let F denote any antiderivative of f. Then,

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = F(g(x))\Big]_{x=a}^{x=b}$$

$\frac{d}{dx}F(g(x))$
$= F'(g(x))g'(x)$
$= f(g(x))g'(x)$

$$= F(g(b)) - F(g(a))$$

$$= F(u)\Big]_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\,du.$$

Fundamental Theorem, Part 2 ■

To use the formula, make the same u-substitution $u = g(x)$ and $du = g'(x)\,dx$ you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 8.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C$$

Integrate with respect to u.

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$ ■

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 8, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 We use the method of transforming the limits of integration.

(a)
$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du)$$

Let $u = \cot\theta$, $du = -\csc^2\theta \, d\theta$, $-du = \csc^2\theta \, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u \, du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b)
$$\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x \, dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= 0$$

Zero width interval ■

Some Important Notes

(1) If $\int_a^b f(x)\mathrm{d}x = 0$, then the equation $f(x) = 0$ has at least one root in (a, b) provided f is continuous in (a, b).

EXAMPLE 3 If $3a + 4b + 6c + 12d = 0$, then prove that the equation $ax^3 + bx^2 + cx + d = 0$ has a root in $(0, 1)$.

Solution Clearly if we assume $f(x) = ax^3 + bx^2 + cx + d$, then it is a continuous function in $(0, 1)$.

$$\text{Now } \int_0^1 (ax^3 + bx^2 + (cx + d))\mathrm{d}x = \left.\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + \mathrm{d}x\right|_0^1$$

$$= \frac{3a + 4b + 6c + 12d}{12} = 0$$

Hence, we can say graph of $f(x) = ax^3 + bx^2 + cx + d$ intersects x-axis at least once in $(0, 1)$. ■

(2) If $g(x)$ is inverse of $f(x)$ such that $f(a) = c$ and $f(b) = d$, then he value of $\int_a^b f(x)\mathrm{d}x + \int_c^d g(y)\mathrm{d}y = bd - ac$

Proof Let $I_1 = \int_a^b f(x)\mathrm{d}x$ and $I_2 = \int_c^d g(y)\mathrm{d}y$

Now, put $y = f(x)$ in I_2

$$\Rightarrow \qquad dy = f'(x)\,dx$$

$$\therefore \qquad I_2 = \int_a^b g(f(x))f'(x)\,dx = \int_a^b xf'(x)\,dx$$

Now, $\quad I_1 + I_2 = \int_a^b (f(x) + xf'(x))dx = xf(x)\Big|_a^b = (b)f(b) - af(a) = bd - ac$

For example, $\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx = e^{\sqrt{e}}$

Clearly, $e^{\sqrt{e^x}}$ and 2ln (ln x) are inverse functions of each other. ■

(3) $\int_{-a}^{a} f(x)dx = \int_0^a (f(x) + f(-x))dx$

Proof Clearly $\int_{-a}^{a} f(x)dx = \underbrace{\int_{-a}^{0} f(x)dx}_{I_1} + \underbrace{\int_0^a f(x)dx}_{I_2}$

Now, put $\quad x = -t$ in I_1

$$\therefore \qquad I_1 = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx$$

$$\therefore \qquad \int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x)dx$$

Now, if $f(x)$ is an even function, then $f(-x) = f(x)$

$$\therefore \qquad \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$$

But if $f(x)$ is an odd function, then $f(-x) = -f(x)$

$$\therefore \qquad \int_{-a}^{a} f(x)dx = 0$$

■

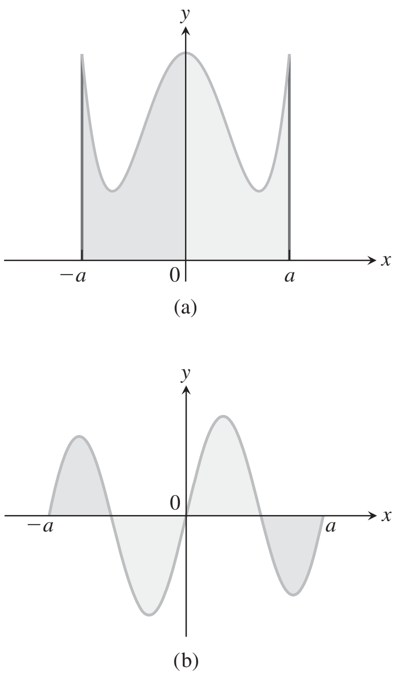


FIGURE 5.30 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

(4) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ or $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof $\quad I = \int_a^b f(x)dx$

Put $\quad x = a + b - t$

$$\Rightarrow \qquad dx = -dt$$

Also

x	a	b
t	b	a

$$\therefore \qquad I = \int_b^a f(a+b-t)(-dt) = \int_a^b f(a+b-t)\,dt = \int_a^b f(a+b-x)\,dx$$

Also, if we put lower limit as 0 and upper limit as "a", then $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

■

(5) $\int_0^{2a} f(x)\,\mathrm{d}x = \int_0^{a} (f(x) + f(2a-x))\,\mathrm{d}x$

Proof $I = \int_0^{2a} f(x)\,\mathrm{d}x = \underbrace{\int_0^{a} (f(x) +}_{I_1} \underbrace{\int_a^{2a} f(x) + \mathrm{d}x}_{I_2}$

Put $x = 2a - t$ in I_2

$$\Rightarrow \quad \mathrm{d}x = -\mathrm{d}t$$

$$I_2 = \int_0^{a} f(2a-t)\,\mathrm{d}t = \int_0^{a} f(2a-x)\mathrm{d}x$$

$$\int_0^{2a} f(x)\mathrm{d}x = \int_0^{a} (f(x) + f(2a-x))\mathrm{d}x$$

If $f(2a - x) = f(x)$

$$\Rightarrow \quad \int_0^{2a} f(x)\mathrm{d}x = 2\int_0^{a} f(x)\mathrm{d}x$$

and if $\quad f(2a - x) = -f(x)$

$$\Rightarrow \quad \int_0^{2a} f(x)\mathrm{d}x = 0$$ ■

(6) $\int_0^{nT} f(x)\mathrm{d}x = n\int_0^{T} f(x)\mathrm{d}x$, (where $n \in I$ and $f(x + T) = f(x)$)

Proof $\int_0^{nT} f(x)\mathrm{d}x = \underbrace{\int_0^{T} f(x)\mathrm{d}x}_{I_1} + \underbrace{\int_T^{2T} f(x)\mathrm{d}x}_{I_2} + \ldots.. + \underbrace{\int_{(n-1)T}^{nT} f(x)}_{I_n}$

Clearly $f(x)$ is a periodic function with period T,

$$\therefore \quad I_1 = I_2 = \ldots = I_{\mathrm{n}}$$

$$\therefore \quad \int_0^{nT} f(x)\mathrm{d}x = n \cdot \int_0^{T} f(x)\mathrm{d}x$$ ■

EXAMPLE 4 $I = \int_0^{3} |4x - 3|\mathrm{d}x$

Clearly $|4x - 3| = \begin{cases} 4x - 3 & x \geq \dfrac{3}{4} \\ 3 - 4x & x < \dfrac{3}{4} \end{cases}$

$$\therefore \quad I = \int_0^{3/4} (3 - 4x)\mathrm{d}x + \int_{3/4}^{3} (4x - 3)\mathrm{d}x$$

$$3x - 2x^2\Big|_0^{3/4} + 2x^2 - 3x\Big|_{3/4}^{3}$$

$$= \frac{9}{4} - \frac{9}{8} + 18 - 9 - \frac{9}{8} + \frac{9}{4}$$

$$= \frac{9}{2} - \frac{9}{4} + 9 = \frac{9}{4} + 9 = \frac{45}{4}$$

Aliter: $I = \int_0^3 |4x - 3|\,dx$

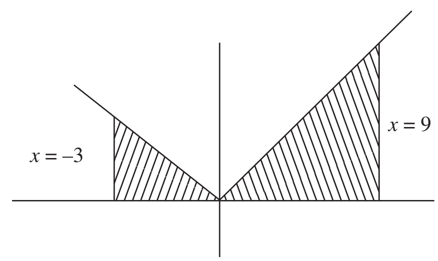
Put $\quad 4x - 3 = t$

$\Rightarrow \quad dx = \frac{1}{4}dt$

x	0	3
t	-3	9

$\therefore \quad I = \int_{-3}^{9} |t|\,dt$

Now, sketch the graph of $|t|$.

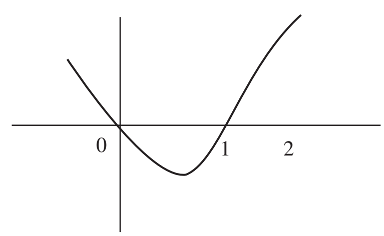


Area of shaded region is $\left(\frac{1}{2}\times 3\times 3 + \frac{1}{2}\times 9\times 9\right) = 45$

$\therefore \quad I = \frac{45}{4}$ ■

EXAMPLE 5 $\quad I = \int_0^2 [x^2 - x]\,dx$ (where $[\cdot]$ denotes greatest integer function).

Solution Let us sketch the graph of $x^2 - x$.



Clearly, when $x \in (0, 1)$

$\Rightarrow \quad x^2 - x \in (-1/4, 0)$

$\Rightarrow \quad [x^2 - x] = -1$

Also when $x \in (1, 2)$

$\Rightarrow \quad x^2 - x \in (0, 2)$

$\Rightarrow \quad [x^2 - x] = 0$ or 1

Now, when $x^2 - x = 1$

$\Rightarrow \quad x^2 - x = 1$

$$x = \frac{1+\sqrt{5}}{2}$$

$\therefore$ when $x \in \left(1, \dfrac{\sqrt{5}+1}{2}\right)$

$\Rightarrow \quad [x^2 - x] = 0$

and $\quad x \in \left(\dfrac{\sqrt{5}+1}{2}, 2\right)$

$\Rightarrow \quad [x^2 - x] = 1$

$$\therefore \quad \int_0^2 [x^2 - x]\mathrm{d}x = 1\times(-1) + \left(2 - \left(\frac{\sqrt{5}+1}{2}\right)\right)\times 1$$

$$= -1 + \left(\frac{3-\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2}$$ ■

EXAMPLE 6 $\quad I = \displaystyle\int_{-1}^{3/2} |x \sin \pi x|\, \mathrm{d}x$

Solution $\quad I = \underbrace{\displaystyle\int_{-1}^{1} |x \sin \pi x| \mathrm{d}x}_{I_1} + \underbrace{\displaystyle\int_{1}^{3/2} |x \sin \pi x| \mathrm{d}x}_{I_2}$

$$I_1 = \int_{-1}^{1} | x \sin \pi x | \mathrm{d}x = 2\int_0^1 | x \sin \pi x | \mathrm{d}x \quad \text{(as } |x \sin \pi x| \text{ is an even function)}$$

$$= 2\int_0^1 x \sin \pi x \, \mathrm{d}x$$

$$= 2\left[x\left(-\frac{\cos \pi x}{\pi}\right)\right]_0^1 + \frac{1}{\pi}\int_0^1 \cos \pi x \, \mathrm{d}x$$

$$= 2\left[\frac{1}{\pi} + \frac{1}{\pi}\frac{\sin \pi x}{\pi}\right]_0^1 = \frac{2}{\pi}$$

Now, $I_2 = -\displaystyle\int_1^{3/2} (x \sin \pi x)\mathrm{d}x = \left[\left.\frac{-x\cos \pi x}{\pi}\right|_1^{3/2} + \left.\frac{1}{\pi^2}\sin \pi x\right|_1^{3/2}\right]$

$$= -\left[\frac{-1}{\pi} + \left(-\frac{1}{\pi^2}\right)\right] = \frac{1}{\pi} + \frac{1}{\pi^2}$$

$\therefore \quad I = \dfrac{3\pi + 1}{\pi^2}$ ■

EXAMPLE 7 $\quad I = \displaystyle\int_{-\pi/2}^{\pi/2} \left(\frac{1}{(2015)^x + 1}\right) \frac{\sin^{2014} x}{\sin^{2014} x + \cos^{2014} x} \mathrm{d}x$

Solution Clearly we can see the integral is of the form $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x$.

So, first we can apply the formulae $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a} f(x)+f(-x)\mathrm{d}x$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2}\left(\frac{1}{(2015)^x+1}\right)\left(\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}\right)\mathrm{d}x$$

$$= \int_{0}^{\pi/2}\left(\frac{1}{(2015)^x+1}\cdot\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}+\frac{1}{(2015)^{-x}+1}\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}\right)\mathrm{d}x$$

$$I = \int_{0}^{\pi/2}\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}\left(\frac{1}{(2015)^x+1}+\frac{(2015)^x}{1+(2015)^x}\right)\mathrm{d}x$$

$$I = \int_{0}^{\pi/2}\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}\mathrm{d}x$$

Now, $$\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)+f(2a-x)\mathrm{d}x$$

$$\therefore \quad \int_{0}^{\pi/2}\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}\mathrm{d}x = \int_{0}^{\pi/4}\left(\frac{\sin^{2014}x}{\sin^{2014}x+\cos^{2014}x}+\frac{\cos^{2014}x}{\sin^{2014}x+\cos^{2014}x}\right)\mathrm{d}x$$

$$= \int_{0}^{\pi/4} 1.\mathrm{d}x = \frac{\pi}{4}$$

$$\therefore \quad I = \frac{\pi}{4}$$ ■

EXAMPLE 8 $I = \int_{0}^{1}\cot^{-1}(1-x+x^2)\,\mathrm{d}x$

Solution $I = \int_{0}^{1}\tan^{-1}\left(\frac{1}{1-x+x^2}\right)\mathrm{d}x$

$$= \int_{0}^{1}\tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)\mathrm{d}x = \int_{0}^{1}(\tan^{-1}x-\tan^{-1}(x-1))\mathrm{d}x$$

$$\therefore \quad I = \underbrace{\int_{0}^{1}\tan^{-1}x\mathrm{d}x}_{I_1} - \underbrace{\int_{0}^{1}\tan^{-1}(x-1)\mathrm{d}x}_{I_2}$$

Now, applying the property

$$\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx \;\; \mathrm{d}x \text{ in } I_2.$$

$$I_2 = \int_{0}^{1}\tan^{-1}(x-1)\mathrm{d}x = \int_{0}^{1}\tan^{-1}(1-x-1)\mathrm{d}x = -\int_{0}^{1}\tan^{-1}x\,\mathrm{d}x$$

$$\therefore \quad I = 2\int_{0}^{1}\tan^{-1}x\mathrm{d}x$$

Applying by parts by taking 1 as second function,

$$I = 2\left[x.\tan^{-1} x\,|_0^1 - \int_0^1 x\frac{1}{1+x^2}\,dx\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\,|_0^1\right] = \frac{\pi}{2} - \ln 2$$ ■

EXAMPLE 9 $I = \int_0^{2n\pi+q} |\cos x|\,dx$ (where $\pi/2 < q < \pi$ and $n \in N$)

Solution $I = \int_0^{2n\pi+q} |\cos x|\,dx = \int_0^{2n\pi} |\cos x|\,dx + \int_{2n\pi}^{2n\pi+q} |\cos x|\,dx$

$$= 2n\int_0^{\pi} |\cos x|\,dx + \int_0^{q} |\cos x|\,dx$$

(Clearly $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\ (n \in N)$)

$$= 2n \times 2 + \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{q} |\cos x|\,dx$$

(Here, $\int_0^{\pi} |\cos x|\,dx = 2$ and since $q \in (\pi/2, \pi)$;

Hence, $\int_0^{q} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{q} |\cos x|\,dx$)

$\therefore$ $I = 4n + 1 - \int_{\pi/2}^{q} \cos x dx = 4n + 1 - \left(\sin x|_{\pi/2}^{q}\right) = 4n + 1 - (\sin q - 1)$

$I = (4n + 2) - (\sin q)$ ■

Derivative of Antiderivative (Leibnitz Rule)

If $f(x)$ is a continuous function then, $\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f(h(x)) \times h'(x) - f(g(x)) \times g'(x)$

Proof Suppose $\int f(t)dt = F(t) + C$

Now $\int_{g(x)}^{h(x)} f(t)\,dt = F(t) + C\Big|_{g(x)}^{b(x)} = F(h(x)) - F(g(x))$

$\therefore$ $\frac{d}{dx}\left(\int_{g(x)}^{h(x)} (t)dt\right) = \frac{d}{dx}\ (F(h(x)) - F(g(x))$

$$= F'(h(x)) \times h'(x) - F'(g(x)) \times g'(x)$$

Now, $\int f(t)dt = F(t) + C$

$\Rightarrow$ $f(t) = F'(t)$

$$\therefore \qquad \boxed{\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(x)dt\right) = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

Now, if the integrand $f(t)$ is function of x as well as t, then we can use the concept of partial derivative.

$$\text{i.e.,} \boxed{\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(x,t)dt\right) = \int_{g(x)}^{h(x)} \frac{d}{dx}(f(x,t))dt + f(x,h(x))\times h'(x) - f(g(x))\cdot g'(x)}$$

EXAMPLE 10 If $x \in \left[0, \frac{\pi}{2}\right]$; prove that $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt = \frac{\pi}{4}$

Solution In such equations generally we start using by parts but if we find the derivative of $f(x)$, it vanishes which implies $f(x)$ is constant.

$$f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}dt$$

$$\Rightarrow \qquad f'(x) = (\sin^{-1}\sqrt{\sin^2 x}\,)\,(2\sin x\cos x) + \cos^{-1}(\sqrt{\cos^2 x}) \times (-2\cos x\sin x)$$

$$= (x) \times (2\sin x\cos x) - x.2\sin x\cos x = 0$$

$\Rightarrow$ $f(x)$ is constant

Now, if we calculate $f(x)$ for any $x \in [0, \pi/2]$, then $f(x)$ will attain the same value in the interval suitably we can see if we put $x = \pi/4$; lower and upper limit will be same and the integrands will add.

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \int_0^{1/2} \sin^{-1}\sqrt{t}dt + \int_0^{1/2} \cos^{-1}\sqrt{t}dt = \int_0^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt = \int_0^{1/2} \frac{\pi}{2}dt = \frac{\pi}{4}$$

$$\therefore \quad f(x) = \frac{\pi}{4}\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$$

■

EXAMPLE 11 Let $f(x)$ is a derivable function satisfying $f(x) = \int_0^x e^t \sin(x-t)dt$ and $g(x) = f''(x) - f(x)$, then find the range of $g(x)$.

Solution $f(x) = \int_0^x e^t \sin(x-t)dt$

Now, we cannot apply Leibnitz rule directly as the integrand consists of x as well as t. Our first priority will be to take x out of the integrand either using certain substitutions or using any property of definite integral the problem can be also tackled by simply integrating to get $f(x)$.

Now if we apply $\int_0^a f(x)dx = \int_0^a f(a-x)dx$; we will get

$$f(x) = \int_0^x e^t \sin(x-t)\,dt = \int_0^x e^{x-t}\sin(x-(x-t))\,dt$$

$$f(x) = \int_0^x e^{x-t}\sin t dt$$

$$\Rightarrow \qquad f(x) = e^x \int_0^x e^{-t}\sin t dt$$

$$\Rightarrow \qquad e^{-x}f(x) = \int_0^x e^{-t}\sin t\, dt$$

Differentiating we will get

$$\Rightarrow \qquad e^{-x}f'(x) - e^{-x}f(x) = e^{-x}\sin x$$

$$\Rightarrow \qquad f'(x) - f(x) = \sin x. \qquad \text{(i)}$$

Differentiating again, we have

$$\Rightarrow \qquad f''(x) - f'(x) = \cos x$$

Adding, we get

$$f''(x) - f(x) = \sin x + \cos x$$

Hence, range of $g(x)$ is $[-\sqrt{2}, \sqrt{2}]$. ■

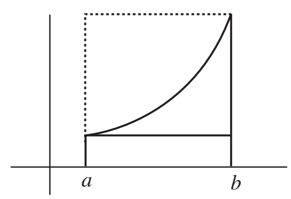
Estimation of Definite Integral and General Inequalities in Integration

Sometimes the integral to be evaluated is not very easy and we need to estimate the value of integral. There can be various techniques to estimate the value of definite integral.

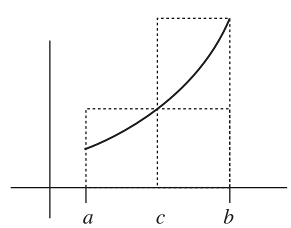
(1) If a function is increasing in (a, b), then

$$\boxed{(b-a)f(a) \le \int_a^b f(x)\,dx \le (b-a)\ f(b)}$$

This can be clearly seen from the following graph.



Clearly $\int_a^b f(x)dx$ represents are under curve and $(b-a)f(a)$ denotes the area of the smaller rectangle clearly the approximation can be improved if we apply two rectangle approach.



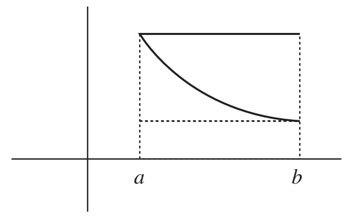
If we select any $x = c \in (a, b)$, we get the following estimation.

$$\boxed{(c-a)f(a)+(b-c)f(c) \le \int_a^b f(x)\,dx \le (c-a)f(c)+(b-c)f(b)}$$

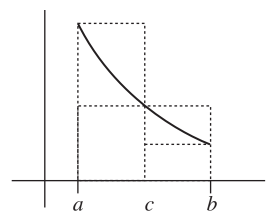
(2) If $f(x)$ is decreasing in (a, b), then

$$\boxed{(b-a)f(b) \le \int_a^b f(x)\mathrm{d}x \le (b-a)f(a)}$$

This can be seen through the following graph.



Clearly $\int_a^b f(x)\mathrm{d}x$ is area under curve bounded by the lines $x = a$, $x = b$ and x-axis; $(b - a)\, f(b)$ denotes the area of rectangle and $(b - a)\, f(a)$ denotes the area of bigger rectangle. Also if we take a point $x = c \in (a, b)$ we get

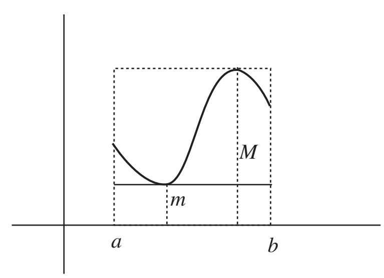


$$\boxed{(c-a)f(c)+(b-c)f(b) \le \int_a^b f(x)\,\mathrm{d}x \le (c-a)f(a)+(b-c)f(c)}$$

(3) If $f(x)$ is non-monotonic in (a, b), then

$$\boxed{(b-a)m \le \int_a^b f(x)\mathrm{d}x \le (b-a)\cdot M}$$

Clearly "m" and "M" denotes, respectively, the smallest and largest value of $f(x)$ in the interval (a, b). Clearly, the following graph gives a fair idea about the above inequality.



(4) The following inequality is also used

$$\boxed{\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |\, f(x)\,|\,\mathrm{d}x}$$

where equality holds if the graph of $f(x)$ lies completely above or below x-axis.

EXAMPLE 12 Prove that $1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$

Solution Let $f(x) = \frac{\sin x}{x}$

$$\therefore \qquad f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$$

Hence $f(x)$ is a decreasing function.

Now,
$$\left(\frac{\pi}{2} - 0\right)\cdot\frac{2}{\pi} < \int_0^{\pi/2} \frac{\sin x}{x} dx < \left(\frac{\pi}{2} - 0\right)\cdot 1$$

$$\Rightarrow \qquad 1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$$ ■

Exercises 5.15

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 8 to evaluate the integrals in Exercises 1–24.

1. a. $\int_0^3 \sqrt{y+1}\, dy$ **b.** $\int_{-1}^0 \sqrt{y+1}\, dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\, dr$ **b.** $\int_{-1}^1 r\sqrt{1-r^2}\, dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\, dx$ **b.** $\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\, dx$ **b.** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$

5. a. $\int_0^1 t^3(1+t^4)^3\, dt$ **b.** $\int_{-1}^1 t^3(1+t^4)^3\, dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ **b.** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ **b.** $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ **b.** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ **b.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\, dx$ **b.** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\, dx$

11. a. $\int_0^1 t\sqrt{4+5t}\, dt$ **b.** $\int_1^9 t\sqrt{4+5t}\, dt$

12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\, dt$ **b.** $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\, dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$ **b.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$ **b.** $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$ **16.** $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$ **18.** $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\, dt$ **20.** $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\, dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\, dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\, dy$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2\left(\theta^{3/2}\right) d\theta$ **24.** $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right) dt$

Theory and Examples

25. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\, dx$$

in terms of F.

26. Show that if f is continuous, then

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

27. Suppose that

$$\int_0^1 f(x)\,dx = 3.$$

Find

$$\int_{-1}^0 f(x)\,dx$$

if **a.** f is odd, **b.** f is even.

28. a. Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x)\,dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

29. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x) + f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

30. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}dt = \int_1^y \frac{1}{t}dt.$$

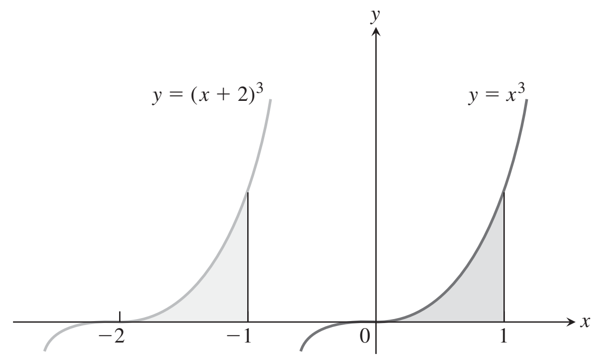
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \tag{1}$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3\,dx = \int_0^1 x^3\,dx$$

because the areas of the shaded regions are congruent.



31. Use a substitution to verify Equation (1).

32. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x-4}, \quad a = 4, \quad b = 8, \quad c = 5$

Chapter 5 Questions to Guide Your Review

1. How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
2. What is sigma notation? What advantage does it offer? Give examples.
3. What is a Riemann sum? Why might you want to consider such a sum?
4. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
5. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
6. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
7. Discuss how the processes of integration and differentiation can be considered as "inverses" of each other.
8. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
9. How is integration by substitution related to the Chain Rule?
10. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
11. How does the method of substitution work for definite integrals? Give examples.
12. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

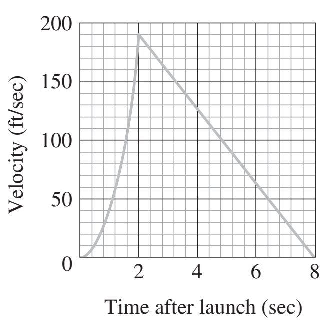
13. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
14. What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?
15. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
16. What is an initial value problem? How do you solve one? Give an example.
17. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
18. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
19. If an integrand is a product of the form $\sin^n x \cos^m x$, where m and n are nonnegative integers, how do you evaluate the integral? Give a specific example of each case.
20. What substitutions are made to evaluate integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$? Give an example of each case.
21. What substitutions are sometimes used to transform integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals that can be evaluated directly? Give an example of each case.
22. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?
23. What is the goal of the method of partial fractions?
24. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
    a. is a product of distinct linear factors?
    b. consists of a repeated linear factor?
    c. contains an irreducible quadratic factor?

    What do you do if the degree of f is *not* less than the degree of g?
25. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
26. What is a reduction formula? How are reduction formulas used? Give an example.

Chapter 5 Practice Exercises

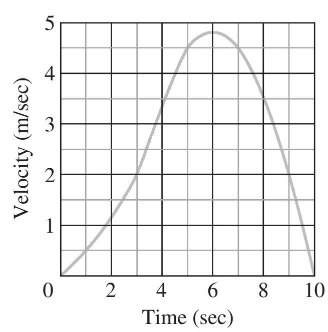
Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.



   a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)
   b. Sketch a graph of the rocket's height above ground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?
   b. Sketch a graph of s as a function of t for $0 \le t \le 10$, assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of
   a. $\sum_{k=1}^{10} \frac{a_k}{4}$
   b. $\sum_{k=1}^{10} (b_k - 3a_k)$
   c. $\sum_{k=1}^{10} (a_k + b_k - 1)$
   d. $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of
   a. $\sum_{k=1}^{20} 3a_k$
   b. $\sum_{k=1}^{20} (a_k + b_k)$
   c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$
   d. $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

5. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a. $\displaystyle\int_{-2}^{2} f(x)\,dx$ **b.** $\displaystyle\int_{2}^{5} f(x)\,dx$

c. $\displaystyle\int_{5}^{-2} g(x)\,dx$ **d.** $\displaystyle\int_{-2}^{5} (-\pi g(x))\,dx$

e. $\displaystyle\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

6. If $\int_{0}^{2} f(x)\,dx = \pi$, $\int_{0}^{2} 7g(x)\,dx = 7$, and $\int_{0}^{1} g(x)\,dx = 2$, find the values of the following.

a. $\displaystyle\int_{0}^{2} g(x)\,dx$ **b.** $\displaystyle\int_{1}^{2} g(x)\,dx$

c. $\displaystyle\int_{2}^{0} f(x)\,dx$ **d.** $\displaystyle\int_{0}^{2} \sqrt{2}\, f(x)\,dx$

e. $\displaystyle\int_{0}^{2} (g(x) - 3f(x))\,dx$

Initial Value Problems

7. Show that $y = x^2 + \displaystyle\int_{1}^{x} \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2 y}{dx^2} = 2 - \frac{1}{x^2};\quad y'(1) = 3,\quad y(1) = 1.$$

8. Show that $y = \int_{0}^{x} \left(1 + 2\sqrt{\sec t}\right) dt$ solves the initial value problem

$$\frac{d^2 y}{dx^2} = \sqrt{\sec x}\tan x;\quad y'(0) = 3,\quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 35 and 36 in terms of integrals.

9. $\dfrac{dy}{dx} = \dfrac{\sin x}{x},\quad y(5) = -3$

10. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x},\quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 11–18.

11. $\displaystyle\int 2(\cos x)^{-1/2} \sin x\,dx$ **12.** $\displaystyle\int (\tan x)^{-3/2} \sec^2 x\,dx$

13. $\displaystyle\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

14. $\displaystyle\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

15. $\displaystyle\int \left(t - \frac{2}{t}\right)\left(t + \frac{2}{t}\right) dt$ **16.** $\displaystyle\int \frac{(t+1)^2 - 1}{t^4}\,dt$

17. $\displaystyle\int \sqrt{t}\,\sin(2t^{3/2})\,dt$ **18.** $\displaystyle\int (\sec\theta\tan\theta)\sqrt{1 + \sec\theta}\,d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 19–40.

19. $\displaystyle\int_{-1}^{1} (3x^2 - 4x + 7)\,dx$ **20.** $\displaystyle\int_{0}^{1} (8s^3 - 12s^2 + 5)\,ds$

21. $\displaystyle\int_{1}^{2} \frac{4}{v^2}\,dv$ **22.** $\displaystyle\int_{1}^{27} x^{-4/3}\,dx$

23. $\displaystyle\int_{1}^{4} \frac{dt}{t\sqrt{t}}$ **24.** $\displaystyle\int_{1}^{4} \frac{\left(1 + \sqrt{u}\right)^{1/2}}{\sqrt{u}}\,du$

25. $\displaystyle\int_{0}^{1} \frac{36\,dx}{(2x + 1)^3}$ **26.** $\displaystyle\int_{0}^{1} \frac{dr}{\sqrt[3]{(7 - 5r)^2}}$

27. $\displaystyle\int_{1/8}^{1} x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$ **28.** $\displaystyle\int_{0}^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

29. $\displaystyle\int_{0}^{\pi} \sin^2 5r\,dr$ **30.** $\displaystyle\int_{0}^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right) dt$

31. $\displaystyle\int_{0}^{\pi/3} \sec^2\theta\,d\theta$ **32.** $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2 x\,dx$

33. $\displaystyle\int_{\pi}^{3\pi} \cot^2\frac{x}{6}\,dx$ **34.** $\displaystyle\int_{0}^{\pi} \tan^2\frac{\theta}{3}\,d\theta$

35. $\displaystyle\int_{-\pi/3}^{0} \sec x\tan x\,dx$ **36.** $\displaystyle\int_{\pi/4}^{3\pi/4} \csc z\cot z\,dz$

37. $\displaystyle\int_{0}^{\pi/2} 5(\sin x)^{3/2}\cos x\,dx$ **38.** $\displaystyle\int_{-\pi/2}^{\pi/2} 15\sin^4 3x\cos 3x\,dx$

39. $\displaystyle\int_{0}^{\pi/2} \frac{3\sin x\cos x}{\sqrt{1 + 3\sin^2 x}}\,dx$ **40.** $\displaystyle\int_{0}^{\pi/4} \frac{\sec^2 x}{(1 + 7\tan x)^{2/3}}\,dx$

Average Values

41. Find the average value of $f(x) = mx + b$

a. over $[-1, 1]$ **b.** over $[-k, k]$

42. Find the average value of

a. $y = \sqrt{3x}$ over $[0, 3]$ **b.** $y = \sqrt{ax}$ over $[0, a]$

43. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

44. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

Differentiating Integrals

In Exercises 45–48, find dy/dx.

45. $y = \displaystyle\int_{2}^{x} \sqrt{2 + \cos^3 t}\,dt$ **46.** $y = \displaystyle\int_{2}^{7x^2} \sqrt{2 + \cos^3 t}\,dt$

47. $y = \displaystyle\int_{x}^{1} \frac{6}{3 + t^4}\,dt$ **48.** $y = \displaystyle\int_{\sec x}^{2} \frac{1}{t^2 + 1}\,dt$

Theory and Examples

49. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

50. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

51. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

52. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 53–68. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

53. $\int (x^3 + 5x - 7)\, dx$

54. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

55. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

56. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

57. $\int \frac{dr}{(r + 5)^2}$

58. $\int \frac{6\, dr}{\left(r - \sqrt{2}\right)^3}$

59. $\int 3\theta\sqrt{\theta^2 + 1}\, d\theta$

60. $\int \frac{\theta}{\sqrt{7 + \theta^2}} d\theta$

61. $\int x^3(1 + x^4)^{-1/4}\, dx$

62. $\int (2 - x)^{3/5}\, dx$

63. $\int \sec^2 \frac{s}{10} ds$

64. $\int \csc^2 \pi s\, ds$

65. $\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\, d\theta$

66. $\int \sec \frac{\theta}{3} \tan \frac{\theta}{3}\, d\theta$

67. $\int \sin^2 \frac{x}{4}\, dx \quad \left(\textit{Hint: } \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

68. $\int \cos^2 \frac{x}{2}\, dx$

Initial Value Problems

Solve the initial value problems in Exercises 69–72.

69. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

70. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

71. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

72. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

Integration by Parts

Evaluate the integrals in Exercises 73–80 using integration by parts.

73. $\int \ln\, (x + 1)\, dx$

74. $\int x^2 \ln x\, dx$

75. $\int \tan^{-1} 3x\, dx$

76. $\int \cos^{-1}\left(\frac{x}{2}\right) dx$

77. $\int (x + 1)^2 e^x\, dx$

78. $\int x^2 \sin\, (1 - x)\, dx$

79. $\int e^x \cos 2x\, dx$

80. $\int x \sin x \cos x\, dx$

Partial Fractions

Evaluate the integrals in Exercises 81–100. It may be necessary to use a substitution first.

81. $\int \frac{x\, dx}{x^2 - 3x + 2}$

82. $\int \frac{x\, dx}{x^2 + 4x + 3}$

83. $\int \frac{dx}{x(x + 1)^2}$

84. $\int \frac{x + 1}{x^2(x - 1)} dx$

85. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$

86. $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + \sin\theta - 6}$

87. $\int \frac{3x^2 + 4x + 4}{x^3 + x} dx$

88. $\int \frac{4x\, dx}{x^3 + 4x}$

89. $\int \frac{v + 3}{2v^3 - 8v} dv$

90. $\int \frac{(3v - 7)\, dv}{(v - 1)(v - 2)(v - 3)}$

91. $\int \frac{dt}{t^4 + 4t^2 + 3}$

92. $\int \frac{t\, dt}{t^4 - t^2 - 2}$

93. $\int \frac{x^3 + x^2}{x^2 + x - 2} dx$

94. $\int \frac{x^3 + 1}{x^3 - x} dx$

95. $\int \frac{x^3 + 4x^2}{x^2 + 4x + 3} dx$

96. $\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx$

97. $\int \frac{dx}{x(3\sqrt{x + 1})}$

98. $\int \frac{dx}{x\left(1 + \sqrt[3]{x}\right)}$

99. $\int \frac{ds}{e^s - 1}$

100. $\int \frac{ds}{\sqrt{e^s + 1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 101–104 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

101. $\int \frac{y\, dy}{\sqrt{16 - y^2}}$

102. $\int \frac{x\, dx}{\sqrt{4 + x^2}}$

103. $\int \frac{x\, dx}{4 - x^2}$

104. $\int \frac{t\, dt}{\sqrt{4t^2 - 1}}$

Evaluate the integrals in Exercises 105–108.

105. $\int \frac{x\, dx}{9 - x^2}$

106. $\int \frac{dx}{x(9 - x^2)}$

107. $\int \frac{dx}{9 - x^2}$

108. $\int \frac{dx}{\sqrt{9 - x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 109–116.

109. $\int \sin^3 x \cos^4 x\, dx$

110. $\int \cos^5 x \sin^5 x\, dx$

111. $\int \tan^4 x \sec^2 x\, dx$

112. $\int \tan^3 x \sec^3 x\, dx$

113. $\int \sin 5\theta \cos 6\theta\, d\theta$

114. $\int \sec^2 \theta \sin^3 \theta\, d\theta$

115. $\int \sqrt{1 + \cos(t/2)}\, dt$

116. $\int e^t \sqrt{\tan^2 e^t + 1}\, dt$

Improper Integrals

Evaluate the improper integrals in Exercises 117–126.

117. $\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$

118. $\int_0^1 \ln x\, dx$

119. $\int_0^2 \frac{dy}{(y - 1)^{2/3}}$

120. $\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$

121. $\int_3^{\infty} \frac{2\, du}{u^2 - 2u}$

122. $\int_1^{\infty} \frac{3v - 1}{4v^3 - v^2}\, dv$

123. $\int_0^{\infty} x^2 e^{-x}\, dx$

124. $\int_{-\infty}^0 x e^{3x}\, dx$

125. $\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 9}$

126. $\int_{-\infty}^{\infty} \frac{4\, dx}{x^2 + 16}$

Which of the improper integrals in Exercises 127–132 converge and which diverge?

127. $\int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2 + 1}}$

128. $\int_0^{\infty} e^{-u} \cos u\, du$

129. $\int_1^{\infty} \frac{\ln z}{z}\, dz$

130. $\int_1^{\infty} \frac{e^{-t}}{\sqrt{t}}\, dt$

131. $\int_{-\infty}^{\infty} \frac{2\, dx}{e^x + e^{-x}}$

132. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1 + e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 133–179. The integrals are listed in random order so you need to decide which integration technique to use.

133. $\int \frac{x\, dx}{1 + \sqrt{x}}$

134. $\int \frac{x^3 + 2}{4 - x^2}\, dx$

135. $\int \sqrt{2x - x^2}\, dx$

136. $\int \frac{dx}{\sqrt{-2x - x^2}}$

137. $\int \frac{2 - \cos x + \sin x}{\sin^2 x}\, dx$

138. $\int \sin^2 \theta \cos^5 \theta\, d\theta$

139. $\int \frac{9\, dv}{81 - v^4}$

140. $\int_2^{\infty} \frac{dx}{(x - 1)^2}$

141. $\int \theta \cos(2\theta + 1)\, d\theta$

142. $\int \frac{x^3\, dx}{x^2 - 2x + 1}$

143. $\int \frac{\sin 2\theta\, d\theta}{(1 + \cos 2\theta)^2}$

144. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x}\, dx$

145. $\int \frac{x\, dx}{\sqrt{2 - x}}$

146. $\int \frac{\sqrt{1 - v^2}}{v^2}\, dv$

147. $\int \frac{dy}{y^2 - 2y + 2}$

148. $\int \frac{x\, dx}{\sqrt{8 - 2x^2 - x^4}}$

149. $\int \frac{z + 1}{z^2(z^2 + 4)}\, dz$

150. $\int x^2(x - 1)^{1/3}\, dx$

151. $\int \frac{t\, dt}{\sqrt{9 - 4t^2}}$

152. $\int \frac{\tan^{-1} x}{x^2}\, dx$

153. $\int \frac{e^t\, dt}{e^{2t} + 3e^t + 2}$

154. $\int \tan^3 t\, dt$

155. $\int_1^{\infty} \frac{\ln y}{y^3}\, dy$

156. $\int y^{3/2}(\ln y)^2\, dy$

157. $\int e^{\ln \sqrt{x}}\, dx$

158. $\int e^{\theta} \sqrt{3 + 4e^{\theta}}\, d\theta$

159. $\int \frac{\sin 5t\, dt}{1 + (\cos 5t)^2}$

160. $\int \frac{dv}{\sqrt{e^{2v} - 1}}$

161. $\int \frac{dr}{1 + \sqrt{r}}$

162. $\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9}\, dx$

163. $\int \frac{x^3}{1 + x^2}\, dx$

164. $\int \frac{x^2}{1 + x^3}\, dx$

165. $\int \frac{1 + x^2}{1 + x^3}\, dx$

166. $\int \frac{1 + x^2}{(1 + x)^3}\, dx$

167. $\int \sqrt{x} \cdot \sqrt{1 + \sqrt{x}}\, dx$

168. $\int \sqrt{1 + \sqrt{1 + x}}\, dx$

169. $\int \frac{1}{\sqrt{x} \cdot \sqrt{1 + x}}\, dx$

170. $\int_0^{1/2} \sqrt{1 + \sqrt{1 - x^2}}\, dx$

171. $\int \frac{\ln x}{x + x \ln x}\, dx$

172. $\int \frac{1}{x \cdot \ln x \cdot \ln(\ln x)}\, dx$

173. $\int \frac{x^{\ln x} \ln x}{x}\, dx$

174. $\int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right] dx$

175. $\int \frac{1}{x\sqrt{1 - x^4}}\, dx$

176. $\int \frac{\sqrt{1 - x}}{x}\, dx$

177. $\int \frac{\sin x}{\sin x + \cos x}\, dx$

178. $\int \frac{\sin^2 x}{1 + \sin^2 x}\, dx$

179. $\int \frac{1 - \cos x}{1 + \cos x}\, dx$

Chapter 5 Single Choice Questions (Indefinite)

1. Let $x^2 \neq n\pi - 1$, $n \in N$. Then the value of

$$\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx \text{ is}$$

a. $\ln\left|\frac{1}{2}\sec(x^2+1)\right|+c$ **b.** $\ln\left|\sec\left(\frac{x^2+1}{2}\right)\right|+c$

c. $\frac{1}{2}\ln\left|\sec(x^2+1)\right|+c$ **d.** $\frac{1}{2}\ln\left|\frac{1}{\sec(x^2+1)}\right|+c$

2. $\int \frac{\sec x.\cos ecx}{2\cot x - \sec x\cos ecx}dx$

a. $\ln|\sec 2x + \tan 2x| + c$ **b.** $\ln|\sec x + \operatorname{cosec} x| + c$

c. $\ln|\sec x + \tan x| + c$ **d.** $\frac{1}{2}\ln|\sec x + \operatorname{cosec} x| + c$

3. If $I = \int \frac{x^5 dx}{\sqrt{1+x^3}}$, then I is equal to

a. $\frac{2}{9}(1+x^3)^{5/2} + \frac{2}{3}(1+x^3)^{3/2} + c$

b. $\ln\left|\sqrt{x} + \sqrt{1+x^3}\right| + c$

c. $\ln\left|\sqrt{x} - \sqrt{1+x^3}\right| + c$

d. $\frac{2}{9}(1+x^3)^{3/2} - (1+x^3)^{1/2} + c$

4. If $f(x) = \int\left(\tan(\ln x) + \frac{1}{2}\right)^2 dx$ and $f(1) = 0$, then $f(e^{\pi/4})$ is

a. $\frac{3-e^{\pi/4}}{4}$ **b.** $-\frac{3-e^{\pi/4}}{4}$

c. $-\frac{(3+e^{\pi/4})}{4}$ **d.** $\frac{e^{\pi/4}+3}{4}$

5. Let $A = \text{diag}(x^2, 6x, 1)$ and $B = \text{diag}\left(\frac{1}{x^2}, \frac{1}{x}, 1\right)$ be two matrices. A function $f(x)$ is defined as trace of $(A^{-1}B)^{-1}$. If $\int \frac{x^2-1}{f(x)}dx = \frac{1}{2}g(x) + c$ and $g(1) = \pi/8$, then range of $g(x)$ is

a. $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ **b.** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c. $\left(0, \frac{\pi}{2}\right)$ **d.** $\left(-\frac{\pi}{2}, 0\right)$

6. $\int \frac{(1-\tan^2 x)\cos ec2x dx}{\sec^2 x}$ is equal to

a. $\frac{\ln|\sin 2x|}{2} + C$ **b.** $\frac{\ln|\sin 2x|}{2} + C$

c. $\frac{\ln|\tan x|}{2} + C$ **d.** $\frac{\ln|\cos x + \sin x|}{2} + C$

7. $\int (2 - x\tan x)\sqrt{\cos x}dx$ is equal to

a. $x\sqrt{\sin x} + C$ **b.** $-x\sqrt{\sin x} + C$

c. $2x\sqrt{\cos x} + C$ **d.** $-x\sqrt{\cos x} + C$

8. If $\int (5 + 6x)(x^2 + x^3)^{1/3} \cdot x^5 dx = kx^{\ell}(1 + x)^m + C$, then km is equal to

a. 1 **b.** 2 **c.** 3 **d.** 4

9. $\int xe^{\tan x}(2 + x\sec^2 x)\,dx$ is equal to

a. $2e^{\tan x} + C$ **b.** $x^2e^{\tan x} + C$

c. $\tan x + xe^{\tan x} + C$ **d.** $x^2 + xe^{\tan x} + C$

10. If $\int e^x(\ln x + x\ln x + 1)dx = f(x) + C$ when $f(1) = 0$, then $f(e)$ is equal to

a. e **b.** e^e **c.** e^{e-1} **d.** e^{e+1}

11. $\int \sqrt{\frac{2-x}{2+x}}\cos^{-1}\left(\frac{\sqrt{2+x}}{2}\right)dx$ equal to

a. $\frac{1}{2}\left[x + \sqrt{4-x^2}\cos^{-1}\frac{x}{2} - \left(\cos^{-1}\frac{x}{2}\right)^2\right] + c$

b. $\frac{1}{2}\left[x + \sqrt{4-x^2}\cos^{-1}\frac{x}{2} - \left(\cos^{-1}\frac{x}{2}\right)^2\frac{1}{2}\right] + c$

c. $\frac{1}{2}\left[x - \sqrt{4-x^2}\cos^{-1}\frac{x}{2} + \left(\cos^{-1}\frac{x}{2}\right)^2\right] + c$

d. $\frac{1}{2}\left[x - \sqrt{4-x^2}\cos^{-1}\frac{x}{2} + \left(\cos^{-1}\frac{x}{2}\right)^2\frac{1}{2}\right] + c$

(where c is integration constant)

12. $\int\left(\frac{g'(x)}{g(x)} - \frac{f'(x)}{f(x)}\right)(\ln(g(x)) - \ln(f(x)))dx$ equal to

a. $\ln\left(\frac{g(x)}{g(x)}\right) + c$ **b.** $\frac{1}{2}\left(\ln\left(\frac{g(x)}{f(x)}\right)\right)^2 + c$

c. $\frac{g(x)}{f(x)}\ln\frac{g(x)}{f(x)} + c$ **d.** $\ln\frac{f(x)}{g(x)} + c$

(where c is integration constant)

13. $\int e^{\sin x}\left(\frac{x\operatorname{cosec} x - \sec^3 x}{\operatorname{cosec} x \sec x}\right)dx$ equals

a. $e^{\sin x} - e^{\sin x}\sec x + c$

b. $xe^{\sin x} + e^{\sin x}\sec x + c$

c. $xe^{\sin x} - e^{\sin x}\sec x + c$

d. $xe^{\sin x} - \sec x + c$

(where c is integration constant)

14. $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$ equals

a. $2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+c$

b. $2\sqrt{x}+3\sqrt[3]{x}-6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+c$

c. $3\sqrt{x}+2\sqrt[3]{x}-6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+c$

d. $3\sqrt{x}+2\sqrt[3]{x}-6\sqrt[6]{x}+\ln|\sqrt[6]{x}-1|+c$

(where c is integration constant)

15. $\int \frac{(x^3-1)}{(x^4+1)(x+1)}dx$, is

a. $\frac{1}{4}\ln(1+x^4)+\frac{1}{3}\ln(1+x^3)+c$

b. $\frac{1}{4}\ln(1+x^4)-\frac{1}{3}\ln(1+x^3)+c$

c. $\frac{1}{4}\ln(1+x^4)-\ln(1+x)+c$

d. $\frac{1}{4}\ln(1+x^4)+\ln(1+x)+c$

(where c is constant of integration)

16. If $g(x)=\left(4\cos^4 x-2\cos 2x-\frac{1}{2}\cos 4x-x^7\right)^{1/7}$, then $\int g(g(x))dx$ is

a. $2x$ + c **b.** $\frac{x^2}{2}+c$ **c.** $-2\cos x$ + c **d.** $2\sin x$ + c

17. If $f(x)=\begin{vmatrix} x^2+4x+5 & 2x^2+2x+1 & 3x^2+4x+2 \\ 1 & 2 & 3 \\ x+2 & 2x+1 & 3x+2 \end{vmatrix}$, then $\int x^2 f(x)dx$ is equal to

a. x^3+c **b.** 2^3x + c **c.** x^5+c **d.** $2x^5+c$

(where c is integration constant)

18. If $\int \frac{(x+1)}{x(1+xe^x)^2}dx = \log|1-f(x)|+f(x)+C$ (where C is constant of integration), then $f(x)$ is equal to

a. $\frac{1}{x+e^x}$ **b.** $\frac{1}{(1+xe^x)}$

c. $\frac{1}{(1+xe^x)^2}$ **d.** $\frac{1}{(x+e^x)^2}$

19. $\int x^{-7}(1+x^4)^{-1/2}dx$ is equal to

a. $\frac{\sqrt{x^4+1}(2x^4-1)}{6x^6}+C$ **b.** $\frac{\sqrt{x^4+1}(2x^4+1)}{6x^6}+C$

c. $\frac{\sqrt{x^4+1}(2x^4+1)}{4x^6}+C$ **d.** $\frac{\sqrt{x^4+1}(2x^4-1)}{4x^6}+C$

20. $\int \frac{(x-3)e^x}{(x-1)^3}dx$ is equal to

a. $\frac{e^x}{(x-1)}+c$ **b.** $\frac{e^x}{(x-1)^2}+c$

c. $\frac{-e^x}{(x-1)^2}+c$ **d.** $\frac{2e^x}{(x-1)^3}+c$

21. $\int |\ln x| dx$ equal $(0<x<1)$:

a. $x+x|\ln x|+c$ **b.** $x|\ln x|-x+c$

c. $x+|\ln x|+c$ **d.** $x-|\ln x|+c$

22. $\int(\sin x\cos x\cos 2x\cos 4x\cos 8x)dx$ equal

a. $\frac{-1}{128}\cos 16x+C$ **b.** $\frac{1}{256}\cos 16x+C$

c. $-\frac{1}{256}\sin 16x+C$ **d.** $-\frac{1}{256}\cos 16x+C$

23. The primitive of the function $x|\sin x|$, when $\frac{3\pi}{2}<x<2\pi$ is given by

a. $x\cos x+\sin x+C$ **b.** $x\cos x-\sin x+C$

c. $x\cos x$ **d.** $x\sin x+\cos x+C$

24. If $\int \frac{dx}{x\sqrt{1-x^{2013}}}=k.\log\left|\frac{\sqrt{1-x^{2013}}-1}{\sqrt{1-x^{2013}}+1}\right|+C$, then the value of $\left(\frac{1}{k}\right)$ is equal to

a. 2010 **b.** 2011 **c.** 2012 **d.** 2013

25. $\int \frac{(x^2+2)dx}{x\sqrt{-x^4+5x-4}\sin^{-1}\left(\frac{x^2-2}{x}\right)}$ is equal to

a. $\ln\left|\left(\sin^{-1}\left(x-\frac{2}{x}\right)\right)\right|+C$ **b.** $\sin^{-1}\left(\frac{x-2}{x}\right)+C$

c. $\ln|\sin^{-1}|+\sqrt{x+2}\ C$ **d.** $\tan^{-1}\left(\frac{x-2}{x}\right)+C$

(where C is constant of integration)

26. If $\int \frac{x^3-8}{x^3\sqrt{x^4-x^3+2}}dx=\frac{k\sqrt{x^4-x^3+2}}{x^2}+C$, then the value of k is

a. 3 **b.** 2 **c.** 7 **d.** 7

27. $\int \frac{\cos^6 x dx}{\sin^4 x(\cos^7 x+\sin^7 x)^{\frac{4}{7}}}$ is equal to

a. $\frac{-1}{3}(1+\cot^7 x)^{3/7}+C$

b. $-\frac{1}{3}(1+\cot^7 x)^{7/3}+C$

c. $-\frac{1}{21}(1+\cot^7 x)^{7/3}+C$

d. $\frac{1}{21}(1+\tan^7 x)^{3/7}+C$

28. $\int \frac{\sec x}{\sqrt{-6\operatorname{cosec}^2 x+9\cot^2 x+5}}dx$ is equal to

a. $\sin^{-1}\left(\frac{\sec x}{2}\right)+C$ **b.** $\cos^{-1}(\tan x)+C$

c. $\frac{1}{2}\sec^{-1}\left(\frac{\sin x}{2}\right)+C$ **d.** $\frac{1}{2}\sin^{-1}\left(\frac{\sec x}{2}\right)+C$

29. $\int \frac{(1+\tan^2 x)dx}{1-(\sin x\sqrt[4]{\tan^2 x(\tan^2 x+2)+1})}$ where $x \in \left(\frac{9\pi}{2}, \frac{11\pi}{2}\right)$ is

- **a.** $\ln|1-\tan x| + C$
- **b.** $\ln|1-\sec x| + C$
- **c.** $\ln|1+\sec x| + C$
- **d.** $\ln|1+\tan x| + C$

30. $\int e^{x^2}.e^x(2x^2+x+1)dx$ is equal to

- **a.** $(x+1)\ e^{x^2} + e^x + c$
- **b.** $xe^{x^2} + (x+1)e^x + c$
- **c.** $(2x+1)\ e^{x^2+x} + c$
- **d.** $xe^{x^2+x} + c$

31. If $\int x^3(\ln x)^2 x dx = \frac{x^4}{32}$ $(a(\ln x)^2 + b(\ln x) + c) + d$, where d is an arbitrary constant, then $(a + b + c)$ is equal to

- **a.** 13
- **b.** 11
- **c.** 5
- **d.** 3

32. If $\int \frac{2\sec x.\tan x}{(\sec x-\tan x)^{10}}dx = (\sec x + \tan x)^{11}\ (m(\sec x - \tan x)^2 + n) + C$, then $\frac{1}{m}+\frac{1}{n}$ is equal to

- **a.** 1
- **b.** –1
- **c.** – 2
- **d.** 2

33. $\int \frac{x^4-1}{\sqrt{2x^7+x^4+6x^3}}dx$ is equal to

- **a.** $\frac{1}{3}\sqrt{\frac{2x^4+x+6}{x}}+C$
- **b.** $\frac{1}{2}\sqrt{2+x^3+\frac{6}{x^4}}+C$
- **c.** $\frac{1}{3}\sqrt{2x^3+\frac{1}{x^2}+\frac{6}{x^3}}+C$
- **d.** $\frac{1}{2}\sqrt{2x^4+x+6}+C$

34. $\int e^{x/2}\operatorname{cosec} 2x(1-4\cot 2x)dx$ is equal to

- **a.** $4e^{x/2}\operatorname{cosec}2x + c$
- **b.** $2e^{x/2}\operatorname{cosec}2x + c$
- **c.** $e^{x/2}\operatorname{cosec}2x + c$
- **d.** $\frac{1}{2}e^{e/x}\operatorname{cosec}2x+c$

35. $\int \frac{e^x(x-1)(x-\ln x)}{x^2}dx$ is equal to

- **a.** $e^x\left(x-\frac{\ln x}{x}\right)+c$
- **b.** $e^x\left(\frac{x-\ln x-1}{x}\right)+c$
- **c.** $e^x\left(\frac{x-\ln x}{x^2}\right)+c$
- **d.** $e^x\left(\frac{x-\ln x}{x^2}\right)+c$

36. Evaluate: $\int \frac{\left(\sqrt[3]{x+\sqrt{2-x^2}}\right)\left(\sqrt[6]{1-x\sqrt{2-x^2}}\right)dx}{\sqrt[3]{1-x^2}}$

- **a.** $2^{\frac{1}{6}}x+C$
- **b.** $2^{\frac{1}{12}}x+C$
- **c.** $2^{\frac{1}{3}}x+C$
- **d.** None of these

37. $\int \frac{dx}{\sqrt{1-\tan^2 x}} = \frac{1}{\lambda}\ \sin^{-1}(\sqrt{2}\sin x) + C$, then $\lambda =$

- **a.** $\sqrt{2}$
- **b.** $\sqrt{3}$
- **c.** 2
- **d.** $\sqrt{5}$

38. $\int \frac{\cos\{\log(f(x))+\log(g(x))\}}{f(x).g(x)}\{f(x).g'(x) + g(x).f'(x)\}dx$ is equal to

- **a.** $\sin\{\log(f(x).\ g(x))\} + C$
- **b.** $\frac{g(x)}{f(x)}\ \sin\{\log(f(x).\ g(x))\} + C$
- **c.** $\frac{f(x)}{g(x)}\ \sin\{\log(f(x).\ g(x))\} + C$
- **d.** None of these

39. $\int \frac{dx}{x^{1/5}(1+x^{4/5})^{1/2}}$ is equal to

- **a.** $\sqrt{1+x^{\frac{4}{5}}}+K$
- **b.** $\frac{5}{2}\sqrt{1+x^{\frac{4}{5}}}+K$
- **c.** $x^{\frac{4}{5}}\left(1+x^{\frac{4}{5}}\right)^{\frac{1}{2}}+K$
- **d.** $\frac{2}{5}\sqrt{1+x^{\frac{4}{5}}}+K$

40. If $\int \sqrt{1+2\tan x(\tan x+\sec x)}dx = a\ln\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right| + c$ $\forall x \in \left(0, \frac{\pi}{2}\right)$, then a is equal to

- **a.** 2
- **b.** –2
- **c.** –1
- **d.** $\frac{1}{2}$

41. If $3f\left(\frac{2x+3}{2x-3}\right) = 2x - 3$, then $\int f(x)dx$ is equal to

- **a.** $\ln(x-1)^2 + c$
- **b.** $\ln|(x-1)^3| + c$
- **c.** $\ln(x-1)^6 + c$
- **d.** $\ln|(x-1)| + c$

42. Let $f(x)$ be a differentiable function such that $f^2(x) + xf(x) = 3$, then $\int \frac{3x^3+6x^2f(x)+2f(x)}{(2f(x)+x)(x^3-2f(x))^2}dx$ equals

- **a.** $\frac{1}{x^3-2f(x)}+c$
- **b.** $\frac{1}{2f(x)-x^3}+c$
- **c.** $\frac{1}{2f(x)+x^3}+c$
- **d.** $\frac{1}{x^3+2f(x)}+c$

43. $\int \frac{x^4+x^3-1}{x^2\sqrt{x^4+1+2x^3}}$ dx is equal to (where C is constant of integration)

- **a.** $\frac{\sqrt{x^4+2x^3+1}}{x^{1/2}}+C$
- **b.** $\frac{\sqrt{x^4+2x^3+1}}{x}+C$
- **c.** $\frac{\sqrt{x^4+2x^3+1}}{x^{3/2}}+C$
- **d.** $\frac{\sqrt{x^4+2x^3+1}}{x^3}+C$

44. $\int e^x\left(\frac{x^2-3}{(x-1)^2}\right)dx$ is equal to

- **a.** $e^x\ \frac{(x+3)}{(x-1)}+C$
- **b.** $e^x\ \frac{(x-3)}{(x-1)}+C$
- **c.** $e^x\left(\frac{x+1}{x-1}\right)+C$
- **d.** $e^x\left(\frac{1}{x-1}\right)^2+C$

45. If $f(x) = \sqrt{1-x^2}$, $x \in (0, 1)$ and $f_n(x) = \underbrace{f \circ f \circ f \ldots \circ f(x)}_{n \text{ times}}$ then for even number "n" $\int \frac{f_n(x)}{f_{n-1}(x)}$ z $\int \frac{f_n(x)}{f_{n-1}(x)}$ dx is equal to

a. $C - \sqrt{1-x^2}$ **b.** $C + \sqrt{1-x^2}$

c. $\frac{1}{\sqrt{1-x^2}} + C$ **d.** $\sqrt{1-x^2} - C$

(where C is constant of integration)

46. $\int 4\cos x.\cos 2x.\cos 4x dx$ is equal to

a. $\sum_{r=1}^{4} \frac{\sin(2r-1)x}{2r-1} + C$ **b.** $\sum_{r=1}^{4} \frac{\cos 2rx}{2r} + C$

c. $\sum_{r=1}^{4} \frac{\cos(2r-1)x}{(2r-1)} + C$ **d.** $\sum_{r=1}^{4} \frac{\sin(2r)x}{2r} + C$

(where C is constant of integration)

47. Integrate $\int \frac{x(x^2+3)+3(\tan^{-1})(1+x^2)^2}{(1+x^2)}$ dx (where C is an integration constant)

a. $\frac{(x^2+3)}{(1+x^2)}\tan^{-1} + C$ **b.** $x(x^2 + 3)\tan^{-1}x + C$

c. $\frac{\tan^{-1}x}{1+x^2} + C$ **d.** $(x^2 + 3)(\tan^{-1}x) + C$

48. $\int \frac{x^3}{(2x^2+1)^3} dx$ is equal to

a. $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2} + C$ **b.** $-\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2} + C$

c. $\frac{1}{2}\left(2+\frac{1}{x^2}\right)^{-2} + C$ **d.** $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{2} + C$

(where C is integration constant)

49. If $\int (e^x + x - |e^x - x|)dx = g(x) + c$, where $g(0) = 0$, then $g(2)$ is equal to

a. 1 **b.** 1/2 **c.** 5 **d.** 4

50. $\int \frac{x^2-1}{(x^2+1)^2}$ dx is equal to (where C is constant of integration)

a. $\frac{-x}{1+x^2} + C$ **b.** $\frac{x}{1+x^2} + C$

c. $-\frac{\sin 2x}{2} + C$ **d.** $-\cos 2x + C$

51. Let $f(x) = \sin x \,.\, e^{\cos x}$, $g(x) = \int f(x)dx$.

Statement-1 : $g(x)$ is an even function.

Statement-2 : For $g : R \to R$, $g(x)$ is a many one function

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

52. $\int (\sec x)^{2/3}(\text{cosec}\, x)^{4/3}$ dx is equal to (where C is constant of integration)

a. $\frac{3}{(\tan x)^{4/3}} + C$ **b.** $\frac{-3}{(\tan x)^{4/3}} + C$

c. $\frac{-3}{(\tan x)^{4/3}} + C$ **d.** $\frac{-3}{(\tan x)^{1/3}} + C$

53. $\int \frac{\sqrt{x}}{(1+x^2)^{7/4}} dx =$

a. $\frac{2}{3}\frac{x^{3/2}}{(x^2+1)^{3/4}} + c$ **b.** $\frac{1}{3}\frac{x^{3/2}}{(x^2+1)^{3/4}} + c$

c. $\frac{1}{2}\frac{x^{1/2}}{(x^2+1)^{3/4}} + c$ **d.** $\frac{x^{3/2}}{(x^2+1)^{3/4}} + c$

54. $\int \frac{(x^2-x)}{(x+x^3)(1+x^2)}$ dx is equal to

a. $\ln|1 + x^3| - \ln(1 + x^2) + C$

b. $\frac{1}{3}\ln|1 + x^3| - \frac{1}{2}\ln(1 + x^2) + C$

c. $\frac{1}{2}\ln|1 + x^3| - \frac{1}{3}\ln(1 + x^2) + C$

d. $\ln(1 + x^2) - \ln|1 + x^3| + C$

(where C is integration constant)

55. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}} dx$ is equal to

a. $\frac{\sqrt{2x^4-2x^2+1}}{x^2} + c$ **b.** $\frac{\sqrt{2x^4-2x^2+1}}{x^3} + c$

c. $\frac{\sqrt{2x^4-2x^2+1}}{x} + c$ **d.** $\frac{\sqrt{2x^4-2x^2+1}}{2x^2} + c$

56. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{f \circ f \circ \ldots \circ f}_{f \text{ occurs } n \text{ times}}(x)$. Then $\int x^{n-2}g(x)\,dx$ equals

a. $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

b. $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$

c. $\frac{1}{n(n-1)}(1+nx^n)^{1+\frac{1}{n}} + K$

d. $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

57. Let $f(x)$ be an indefinite integral of $\sin^2 x$.

Statement-1: The function $f(x)$ satisfies $f(x + \pi) = f(x)$ for all real x.

Statement-2 : $\sin 2(x + \pi) = \sin^2 x$ for all real x.

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

58. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant c, the value of J – I equals

a. $\frac{1}{2}\ln\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$ **b.** $\frac{1}{2}\ln\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

c. $\frac{1}{2}\ln\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$ **d.** $\frac{1}{2}\ln\left(\frac{e^{2x}+e^{x}+1}{e^{4x}-e^{x}+1}\right)+c$

59. The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ n equal (for some arbitrary constant K)

a. $-\frac{1}{(\sec+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

b. $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

c. $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

d. $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

Chapter 5 Multiple Choice Questions (Indefinite)

1. $\int \frac{\sec^2 x}{\sqrt{a\sec^2 x-b\tan^2 x}}dx$ is

a. $\frac{1}{\sqrt{b-a}}\sin^{-1}\left(\tan x\sqrt{\frac{b-a}{a}}\right)+c$ is $b>a>0$

b. $\frac{1}{\sqrt{b-a}}\log_2(\tan x\sqrt{b-a}+\sqrt{a\sec^2 x-b\tan^2 x})+c$ if $b>a>0$

c. $\frac{1}{\sqrt{a-b}}\sin^{-1}\left(\tan x\sqrt{\frac{a-b}{a}}\right)+c$ if $a>b>0$

d. $\frac{1}{\sqrt{a-b}}\log_e(\tan x\sqrt{a-b}+\sqrt{a\sec^2 x-b\tan^2 x})+c$ if $a>b>0$

(where c is integration constant) $\frac{1}{1+xe^x}$

2. $\int(\tan x+2\tan 2x+4\tan 4x+8\cot 8x)dx$ has the value

a. $\ln(\sin x)+c$

b. $-\ln(\operatorname{cosec} x)+c$

c. $\ln(\cos x)+c$

d. $\ln(\sin 2x)-\ln(2\cos x)+c$

3. If $\int \cos x\, d(\sin x) = f(x)+g(x)+C$ (where $f(x)$ is aperiodic function and $g(x)$ is periodic function)

a. $f(x)=x$ **b.** $f(x)=\frac{x}{2}$

c. $g(x)=\sin 2x$ **d.** $g(x)=\frac{\sin 2x}{4}$

4. $\int\left(1+\tan x\tan\frac{x}{2}\right)dx$ is equal to

a. $\int\left(\tan x\cot\frac{x}{2}-1\right)dx$ **b.** $\int \sec x\,dx$

c. $-\ln|\sec x-\tan x|+C$ **d.** $-\ln\left|\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right|+c$

5. For $a>0$, if $I=\int\sqrt{\frac{x}{a^3-x^3}}\,dx = A\sin^{-1}\left(\frac{x^{3/2}}{B}\right)+C$, where C is any arbitrary constant, then

a. $A=\frac{2}{3}$ **b.** $B=a^{3/2}$ **c.** $A=\frac{1}{3}$ **d.** $B=a^{1/2}$

6. If $\int \frac{\ln(2-\cos^2 x)}{\cos^2 x}dx = f(x)+g(x)+h(x)+c$ (where 'c' is constant of integration)

a. $f(0)+g(0)+h(0)=0$ **b.** $f(0)+g(0)=0$

c. $h(0)-3g(0)+f(0)=0$ **d.** $f(0)-2g(0)-3h(0)=0$

7. If $\int \frac{px\cos 5x-q\sin 5x}{x^2}dx=\frac{q\sin 5x}{x}+C$, then p and q may be (where C is constant of integration)

a. $p=-1, q=-1/5$ **b.** $p=5\sqrt{2}, q=\sqrt{2}$

c. $p=1, q=1/5$ **d.** $p=10065, q=2013$

8. The value of $\int \frac{x^4-7x^2-1}{x^2-3x+1}dx$ is equal to (where C_k, $k=1, 2, 3, 4$ represents integration constants)

a. $\frac{x^3}{3}+\frac{3x^2}{2}+x+C_1$ **b.** $\frac{x^3}{3}+\frac{3(x+1)^2}{2}-2x+C_2$

c. $\frac{x^3}{3}+\frac{3x^2}{2}+(x-\pi)+C_3$ **d.** $\frac{x^3}{3}+\frac{3x^2}{2}+(x+\pi)+C_4$

9. If $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = A.x^B+C\tan^{-1}(x^D)+E$, then which of the following is/are true (where E is constant of integration)

a. $A=\frac{2}{3}$

b. $AB=1$

c. $(AB+C)$ is a prime number

d. Sum of all coefficients of even power of x in the expansion of $\left(AB+\frac{x}{D}\right)^C$ is equal to $\frac{(C+1)^C+(C-1)^C}{2}$

10. If $\int \frac{x^6 - x^3 - 2}{x^3\sqrt{(x^3+1)^2 + 3x^4}}\,dx = f(x) + C$, then

a. $f(1) < f(-1)$ **b.** $f(2) > f(3)$
c. $f(1) > f(-1)$ **d.** $f(2) < f(3)$

11. $\int \frac{2(x+1)}{\sqrt{e^{-2x} + 4x^2}}\,dx$ is equal to

a. $\cos^{-1}(2x^2e^{-x}) + C$ **b.** $-\cos^{-1}(2xe^x) + C$
c. $\sin^{-1}(xe^x) + C$ **d.** $\sin^{-1}(2xe^x) + C$

12. $\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}}\,dx = Ax + B\ln|4e^{2x} - 5| + C$ then

a. A is -1 **b.** B is $\frac{7}{8}$
c. C can be -1 **d.** $A = 1$ and $B = -\frac{7}{8}$

13. If $\sin(ax)\sin(bx) = \cos(ax)\cos(bx)$ ($a, b \neq 0$ and $a \neq b$), then $\int \frac{\sin(ax) + \cos(bx)}{\cos(ax) + \sin(bx)}\,dx$ is

a. $\frac{1}{a}\ln|\sec ax| + C$ **b.** $\frac{1}{b}\ln|\sin bx| + C$
c. $\frac{1}{a}\ln|\text{cosec } bx| + C$ **d.** $\frac{1}{b}\ln|\cos ax| + C$

14. If $\int \frac{x^7}{(x^4-1)^3}\,dx = -\frac{1}{8}\frac{x^a}{(x^b-1)^2} + c$, where c is constant of integration, then

a. $a + b = 12$ **b.** $a - b = 4$
c. $a + b = 6$ **d.** $2a + 3b = 40$

15. If $\int \sec 2x\,dx = f(g(x)) + c$, where $f(x) = \ln|x|$ and c is constant of integration, then correct statements are

a. domain of $f(x) = R - \{0\}$
b. range of $g(x) = R$
c. $f'(x) = \frac{1}{2x}$, $x \neq 0, x \in R$
d. $g'(x) = 2\,\text{cosec}^2\left(\frac{\pi}{4} - x\right)$

16. If $\int \frac{dx}{\sqrt[3]{\cos^{12}x + 3\cos^{10}x + 3\cos^8 x + \cos^6 x}}$ $= f(x) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{f(x)}{\sqrt{2}}\right) + c$ (where c is constant of integration, then $f(x)$ is

a. many-one function **b.** bounded and aperiodic
c. unbounded and periodic **d.** unbounded and aperiodic

17. If antiderivative of $\frac{x^3}{\sqrt{1+2x^2}}$ which passes through (1, 2) is $\frac{1}{m}$ $(1 + 2x^2)^{1/2}(x^2 - 1) + n$. Then

a. $f(0) = \frac{11}{6}$ **b.** $f(0) = 2$
c. $m + n = 7$ **d.** $m + n = 8$

18. $I = \int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx = \frac{3}{4}\sqrt[3]{f^2(x)} + c$ (where $f(x) < 0$ $x > 2$), then the correct statements are

a. $f(x)$ attains all real values **b.** $f'(1) = -4$
c. $f'(1) = 4$ **d.** $f(x)$ is one-one function

19. If $\int \frac{\cos x}{3 + 3\sin x - \cos^2 x}\,dx = \ln(1 - f(x)) + C$, then which of the following is correct?

a. $f(x) = \frac{1}{2 + \sin x}$
b. $f(x) = \frac{1}{2 + \cos x}$
c. Number of integers in the range of $f(x)$ is 1
d. Number of integers in the range of $f(x)$ is 0

20. If $f(x) = \lim\limits_{n\to\infty} n^2(x^{1/n} - x^{1/n-1})$, $x > 0$ then

a. $f(x)$ is increasing function
b. $f(x)$ is a non-monotonic function
c. $\int xf(x)\,dx = \frac{x^3}{3} + x\ln x + c$
d. $\int xf(x).dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

Chapter 5 Passage Type Questions (Indefinite)

Passage 1

Let f, g, h be three functions such that $f'(x) > 0$ and $g(x) \geq 0$ $x \in R$, where $\int f(x)\cdot g(x)\,dx = \frac{x^4}{4} + C$ and $\int \frac{f(x)}{g(x)}dx = \int \frac{g(x)}{h(x)}dx = \ln|x| + C$, $x \neq 0$.

On the basis of above information, answer the following questions:

1. $\int f(x).g(x).h(x)\,dx$ is equal to

a. $\frac{x^5}{5} + C$ **b.** $\frac{x^6}{6} + C$
c. $\frac{x^7}{7} + C$ **d.** $\frac{x^8}{8} + C$

2. $f(x)$, $g(x)$, $h(x)$ are in

a. A.P. **b.** G.P.

c. H.P. **d.** None of these

3. $\int \frac{h(x)\mathrm{d}x}{\sqrt{1+g(x)}}$ is equal to

a. $\frac{\sqrt{x^2+1}(x^2-2)}{3}+C$ **b.** $\frac{\sqrt{x^2+1}(x^2-1)}{3}+C$

c. $\frac{\sqrt{x^2+1}(x^2+1)}{3}+C$ **d.** $\frac{\sqrt{x^2+1}(x^2-1)}{3}+C$

Passage 2

Let $f(x) = \int x^2 \cos^2 x(2x+6\tan x-2x\tan^2 x)\, \mathrm{d}x$ and $f(x)$ passes through the point $(\pi, 0)$

1. If f: $R-(2n+1)\frac{\pi}{2} \to R$ then $f(x)$ is equal to

a. even function

b. odd function

c. neither even nor odd

d. even as well as odd both

2. The number of solutions of the equation $f(x) = x^3$ in $[0, 2\pi]$ be

a. 0 **b.** 3

c. 2 **d.** None of these

Passage 3

Let $\int \frac{x^2(x\sec^2 x+\tan x)}{(x\tan x+1)^2}\mathrm{d}x = 2\ln|f(x)| - \frac{x^2}{g(x)} + k$ (where k is constant of integration)

On the basis of above information, answer the following questions:

1. $f(x)$ is a/an:

a. bounded function **b.** even function

c. odd function **d.** periodic function

2. $\lim\limits_{x\to 0}\frac{g(x)-1}{f(x)-1}$ is equal to

a. 0 **b.** 1

c. 2 **d.** does not exist

Passage 4

Let $f(x)$, $g(x)$, $h(x)$_ are three differentiable functions satisfying

$$\int (f(x)+g(x))\mathrm{d}x = \frac{x^3}{3}+C_1,\ \int (f(x)-g(x))\mathrm{d}x = x^2-\frac{x^3}{3}+C_2$$

and $\int \frac{f(x)}{h(x)}\mathrm{d}x = -\frac{1}{x}+C_3$

(where C_1, C_2, C_3 are integration constants)

1. The value of $\int (f(x)+g(x)+h(x))\mathrm{d}x$ is equal to

a. $\frac{x^3}{3}+\frac{x^4}{4}+C$ **b.** $\frac{x^2}{3}-x^2+\frac{x^4}{4}+C$

c. $\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+C$ **d.** None of these

(where C is integration constant)

2. Number of points of non-differentiability of function $\phi(x) = \min\{f(x), f(x)+g(x), h(x)\}$ is equal to

a. 0 **b.** 1

c. 2 **d.** 3

3. If number of distinct terms in the expansion of $\left((1+f(x))+\left(\frac{f(x)+g(x)}{h(x)}\right)\right)^{\Sigma n}$, $(n \in N)$ is 31, then the value of n is equal to

a. 17 **b.** 5

c. 16 **d.** 15

Passage 5

Let $f(x)$ be a polynomial function of degree 2 satisfying

$$\int \frac{f(x)}{x^3-1}dx = \ln\left|\frac{x^2+x+1}{x-1}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C,$$

where C is indefinite integration constant.

1. The value of $f(1)$ is equal to

a. 1 **b.** 2

c. –1 **d.** –3

2. Let $\int \frac{1-6\,cosec\,x}{6+f(\sin x)}d(\sin x) = g(x)+K$, where $g(x)$ contains no constant term.

Then $\lim\limits_{t\to\frac{\pi}{2}} g(t)$ is equal to (where K is indefinite integration constant.)

a. ln 1 **b.** ln 2

c. ln 3 **d.** ln 4

3. Let $\int \frac{5+f(\sin x)+f(\cos x)}{\sin x+\cos x}\mathrm{d}x = h(x)+\lambda$, where $h(1) = -1$.

The value of $\tan^{-1}(h(2)) + \tan^{-1}(h(3))$ is equal to (where λ is indefinite integration constant.)

a. $\frac{\pi}{4}$ **b.** $\frac{-\pi}{4}$

c. $\frac{3\pi}{4}$ **d.** $\frac{-3\pi}{4}$

Chapter 5 Matrix Match Type Questions (Indefinite)

1. If $y^2 = 3x^2 + 2x + 1$ and integration I_n is defined as $I_n = \int \frac{x^n}{y} dx$, where $AI_{10} + BI_9 + CI_8 = x^9 y$ then

Column-I	Column-II
(a) A is equal to	(p) 9
(b) B is equal to	(q) 10
(c) C is equal to	(r) 19
(d) Absolute value of thrice of x-intercept of the line	(s) 30
$Cx + By + A = 0$ is	(t) A composite number

2.

Column-I	Column-II
(a) If $\int \frac{dx}{x^{1/3}+2} = g(x) + C$, where $g(0) =$ 12 ln 2, then $g(-1)$ is equal to	(p) 1
(b) If $\int (\sec x)^{9/5} (\operatorname{cosec} x)^{1/5} dx = k \tan^m x + C$, then km is equal to	(q) 3
(c) Let $\int \frac{dx}{\cot^2 x - 1} = \frac{1}{\ell} \ln\lvert\sec 2x + \tan 2x\rvert - \frac{x}{m} + C$, then $\ell + m$ is equal to	(r) 6
(d) Let $\int \frac{\left(1+\frac{1}{x}\right)dx}{\sqrt{1+xe^x}} = \ln\left\lvert\frac{g(x)-1}{g(x)+1}\right\rvert + C$, then $[g^2(1)]$ is equal to (where $[\cdot]$ denotes greatest integer function)	(s) Rational number

3.

Column-I	Column-II
(a) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$	(p) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$
(b) $\int \frac{x \sin^{-1} x}{(1-x^2)^{3/2}} dx$	(q) $-2\sqrt{\left\{-\left(\frac{x-2}{x-1}\right)\right\}} + c$
(c) $\int \frac{x}{x^4+x^2+1} dx$	(r) $\frac{\sin^{-1} x}{\sqrt{1-x^2}} - \frac{1}{2} \log\left\lvert\frac{1+x}{1-x}\right\rvert + c$
(d) $\int \frac{dx}{(x-1)\sqrt{(-x^2+3x-2)}}$	(s) $\frac{e^{2x}}{2} \tan x + c$

4. Let f, g, h be three functions such that $f'(x) > 0$ $g(x) \geq 0$ $\forall$ $x \in R$, where $\int f(x).g(x)dx = \frac{x^4}{4} + C$ and $\int \frac{f(x)}{g(x)} dx = \int \frac{g(x)}{h(x)} dx$ = ln $|x| + C$, $x \neq 0$

Column-I	Column-II
(a) $f(x)$ is	(p) x^2
(b) $g(x)$ is	(q) x
(c) $h(x)$ is	(r) x^6
(d) $f(x).g(x).h(x)$ is	(s) x^3

Chapter 5 Integer Type Questions (Indefinite)

1. Let $\int \frac{(x^6-4)dx}{(x^6+2)^{1/4}.x^4} = \frac{\ell(x^6+2)^m}{x^n} + C$, then $\frac{n}{\ell m}$ is equal to

2. If $\int (\sin x)^{-\frac{3}{4}} (\cos x)^{-\frac{21}{4}} dx = \frac{(\tan x)^{a_1}}{b_1} + \frac{(\tan x)^{a_2}}{b_2} + \frac{(\tan x)^{a_3}}{b_3} + C$, where C is integration constant, then $\left[\frac{a_1+a_2+a_3+b_1+b_2+b_3}{2}\right]$ equals (where [·] denotes greatest integer function)

3. If $\int \frac{e^x(x-1)(x-\ln x)}{x^2} dx = \frac{e^x(\ell x + m\ell nx + n)}{x} + c$ (where c is integration constant and $\ell, m, n \in N$), then $(\ell - m - n)$ is equal to

4. If $\int \frac{x^5 dx}{x^3+1} = \frac{x^3}{\ell} + \frac{1}{m}$ ln $|x^3 + n| + c$ (where c is integration constant and $\ell, m, n \in N$, then $(\ell - m - n)$ is equal to

5. Let $\int \frac{\sin^3 x dx}{1+\sin^2 x} = \frac{1}{p} \ln\left\lvert\frac{\cos x + q}{r - \cos x}\right\rvert - \cos x + c$ (where c is integration constant), then $(pq + pr)$ is equal to

6. If $\int \frac{(1 + x\cos x - \sin x)dx}{(x + \sin x + \cos x)^{1/2}.(x - \sin x + \cos x)^{3/2}} = (x + l \sin x + m\cos x)^n, (x + p\sin x + q\cos x)^r + C$ (where C is constant of integration and $l, n > 0$), then $\left(\frac{\ell+m}{n}\right)(p - q)r$ is equal to

7. If $\int \frac{e^{9x}+e^{11x}}{e^{x}+e^{-x}}\,dx = \frac{e^{k_1 x}}{k_2}+C$, where C denotes constant of integration, then value of $(k_1 + k_2)$ is

8. If $\int\sqrt{\frac{\tan x+2\tan 2x+4\tan 4x+8\cot 8x}{\sin^4 x}}\,dx = -k(\cot x)^{1/k} + C$, then "$6k$" is equal to

9. If $f(x) = \int x.\sqrt{\frac{2\sin(x^2-1)-\sin(2(x^2-1))}{2\sin(x^2-1)+\sin(2(x^2-1))}}\,dx$, then $(f(2) - f(-2))$ is equal to

10. If $\int \sin^{-1}\left(\frac{2x+1}{\sqrt{4x^2+4x+10}}\right)dx = \left(\frac{2x+1}{2}\right).\tan^{-1}\left(\frac{2x+1}{3}\right) + \lambda \ln(4x^2+4x+10) + C$

(where C is constant of integration), then the value of $|100\,\lambda|$ is

11. If $\int \frac{\ln(ex^{x+1})+(\ln(x^{\sqrt{x}}))^2}{1+(x\ln x)(\ln(e^2x^x))}\,dx = f(x) + C$, where $f(1) = 0$, then $e^{(e^{f(2)}-1)}$ is equal to

12. Let $f(x)$ be a quadratic function such that $f(0) = 8$, $f(1) = 25$ and $\int \frac{f(x)}{x^2(x+2)^3}\,dx = g(x) + C$

where $g(x)$ is a rational function and C is constant of integration, then the value of $f(10)$.

13. If $\int \frac{\sqrt{e^{7x}}}{\sqrt[4]{e^{2x}+e^{-2x}}}\,dx = \frac{1}{k}(e^{\alpha x}+1)^{\beta/\gamma} + C$ (where C is constant of integration and k, α, β, γ are positive integers and β, γ are relatively prime), then $\left(\frac{k}{\beta}+\frac{\gamma}{\alpha}\right)$ is

14. If $\int \frac{3x+4}{(x-2)(x^2+2x+2)}\,dx = P\log|x-2| - \frac{1}{2}\log(f(x)) + C$, then the value of $(P + \min(f(x)))$ is (where C is constant of integration)

15. Let $\int \left(x^{2010}+x^{804}+x^{402}\right)\left(2x^{1608}+5x^{402}+10\right)^{\frac{1}{402}}dx$

$= \frac{1}{10a}\left(2x^{2010}+5x^{804}+10x^{402}\right)^{\frac{a}{402}} + C$, where C is indefinite integration constant. Find a.

16. If $\int \frac{dx}{(x-1)(x^2+1)} = A\log_e(x-1) - B\log_e(x^2+1) - C\tan^{-1}x + D$, then the value of $(2A + 4B + 2C)$ is equal to

17. Let $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = f(x)+C$ where $f(0) = -\ln 2$ and $f\left(-\frac{1}{2}\right) = \ln\left(\sqrt{k}-\frac{3}{2}\right)$, then k is equal to

18. Let $\int \frac{(4+3x)dx}{(x^3-2x-4)} = \ln\left|\frac{x-k}{\sqrt{x^2+kx+\ell}}\right| + C$, then $k^2 + \ell^2$ is equal to

19. If $\int \frac{(x^2+x)}{\sqrt{2x+1}}\,dx = \frac{\sqrt{2x+1}}{k}(x^2+mx-1) + c$, then k is equal to

20. Let $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = k(1+\sqrt[3]{x})^m + C$, then $(k + 2m)$ is equal to

21. Let $f(x) = \int (x - {}^{10}C_1x^2 + {}^{10}C_2x^3 - {}^{10}C_3x^4 + \ldots.. + {}^{10}C_{10}x^{11})\,dx$ and $f(0) = \frac{263}{132}$, then $|f(1)|$ is

22. Let $\int \frac{(5x^4+1)dx}{3x^{4/3}\sqrt{x^4-1}} = \frac{\sqrt{f(x)}}{x^m} + C$ (where $f(0) = -1$), then $\frac{3}{m}$ is equal to

23. If $\int \frac{x+\cos 2x+1}{x\cos^2 x}\,dx = f(x) + k\ln|x| + c$, where $f\left(\frac{\pi}{4}\right) = 1$v, then "$f(0) + 10\,k$" is equal to

Chapter 5 Single Choice Questions (Definite)

1. Let $\int_0^x \left(\frac{mt\cos 4t - \ell\sin 4t}{t^2}\right)dt =$, where $\left(0 < x < \frac{\pi}{4}\right)$ then

a. $\ell + m = \frac{5}{4}$ **b.** $m = 5\ell$

c. $\ell - m = \frac{3}{4}$ **d.** $m = \ell + \frac{1}{4}$

2. $\int_{1/3}^{3} \frac{\tan\left(x^2-\frac{1}{x^2}\right)}{\sin\left(x+\frac{1}{x}\right)}\frac{dx}{x}$

a. 0 **b.** $\frac{3}{2}$ **c.** $\frac{1}{2}$ **d.** $\frac{4}{3}$

3. The value of the integral $\int_0^1 \sin^{-1}(3x-4x^3)\,dx$ is

a. $3(\sqrt{3}+1)-\frac{\pi}{2}$ **b.** $3(\sqrt{3}+1)+\frac{\pi}{2}$

c. $3(\sqrt{3}-1)-\frac{\pi}{2}$ **d.** $3(\sqrt{3}-1)+\frac{\pi}{2}$

4. $\lim\limits_{x\to 0}\frac{1}{x}\int_{-x}^{x}(t\sin(2010t)+2009t+1006)\,dt$ is equal to

a. 2009 **b.** 2010

c. 2011 **d.** 2012

5. If $f(y)$ is differentiable at $y = 5$ and $f(5) = 1, f'(5) = \dfrac{1}{12e}$, then the value of $\lim\limits_{y\to 0}(y-5)^{-1}\int_1^{f^2(y)}(y+t)e^t\,dt$ is

a. 0 **b.** 1 **c.** $\dfrac{5}{6}$ **d.** None

6. $\int_{-3}^{3}\dfrac{\ln|x|\,dx}{1+e^x}$ value is

a. 3 ln 3 **b.** 3 (ln 3 – 1)
c. 3 (ln 3 – 2) **d.** 3 (ln 3 + 1)

7. If $f(x)$ is a linear function satisfying $f(x + f(x)) = x + f(x)$ then $\int_{-5}^{5}|f(x)|\,dx$ equals

a. 0 **b.** 25 **c.** 5 **d.** 10

8. $\int_{(4-\alpha)}^{(\alpha-10)}(x^3+9x^2+29x+33+6x\cos(x+3)+18(x+3))dx$ equals

a. α **b.** $\alpha + 3$ **c.** $-\alpha$ **d.** 0

9. $\int_{\pi/4}^{\pi/2}[|\sin x|+|\cos x|+[\operatorname{cosec} x]]dx$ is equal to (where [·] denotes greatest integer function)

a. $\dfrac{\pi}{4}$ **b.** $\dfrac{\pi}{2}$ **c.** π **d.** $\dfrac{3\pi}{2}$

10. $\int_{-\pi/2}^{\pi/2}\dfrac{dx}{(1+\tan^2 x)(1+e^x)}$ is equal to

a. $\dfrac{\pi}{2}$ **b.** $\dfrac{\pi}{4}$ **c.** $\dfrac{\pi}{4e}$ **d.** $\dfrac{\pi}{2e}$

11. If $\int_a^b\dfrac{x^n}{x^n+(12-x)^n}dx = 4$, then which of the following is correct?

a. $a = 10, b = 2, n \in R$ **b.** $a = 2, b = 10, n \in R$
c. $a = -4, b = 16, n \in R$ **d.** $a = 4, b = 8, n \in R$

12. If $\int_a^b|\sin x|\,dx = 8$ and $\int_a^{a+b}|\cos x|\,dx = 9$, then the value of [sin a + sin b] is (where [×] denotes the greatest integer function)

a. 0 **b.** 1 **c.** 2 **d.** 3

13. The value of $\int_0^2\dfrac{(x-1)^{2n+1}}{1+(x-1)^{2n}+(x-1)^{2n+4}}dx$ ($n \in N$) is equal to

a. $n!$ **b.** $\dfrac{n(n+1)}{2}$ **c.** 0 **d.** n

14. If $A = \int_0^{\pi}\dfrac{\sin x}{x^2}dx$, then $\int_0^{\pi/2}\dfrac{\cos 2x}{x}dx$ is equal to

a. $1 - A$ **b.** $\dfrac{3}{2} - A$
c. $A - 1$ **d.** $1 + A$

15. $\int_{\ln\pi-\ln 2}^{\ln\pi}\dfrac{e^x dx}{1-\cos\left(\frac{2}{3}e^x\right)}$ is equal to

a. $\sqrt{3}$ **b.** $-\sqrt{3}$ **c.** $\dfrac{1}{\sqrt{3}}$ **d.** $-\dfrac{1}{\sqrt{3}}$

16. If [·] denotes greatest integer function and f: $R \to R$ is a non-zero odd function, then $\int_{[-0.15]}^{[1.3]}x^{2/3}f(x^{1/3})\,dx$ is equal to

a. cannot be evaluated **b.** $f(1.3) - f(-0.15)$
c. $3[(f(1.3) - f(-0.15))]$ **d.** 0

17. The value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[x\cos^4 x+\ln\left(\dfrac{1+\tan x}{1-\tan x}\right)\right]dx$ is

a. $\dfrac{\pi^2}{2}$ **b.** $\dfrac{\pi^2}{4}$ **c.** $\dfrac{\pi^2}{8}$ **d.** 0

18. If $I_m = \int_0^{\pi}x\cos^m x\,dx$; $m \in N$, then choose the correct option

a. $I_2 = \dfrac{\pi^2}{2}$ **b.** $I_4 = \dfrac{\pi^2}{8}$

c. $I_2 = \dfrac{3\pi^2}{4}$ **d.** $I_4 = \dfrac{3\pi^2}{16}$

19. The value of $\int_0^2(\sin[x]+\cos[x])dx$ is equal to (where [·] denotes greatest integer function)

a. $1+\sqrt{2}\sin\left(1+\dfrac{\pi}{4}\right)$ **b.** $1+\sqrt{2}\sin\left(1-\dfrac{\pi}{4}\right)$

c. $1-\sqrt{2}\sin\left(1+\dfrac{\pi}{4}\right)$ **d.** $1-\sqrt{2}\sin\left(1-\dfrac{\pi}{4}\right)$

20. If $g(x) = \int_0^x(\sec(\sec t)+\sec(\operatorname{cosec} t)\,dt$, then $g(x + \pi)$ is

a. $g(x) + 2g(\pi)$ **b.** $g(x) - g(\pi)$
c. $g(x).g(\pi)$ **d.** $g(x) + g(\pi)$

21. Statement-1: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\dfrac{1}{1+\tan^n x} = \dfrac{\pi}{6}(n \in R)$

Statement-2: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1
b. Statement-1 is false, statement-2 is true.
c. Statement-1 is true, statement-2 is false
d. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

22. $\int_0^9 \left[\sqrt{x}+2\right] dx$ is equal to (where [·] is G.I.F.)

a. 31 **b.** 23 **c.** 22 **d.** 27

23. $\int_0^{\pi/4} (\tan^6(x-[x])+\tan^4(x-[x]))dx$ is equal to (where [·] is G.I.F.)

a. 5 **b.** 1/5 **c.** – 5 **d.** $-\frac{1}{5}$

24. If $f(x)$ is integrable function in $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ and $I_1 = \int_{\pi/6}^{\pi/3} \tan^2\theta f(2\operatorname{cosec}2\theta)d\theta$ and $I_2 = \int_{\pi/6}^{\pi/3} \cot^2\theta f(2\operatorname{cosec}2\theta)d\theta$ then $\frac{I_1}{I_2}$

a. 2 **b.** – 2 **c.** 1 **d.** – 1

25. If I denotes the value of integral $\int_0^{\pi/2} \frac{\sin x}{[e^{\{x\}}]} dx$ (where [·], {·} denotes greatest integer function and fractional part function respectively) then the value of $[I]$ is

a. 0 **b.** 1
c. 2 **d.** an integer more than 2

26. Let f be a continuous function such that sgn $((f(x)-1)(f(x)-2)) < 0\ \forall\ x \in (4, 8)$, then $\left[\frac{1}{4}\int_4^8 f(x)dx\right]$ is (where sgn is signum function and [·] is greatest integer function)

a. 1 **b.** 2 **c.** 4 **d.** 8

27. If $A = \int_0^{3\pi} \cos^{-1}(\cos x)dx$, then the value of $\left(\frac{2A}{\pi^2}\right)$ is

a. a rational number which is not an integer
b. a prime number
c. an even integer
d. composite number

28. If $I_1 = \int_0^1 \frac{1+x^8}{1+x^4}dx$ and $I_2 = \int_0^1 \frac{1+x^9}{1+x^3}dx$, then

a. $I_1 > 1, I_2 < 1$ **b.** $I_1 < 1, I_2 > 1$
c. $1 < I_1 < I_2$ **d.** $I_2 < I_1 < 1$

29. The value of definite integral $\int_{1/3}^{2/3} \frac{\ln x}{\ln(x-x^2)}dx$ is equal to

a. $\frac{1}{3}$ **b.** $\frac{1}{6}$ **c.** $\frac{1}{9}$ **d.** $\frac{1}{18}$

30. The value of $\lim_{x\to\infty} \frac{1}{x^6}\int_0^{x^2} \frac{t^6+5t^4+55}{2t^4+t+5}dt$ is equal to

a. $\frac{1}{2}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{12}$ **d.** $\frac{1}{6}$

31. If $f(\theta) = \frac{4}{3}(1-\cos^6\theta-\sin^6\theta)$, then

$$\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{f\left(\frac{1}{n}\right)}+\sqrt{f\left(\frac{2}{n}\right)}+\sqrt{f\left(\frac{3}{n}\right)}+.....+\sqrt{f\left(\frac{n}{n}\right)}\right]=$$

a. $\frac{1-\cos 1}{2}$ **b.** $1-\cos 2$
c. $\frac{\sin 2}{2}$ **d.** $\frac{1-\cos 2}{2}$

32. The value of the integral $\int_0^2 \frac{\log(x^2+2)}{(x+2)^2}dx$ is

a. $\frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2}+\frac{5}{12}\log 2-\frac{1}{4}\log 3$
b. $\frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2}-\frac{5}{12}\log 2-\frac{1}{12}\log 3$
c. $\frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2}+\frac{5}{12}\log 2+\frac{1}{12}\log 3$
d. $\frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2}-\frac{5}{12}\log 2+\frac{1}{12}\log 3$

33. The value of $\int_0^1 \frac{(x^6-x^3)}{(2x^3+1)^3}dx$ is equal to

a. $-\frac{1}{6}$ **b.** $-\frac{1}{12}$ **c.** $-\frac{1}{18}$ **d.** $-\frac{1}{36}$

34. The minimum value of $f(x) = \int_0^4 e^{|x-t|}dt$ where $x \in [0, 3]$ is

a. $2e^2 - 1$ **b.** $e^4 - 1$
c. $2(e^2 - 1)$ **d.** $e^2 - 1$

35. The value of $\int_0^{4/\pi}\left(3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}\right)dx$

a. $\frac{8\sqrt{2}}{\pi^3}$ **b.** $\frac{24\sqrt{2}}{\pi^3}$
c. $\frac{32\sqrt{2}}{\pi^3}$ **d.** None of these

36. The value of $\dfrac{\int_0^{x^2}\ln\left(1+t+t^2\right)dt}{\int_0^{x}\ln\left(1+t^3\right)dt}$ equals

a. 0 **b.** $\frac{1}{2}$ **c.** 1 **d.** 2

37. Let $f(x)$ be a continuous and periodic function such that $f(x) = f(x+T)$ for all $x \in R$, $T > 0$. IF $\int_{-2T}^{a+5T} f(x)dx = 19$ $(a > 0)$ and $\int_0^T f(x)dx = 2$, then $\int_0^a f(x)dx$ is equal to

a. 3 **b.** 5 **c.** 7 **d.** 9

38. Let P_k be a point in xy plane whose x coordinate is 1 + (k = 1, 2, 3, … n) on the curve $y = \ln x$. If A is (1, 0), then $\underset{n\to\infty}{\text{lt}} \frac{1}{n}\sum_{k=1}^{n}(AP_k)^2$ equals

a. $\frac{1}{3} + 2\ln^2 1$ **b.** $\frac{1}{3} + 2\ln^2\left(\frac{2}{e}\right)$

c. $\frac{1}{3} + \ln^2\left(\frac{2}{e}\right)$ **d.** $\frac{1}{3} + 2\ln\left(\frac{2}{e}\right)$

39. Given a function g continuous on R such that $\int_0^1 g(t)\mathrm{d}t = 2$ and $g(1) = 5$. If $f(x) = \frac{1}{2}\int_0^x (x-t)^2 \; g(t)\mathrm{d}t$, then the value of $(f'''(1) - f''(1))$ is equal to

a. 0 **b.** 3 **c.** 5 **d.** 7

40. If $\int_{\sin x}^{1} t^2(f(t))\,\mathrm{d}t = (1 - \sin x)$ then $f\left(\frac{1}{\sqrt{3}}\right)$ is

a. 1/3 **b.** $\frac{1}{\sqrt{3}}$ **c.** 3 **d.** $\sqrt{3}$

41. $\int_{-2}^{0}(x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1))\mathrm{d}x$ is equal to

a. – 4 **b.** 0 **c.** 4 **d.** 6

42. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_2^{\sec^2 x} f(t)\,\mathrm{d}t}{x^2 - \frac{\pi^2}{16}}$ equals

a. $\frac{8}{\pi}f(2)$ **b.** $\frac{2}{\pi}f(2)$

c. $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ **d.** $4f(2)$

43. Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\mathrm{d}t$, $0 \le x \le 1$, and $f(0) = 0$, then

a. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

b. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

c. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

d. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

44. The value of $\lim\limits_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}\mathrm{d}t$ is

a. 0 **b.** $\frac{1}{12}$ **c.** $\frac{1}{24}$ **d.** $\frac{1}{64}$

45. The values of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x$ is (are)

a. $\frac{22}{7} - \pi$ **b.** $\frac{2}{105}$

c. 0 **d.** $\frac{71}{15} - \frac{3\pi}{2}$

46. Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,\mathrm{d}t$, for all $x \in (-1, 1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

a. 1 **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{e}$

47. The value of $\int_{\sqrt{\ell n 2}}^{\sqrt{\ell n 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\ell n 6 - x^2)}\mathrm{d}x$ is

a. $\frac{1}{4}\ln\frac{3}{2}$ **b.** $\frac{1}{2}\ln\frac{3}{2}$

c. $\ln\frac{3}{2}$ **d.** $\frac{1}{6}\ln\frac{3}{2}$

48. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ln\frac{\pi+x}{\pi-x}\right)\cos x\,\mathrm{d}x$ is

a. 0 **b.** $\frac{\pi^2}{2} - 4$

c. $\frac{\pi^2}{2} + 4$ **d.** $\frac{\pi^2}{2}$

49. The following integral $\int_{\pi/4}^{\pi/2}(2\operatorname{cosec} x)^{17}\mathrm{d}x$ is equal to

a. $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\mathrm{d}u$ **b.** $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17}\mathrm{d}u$

c. $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17}\mathrm{d}u$ **d.** $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}\mathrm{d}u$

50. Let f: $\int_0^{x^2} f(\sqrt{t})\mathrm{d}t$: [0, 2] → R be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0) = 1$. Let $F(x) =$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

a. $e^2 - 1$ **b.** $e^4 - 1$

c. $e - 1$ **d.** e^4

Chapter 5 Multiple Choice Questions (Definite)

1. $\int_3^4 \frac{dx}{(\ln x)^{1/3}}$ is

a. less than one **b.** greater than 1/2

c. less than two **d.** greater than one

2. If $f(x)$ = sin + $\int_{-\pi/2}^{\pi/2} (\sin x + t\cos x) f(t)dt$ can be expressed as $-\frac{1}{k}\sin x - \frac{2}{k}\cos x$, where k is a constant, then k cannot be more than

a. 2 **b.** 3 **c.** 4 **d.** 5

3. If $f(x - y)$, $f(x) \cdot f(y)$ and $f(x + y)$ are in A.P. $x, y \in R$ and $f(0) \neq 0$, if $f(x)$ has continuous derivative for $x \in R$, then

a. $\int_{-a}^{a} f(x)dx = 0$ **b.** $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

c. $\int_{-a}^{a} f'(x)dx = 0$ **d.** $f'(3) + f'(-3) = 0$

4. If $g(x) \,.\, g(\sqrt{2} - x) = 1$ then $\int_0^{\sqrt{2}} \ln(g(x))dx$ is

a. $\frac{1}{\sqrt{2}}$ **b.** an irrational number

c. a rational number **d.** an even integer

5. Let f: $(0, \infty) \to R$ and $f(x) = \int_0^x tf(t)dt$, $x > 0$. If $f(x^2) = x^4 + x^5$, $x > 0$ then

a. $f(x)$ is onto function **b.** $f(4) = 7$

c. $f(x)$ is continuous everywhere **d.** $\int_0^4 f(x)dx = \frac{64}{3}$

6. The value of the integral $\int_0^1 \left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \left[\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \left[\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]\right]\right]dx$ is less than (where [.] denotes greatest integer function)

a. 1 **b.** 2 **c.** 3 **d.** 4

7. Which of the following definite integral vanishes?

a. $\int_{1/2}^{2} \frac{x^n - 1}{x^{n+2} + 1} dx (x \in N)$

b. $\int_2^4 \left(\log_x 2 - \frac{(\log_x 2)^2}{\ln 2}\right)dx$

c. $\int_{1/2}^{2} x\sin\left(x + \frac{1}{x}\right)dx$

d. $\int_0^{\pi} \cos mx.\sin nx.dx$, where $m, n \in I$ and $(m - n)$ is an even integer

8. If $P = \frac{1}{n^4}\prod_{r=1}^{2n}(n^2 + r^2)^{1/n}$, then $\lim_{n\to\infty} \ln P$ is equal to

a. $\int_0^2 \ln(1+x^2)dx$ **b.** $\int_0^1 \ln(x^2 - 4x + 5)$

c. $\int_0^2 \ln(x^2 - 4x + 5)\,dx$ **d.** $2\int_0^2 \ln(1+x^2)dx$

9. If $\int_{-\alpha}^{\alpha} (e^x + \cos x \ln(x + \sqrt{1+x^2}))dx > \frac{3}{2}$, then α can be

a. 1 **b.** 2 **c.** 3 **d.** 4

10. Let $f(x) = \begin{cases} 5x+1, & x \le 2 \\ \int_0^x (5+|1-t|)dt, & x > 2 \end{cases}$ then which of the following statements is/are incorrect?

a. $f(x)$ is continuous but not differentiable at $x = 2$

b. $f(x)$ is not continuous at $x = 2$.

c. $f(x)$ is differentiable for all $x \in R$.

d. The right-hand derivative of $f(x)$ at $x = 3$ does not exist.

11. Which of the following statement(s)/are true?

a. If function $y = f(x)$ is continuous at $x = c$ such that $f(c) \neq 0$, then $f(x) f(c) > 0$ $x \in (c - h, c + h)$, where h is sufficiently small positive quantity.

b. $\lim_{n\to\infty} \frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right) = 1 + 2\ln 2$

c. Let f be a continuous and non-negative function defined on $[a, b]$. If $\int_a^b f(x)dx = 0$ then $f(x) = 0 \;\forall\; x \in [a, b]$

d. Let f be a continuous function defined on $[a, b]$ such that $\int_a^b f(x)dx = 0$, then there exists at least one $c \in (a, b)$ for which $f(c) = 0$

12. The value of the definite integral

$\int_{-\infty}^{a} \frac{(\sin^{-1} e^x + \sec^{-1} e^{-x})dx}{(\tan^{-1} e^a + \tan^{-1} e^x)(e^x + e^{-x})} (a \in R)$ is

a. Independent of a **b.** dependent on a

c. $\frac{\pi}{2}\ln 2$ **d.** $\frac{\pi}{2}\ln(2\tan^{-1} e^a)$

13. If $I_n = \int_0^1 (1 - x^2)^n dx$, then

a. $I_n = \frac{2n}{2n+1} I_{n-1}$ **b.** $I_n = \frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)}$

c. $I_n = \frac{2^n n!}{3.5.7\ldots(2n+1)}$ **d.** $I_n = \frac{2n+1}{2n} I_{n-1}$

14. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$ for $n = 1, 2, 3, \ldots$ Then,

a. $S_n < \frac{\pi}{3\sqrt{3}}$ **b.** $S_n > \frac{\pi}{3\sqrt{3}}$

c. $T_n < \frac{\pi}{3\sqrt{3}}$ **d.** $T_n > \frac{\pi}{3\sqrt{3}}$

15. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx$, $n = 0, 1, 2, \ldots$, then

a. $I_n = I_n + 2$ **b.** $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

c. $\sum_{m=1}^{10} I_{2m} = 0$ **d.** $I_n = I_n + 1$

16. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$. Then

a. $S \le \frac{1}{e}$ **b.** $S \ge 1 - \frac{1}{e}$

c. $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ **d.** $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

17. For $a \in R$ (the set of all real numbers), $a \ne -1$.

$$\lim_{n\to\infty} \frac{(1^a + 2^a + \ldots\ldots + n^a)}{(n+1)^{a-1}[(na+1) + (na+2) + \ldots\ldots + (na+n)]} = \frac{1}{60}$$

then $a =$

a. 5 **b.** 7 **c.** $\frac{-15}{2}$ **d.** $\frac{-17}{2}$

18. Let f: $[a, b] \to [1, \infty)$ be a continuous function and let g: $R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)dt & \text{if } a \le x \le b, \\ \int_a^b f(t)dt & \text{if } x > b \end{cases}$$

then

a. $g(x)$ is continuous but not differentiable at a

b. $g(x)$ is differentiable on R

c. $g(x)$ is continuous but not differentiable at b

d. $g(x)$ is continuous and differentiable at either a or b but not both.

19. Which of the following definite integral(s) has/have their value(s) equal to at least two of the remaining other three?

a. $\int_0^{2\pi} \sqrt{1 - \sin x}\, dx$ **b.** $\int_0^{2\pi} \sqrt{1 + \sin x}\, dx$

c. $\int_0^{2\pi} \sqrt{1 - \sin 2x}\, dx$ **d.** $\int_0^{2\pi} \sqrt{1 + \sin 2x}\, dx$

20. Let f be a continuous function on R and satisfies $f(x) = e^x + \int_0^1 e^x f(t)\, dt$, then which of the following is(are) correct?

a. $f(0) < 0$

b. $f(x)$ is decreasing function on R

c. $f(x)$ is an increasing function on R

d. $\int_0^1 f(x)dx > 0$

21. Let $f(x) = \int_1^x \frac{3^t}{1+t^2} dt$, $x > 0$ then

a. for $0 < \alpha < \beta$, $f(\alpha) < f(\beta)$

b. for $0 < \alpha < \beta$, $f(\alpha) > f(\beta)$

c. for all $x > 0$, $f(x) + \frac{\pi}{4} < \tan^{-1}x$

d. for all $x > 0$, $f(x) + \frac{\pi}{4} > \tan^{-1}x$.

22. Which of the following statement(s) is(are) correct?

a. If f: $[0, 1] \to R$ is continuous and $\int_0^x f(x)dx = \int_x^1 f(x)dx$ for all $x \in [0, 1]$ then $f(x) = 0$ for all $x \in [0, 1]$.

b. Let f be a function defined in $[a, b]$ such that $f(x) \ge 0$ $\forall$ $x \in [a, b]$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0$ $\forall$ $x \in [a, b]$.

c. If $f(x)$ is continuous on R then $\int_0^{\frac{\pi}{2}} f(\cos x)dx = \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\sin 2x)dx$.

d. For $a > 0$, if f is continuous in $[-a, a]$, then $\int_{-a}^{a} f(x^2)dx = 2\int_0^a f(x^2)dx$.

23. Let $f(x) = \tan x - \tan^3 x + \tan^5 x - \tan^7 x + \ldots \infty$, where $x \in \left(0, \frac{\pi}{4}\right)$, then which of the following is/are correct?

a. $\int_0^{\frac{\pi}{6}} f(x)dx = \frac{1}{8}$ **b.** $f'\left(\frac{\pi}{12}\right) = \frac{1}{2}$

c. $\lim_{x\to 0^+} \frac{f(x)}{x} = 1$ **d.** $f(x)$ is an odd function.

24. Let $l_1 = \lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t(e^{3t}-1)\ln(1+2t)}{(t^3+3)(1-\cos\sqrt{t})} dt$ and

$$l_2 = \lim_{x\to 0} \frac{1}{x}\left(\int_{\frac{\pi}{4}}^{v} \log_{\frac{1}{2}} \sin^2 t\, dt - \int_{\left(\frac{\pi}{4}+x\right)}^{v} \log_{\frac{1}{2}} \sin^2 t\, dt\right)$$

where $\pi < v < 2\pi$. Then which of the following is(are) correct?

a. $9\,l_1^2 + l_2^2 = 18$ **b.** $3l_1 + 4l_2 = 8$

c. $l_1 > 0$ and $l_2 < 0$ **d.** $l_1 > l_2$

25. If $f(x) = \frac{1}{x^3\sqrt{1+x^4}}$ then which of the following statement(s) is/are correct?

a. $\int_{-1}^{1} f(x)\, dx = 0$ **b.** $\int_1^2 f(x)\, dx = \frac{4\sqrt{2} - \sqrt{17}}{8}$

c. $\int_{-2}^{2} f(x)\, dx = 0$ **d.** $f'(x)$ is even function

Chapter 5 Passage Type Questions (Definite)

Passage 1

Consider $f(x) = x^2 + x(1 - 2[x]) + [x]^2 - [x]$ (where $[\cdot]$ denotes greatest integer function). On the basis of above information, answer the following questions.

1. $\int_0^1 f(x)dx$ is equal to

a. $\frac{5}{6}$ **b.** $\frac{5}{3}$ **c.** $\frac{2}{3}$ **d.** $\frac{1}{2}$

2. $\int_0^{20} f(x)dx$ is equal to

a. $\frac{100}{3}$ **b.** $\frac{40}{3}$ **c.** 10 **d.** $\frac{50}{3}$

3. In n denotes the number of solutions of $f(x) = 0$ in $(-3, 3)$ then $\int_0^n f(x)dx$ is equal to

a. $\frac{10}{3}$ **b.** $\frac{25}{6}$ **c.** $\frac{5}{2}$ **d.** $\frac{5}{3}$

Passage 2

Let $f(x)$ be cubic polynomial with leading coefficient unity such that $f(a) = b$ and $f'(a) = f''(a) = 0$. Suppose $g(x) = f(x) - f(a) + (a - x) f'(x) + 3(x - a)^2$ for which conclusion of Rolle's theorem in $[a, b]$ holds at $x = 2$, where $2 \in (a, b)$

1. The value of $f''(2)$ is equal to

a. 2 **b.** 3 **c.** 4 **d.** 6

2. The value of definite integral $\int_a^b f(x)dx$ is equal to

a. $\frac{123}{64}$ **b.** $\frac{213}{64}$ **c.** $\frac{321}{64}$ **d.** $\frac{481}{64}$

Passage 3

Consider a polynomial $P(x) = x^{102} - 3x^{101} + 5x^{97} - x + 2$. If $P(x)$ is divided by $x(x^2 - 1)$, then quotient is $Q(x)$ and remainder is $R(x)$.

1. The value of $\int_0^1 \frac{dx}{R(x)-1}$ is equal to

a. $\frac{\pi}{\sqrt{3}}$ **b.** $\frac{\pi}{3\sqrt{3}}$ **c.** $\frac{2\pi}{\sqrt{3}}$ **d.** $\frac{2\pi}{3\sqrt{3}}$

2. If $\int\left(\frac{P(x)}{x(x^2-1)} - Q(x)\right)dx = \ln(f(x)) + C$, where C is constant of integration and $f(2) = \frac{3}{4}$, then $f(x)$ is

a. $\frac{x+1}{2x}$ **b.** $\frac{(x+1)(x-1)}{2x}$

c. $\frac{(x+1)(x-1)^2}{x^2}$ **d.** $\frac{x^2-x+1}{x^2}$

Passage 4

Let $y = f(x)$ be a twice differentiable, non-negative function defined on $[a, b]$. The area $\int_a^b f(x)dx$, $b > a$ bounded by $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ can be approximated as $\int_a^b f(x)dx \cong \frac{(b-a)}{2}\{f(a) + f(b)\}$. Since $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $c \in (a, b)$, a better approximation is

$$\int_a^b f(x)dx \cong \frac{(c-a)}{2}\{f(a) + f(c)\} + \frac{b-c}{2}\{f(c) + f(b)\} \equiv F(c).$$

If $c = \frac{a+b}{2}$, then this gives: $\int_a^b f(x)dx \cong \frac{b-a}{4}\{f(a) + 2f(c) + f(b)\}$ (i)

1. The approximate value of $\int_0^{\pi/2} \sin x dx$ using rule (1) given above is

a. $\frac{\pi}{8\sqrt{2}}(1+\sqrt{2})$ **b.** $\frac{\pi}{4\sqrt{2}}(1+\sqrt{2})$

c. $\frac{\pi}{8}(1+\sqrt{2})$ **d.** $\frac{\pi}{4}(1+\sqrt{2})$

2. If $\lim_{x\to a}\left\{\frac{\int_a^t f(x)\,dx - \frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3}\right\} = 0$, for each fixed a, then $f(x)$ is a polynomial of degree utmost

a. 4 **b.** 3 **c.** 3 **d.** 1

3. If $f''(x) < 0$, $x \in (a, b)$, then at the point $C(c, f(c))$ on $y = f(x)$ for which $F(c)$ is a maximum, $f'(c)$ is given by

a. $f'(c) = \frac{f(b)-f(a)}{b-a}$ **b.** $f'(c) = \frac{f(b)-f(a)}{a-b}$

c. $f'(c) = \frac{2(f(b)-f(a))}{b-a}$ **d.** $f'(c) = 0$

Passage 5

Given that for each $a \in (0, 1)$, $\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$.

1. The value of $g\left(\frac{1}{2}\right)$ is

a. π **b.** 2π **c.** $\frac{\pi}{2}$ **d.** $\frac{\pi}{4}$

2. The value of $g'\left(\frac{1}{2}\right)$

a. $\frac{\pi}{2}$ **b.** π

c. $-\frac{\pi}{2}$ **d.** 0

Passage 6

Consider the function $f(x) = \frac{x}{2^x}$ and $g(x) = \max. \{f(t) : x \le t \le x + 1\}$

1. Which of the following statement is incorrect?

a. $f(x)$ is increasing in $\left(-\infty, \frac{1}{\ln 2}\right)$ and decreasing in $\left(\frac{1}{\ln 2}, \infty\right)$

b. local maximum value of $f(x)$ exists

c. local minimum value of $f(x)$ does not exist

d. $f(x)$ changes its concavity at 2 points

2. If $f(x) = k$ has 2 distinct real roots then range of k is equal to

a. $\left(0, \frac{1}{e}\right)$ **b.** $\left(0, \frac{1}{e\ln 2}\right)$

c. $\left(\frac{1}{e\ln 2}, \infty\right)$ **d.** $(-\infty, 0)$

3. The value of the definite integral $\int_0^{\frac{1}{\ln 2}-1} g(x)\,dx$ is equal to

a. $\frac{1}{2\ln^2 2} + \frac{1}{2\ln 2} - \frac{2}{e\ln^2 2}$ **b.** $\frac{1}{2\ln^2 2} + \frac{1}{2\ln 2}$

c. $\frac{1}{2\ln^2 2} + \frac{1}{2\ln 2} + \frac{2}{e\ln^2 2}$ **d.** $\frac{1}{2\ln^2 2} - \frac{1}{2\ln 2}$

Passage 7

Let $g: R \to R$ be a differentiable function which satisfies $g(x) = 1 + \int_0^x g(t)\,dt$ and $g'(0) = 1$

1. The value of $g(\ln 10) + g'(\ln 10) + g''(\ln 10)$ is equal to

a. 0 **b.** $\frac{1}{10}$

c. 30 **d.** $\frac{1}{30}$

2. The value of definite integral $\int_{-3}^{-1}\left(\sum_{r=1}^{\infty} g(rx)\right)dx$ is equal to

a. $\ln(1 + e + e^{-1})$

b. $\ln(1 + e^{-1} + e^{-2})$

c. $\ln(1 + e + e^{2})$

d. $(1 + e^{-1} + e^{2})$

3. Number of solution of the equation $f(-x) = f(x)$ is equal to

a. 0 **b.** 1

c. 2 **d.** 3

Chapter 5 Matrix Match Type Questions (Definite)

1.

Column-I	Column-II
(a) $\int_1^5 \sqrt{x - 2\sqrt{x-1}}\,dx$	(p) $\frac{\pi}{2}$
(b) $\int_{-1}^{2}\left[\frac{[x]}{1+x^2}\right]dx$, where $[\cdot]$ denotes greatest integer function	(q) 2
(c) $\int_{1/2}^{2} \frac{x\ln x}{(1+x^2)^2}\,dx$	(r) – 1
(d) $\int_0^{\pi}\left\lvert\sin x - \frac{2x}{\pi}\right\rvert dx$	(s) 0
	(t) a rational number

2.

Column-I	Column-II
(a) $\int_0^6 \{x\}dx$, (where $\{\cdot\}$ is fractional part function) is equal to	(p) 1
(b) $\int_{-2}^{0} (x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1))dx$ is equal to	(q) 2
(c) $\frac{8}{\pi^2}\int_{-\pi/2}^{\pi} \sin^{-1}(\sin x)dx$ is equal to	(r) 3
(d) $\int_0^{1/2} 2[\cos^{-1} x]dx$ is equal to (where $[\cdot]$, is greatest integer function	(s) 4
	(t) 5

3.

Column-I	Column-II
(a) Let $f(x-3) = f(x+3)$ for all $x \in R$ and $f(x) = \begin{cases} x, & 0 \le x < 3 \\ 6-x, & 3 \le x < 6 \end{cases}$ If $\int_0^{54} f(x)dx = 9^k$ then k is equal to	(p) 1
(b) If $\lim_{x\to\frac{\pi}{2}} \dfrac{\left(\int_0^x \sin^2\theta d\theta\right)\left(\int_0^x \cos^2\theta\, d\theta\right) - \frac{\pi^2}{16}}{x - \frac{\pi}{2}} = \dfrac{\pi^2}{\lambda}$ then λ is equal to	(q) 1
(c) If $L = \int_0^{\frac{9\pi}{2}} (1-\sin x).\sin(x+\cos x)\, dx$ then $[L]$ is equal to **Note:** $[y]$ denotes greatest integer less than or equal to y.	(r) 2
(d) If $M = \int_0^1 \sin^{-1}(1-x)dx + \int_2^3 \cos^{-1}(x-2)dx$, then $[M]$ is equal to **Note:** $[y]$ denotes greatest integer less than or equal to y.	(s) 4

4. Match the integrals in **Column-I** with the values in **Column-II**

Column-I	Column-II
(a) $\int_{-1}^{1} \dfrac{dx}{1+x^2}$	(p) $\dfrac{1}{2}\log\left(\dfrac{2}{3}\right)$
(b) $\int_0^1 \dfrac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\dfrac{2}{3}\right)$
(c) $\int_2^3 \dfrac{dx}{1-x^2}$	(r) $\dfrac{\pi}{3}$
(d) $\int_1^2 \dfrac{dx}{x\sqrt{x^2-1}}$	(s) $\dfrac{\pi}{2}$
	(t)

5.

List-I	List-II
(a) The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	(p) 8
(b) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value is	(q) 2
(c) $\int_{-2}^{2} \dfrac{3x^2}{(1+e^x)}dx$ equals	(r) 4
(d) $\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_0^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals	(s) 0

6.

Column-I	Column-II
(a) If $I = \int_2^3 \left((x-1)^3 + (4-x)^3 + x\right) \cos\pi x\, dx$, then $\left|50\pi^2 I\right|$ is equal to	(p) 0
(b) If $J = \int_0^{10} \operatorname{sgn}(\sin \pi x)dx$, then 10 J is equal to, where sgn x denotes signum function of x	(q) 100
(c) If $K = \int_0^{102} \left[\cot^{-1} x\right] dx$, then $[K]$ is equal to, where $[y]$ denotes largest integer less than or equal to y	(r) 50
(d) If $L = \dfrac{\int_0^{51}[x+25]\,dx}{\int_0^{51}\{x+25\}\,dx}$, then $\dfrac{L}{2}$ is equal to, where $[y]$ and $\{y\}$ denote greatest integer function and fractional part function, respectively.	(s) 70

Chapter 5 Integer Type Questions (Definite)

1. If $\int_{-5}^{-2}\left(\dfrac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{1/6}^{1/3}\left(\dfrac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{6/5}^{3/2}\left(\dfrac{x^2-x}{x^3-3x+1}\right)^2 dx$ $= \dfrac{p}{q}$, where p and q are co-primes, then $\dfrac{6q-p}{6}$ is equal to

2. If $8\int_0^{\pi/2} \dfrac{x^2(\cos x - \sin x)dx}{(\cos x + \sin x)^3} = \pi(k-\pi)$, then k is equal to

3. The value of the integral $\int_0^{\sqrt{2}} \left([\sqrt{2-x^2}] + 2x\right)dx$ (where $[\cdot]$ denotes greatest integer function) is

4. If $k = \int_{-1}^{3/2} \max_{-1\le t \le x}\left\{|t| - \left|t - \dfrac{1}{2}\right| + |t-1|\right\}dx$, then the value of $4k - 6$ is equal to

5. A function $f(x)$ satisfies $f(x) = f\left(\frac{c}{x}\right)$ for some real number c ($c > 1$) and all positive numbers. If $\int_1^{\sqrt{c}} \frac{f(x)}{x} dx = 3$, then the value of $\int_1^{c} \frac{f(x)}{x} dx$ is:

6. Let $f(x)$ be a differentiable function on R such that $f'(5 - x) = f'(x)$ $x \in [0, 5]$ with $f(0) = -10$ and $f(5) = 50$, then the value of $\left(5\int_0^5 f(x)dx\right)$ is

7. If $g(x) = \int_{-1}^{x} f(t)dt$, where $f(t) = |t + 1| + |t| + |t - 1|$ and $x \in [-1, -2]$, then the greatest integer less than $g(2)$ is

8. If $f(x) = x^8 - x^6 + x^4 + \frac{x}{2011} - 2012$, then $\int_{-2011}^{2011} \frac{f'(x) + f'(-x)}{(2011)^x + 1} dx$ is equal to

9. If $f(x) = \int_{x^2}^{x^3} \ln t\, dt$ and $f'\left(\frac{4}{9}\right) = f'(\alpha)$, then possible integral value of α is

10. For a positive integer n, let $I_n = \int_{-\pi}^{\pi} \left(\frac{\pi}{2} - |x|\right) \cos(nx)\, dx$, then the value of $[I_1 + I_2 + I_3 + I_4]$ is (where [.] denotes greatest integer function)

11. If $I_k = \int_{-1}^{1} \frac{k \sin x + 5\sqrt[5]{x} + 3x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$ and $\ln 16 = 2.7726$ then $\left[\sum_{k=1}^{4} I_k\right]$, where [.] denotes greatest integer function is

12. If $f: [0, 1] \to R$ is a continuous function satisfying $\int_0^1 f(x)dx = \frac{1}{3} + \int_0^1 (f(x^2))^2 dx$, then the reciprocal of $f\left(\frac{1}{4}\right)$ is

13. If $\int_0^{\pi/4} [[\sec x] + |\cos x| + |\sin x|]dx$, where $[\cdot]$ denotes G.I.F is equal to $\frac{k\pi}{8}$, then k is equal to

14. The value of $\left|\int_0^1 \frac{(x + \sqrt{4 - x^2})^{1/3}(2 - x\sqrt{4 - x^2})^{1/10}}{(x^2 - 2)^{1/5}} dx\right|^{10}$ is equal to

15. If $f(x) = \begin{vmatrix} \ln(x + \sqrt{x^2 + 1}) & \ln\left(\frac{1 + x^2}{1 - x^2}\right) & \sec^2 x \\ \sin x & 1 & \tan^2 x \\ \ln\left(\frac{1 + x}{1 - x}\right) & \sin^2 x & 2 \end{vmatrix}$, then the value of $\int_{-1}^{1} f(x)dx$ is equal to

16. Let $f: (0, 1) \to (0, 1)$ be a differentiable function such that $f'(x) \leq 0$ 0 for all $x \in (0, 1)$ and $f\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$. If $f(x) = \lim_{t \to x} \frac{\int_0^t \sqrt{1 - f^2(s)}\, ds - \int_0^x \sqrt{1 - f^2(s)}\, ds}{f(t) - f(x)}$, then the value of $f\left(\frac{1}{4}\right)$ equals $\frac{\sqrt{m}}{4}$, where $m \in N$. Find the value of m.

17. Let $f(x) = x \cos x$, $x \in R\left[\frac{3\pi}{2}, 2\pi\right]$ and g is the inverse function of f. If $\int_0^{2\pi} g(x)dx = a\pi^2 + b\pi + c$, where $a, b, c \in R$, then find the value of $2(a + b + c)$.

18. For a positive constant t, let a, b be the roots of the quadratic equation $x^2 + t^2x - 2t = 0$. If the minimum value of $\int_{-1}^{2}\left(\left(x + \frac{1}{\alpha^2}\right)\left(x + \frac{1}{\beta^2}\right) + \frac{1}{\alpha\beta}\right)dx$ is $\sqrt{\frac{a}{b}} + c$, where $a, b, c \in N$, then find the least value of $(a + b + c)$.

19. Let $f(x) = x^3 - \frac{3x^2}{2} + x + \frac{1}{4}$. Find the value of $\left(\int_{1/4}^{3/4} f(f(x))dx\right)^{-1}$.

20. The value of $\frac{5050\int_0^t (1 - x^{50})^{100}\, dx}{\int_0^t (1 - x^{50})^{101}\, dx}$ is

21. Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of $f(\ln 5)$ is

22. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real-valued function defined on the interval $[-10, 10]$ by $f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$. Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x) \cos \pi x\, dx$ is

23. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1 - x^2)^5\right\} dx$ is

24. Let $f(t)$ be a cubic polynomial such that $\cos 3x = f(\cos x)$ holds $\forall x \in R$ and $J = \int_0^1 f^2(t)\sqrt{1 - t^2}\, dt$. Find the value of $\frac{(2016)J}{\pi}$.

25. If $\int_0^{\frac{\pi}{2}} \frac{\sin^2(10)\theta}{\sin^2\theta} d\theta = k\pi$, where $k \in N$, then find the value of k.

26. Let $f(x) = \int_x^{\frac{\pi}{4} - x} \log_4(1 + \tan t)\, dt$ $\left(0 < x \leq \frac{\pi}{8}\right)$. Find the value of $\frac{\pi - 16f(x)}{x}$.

[**Ans.** 8]

27. Let g be a real-valued differentiable function on R such that $g(x) = 3e^{x-2} + 4\int_2^x \sqrt{2t^2+6t+5}\, dt \;\; \forall x \in R$ and let g^{-1} be the inverse function of g.

If $(g^{-1})'(3)$ is equal to $\frac{p}{q}$, where p and q are relatively prime then find $(p + q)$.

28. Let x_1 and x_2 $(x_1 < x_2)$ be two values of x, satisfying $\left|\,|x|^2 - 4|x| + 5\right| + \left|\,|x^3| - 4|x| + 3\right| = 4$ and $f(x)$ is a continuous periodic function with period $\frac{1}{2}$ such that $\int_0^{\frac{3}{2}} f(x)\mathrm{d}x = 6$.

Find the value of $\int_{x_1}^{x_2} f(x)\mathrm{d}x$.

29. If the value of definite integral $\int_0^{\ln 2} \frac{(x-\ln 2)e^{x+\ln 2}}{e^{\ln(1+e^x)^2}}\mathrm{d}x$ can be expressed in the form $\ln\left(\frac{a}{b}\right)$, where $a, b \in N$ and are relatively prime, then find the value of $(a + b)$.

30. Given that $U_n = \left(x(1-x)\right)^n$ for $n \in N, n \geq 2$ and $\frac{d^2U_n}{dx^2} = n(n-1)U_{n-2} - 2n(2n-1)U_{n-1}$.

Further if $V_n = \int_0^1 e^x . U_n\, \mathrm{d}x$, then for $n \geq 2$, $V_n + K_1 n(2n-1)V_{n-1} + K_2 n(n-1)V_{n-2} = 0$ where $K_1, K_2 \in I$. Find $(K_1 + K_2)$.

Chapter 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

b. If $\int_0^1 f(x)\,dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^{2} f(x)\,dx = 4, \quad \int_2^5 f(x)\,dx = 3, \quad \int_{-2}^{5} g(x)\,dx = 2.$

Which, if any, of the following statements are true?

a. $\int_5^2 f(x)\,dx = -3$ **b.** $\int_{-2}^{5} (f(x) + g(x)) = 9$

c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. Initial value problem Show that

$$y = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \qquad \frac{dy}{dx} = 0 \text{ and } y = 0 \text{ when } x = 0.$$

(*Hint:* $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. Proportionality Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

a. $\int_0^{x^2} f(t)\,dt = x\cos \pi x$ **b.** $\int_0^{f(x)} t^2\,dt = x\cos \pi x$.

6. Find $f(\pi/2)$ from the following information.

i) f is positive and continuous.

ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a.$$

7. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2+1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt\right) du = \int_0^x f(u)(x-u)\,du.$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

9. Finding a curve Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. Shoveling dirt You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval *I*** if f has only finitely many discontinuities in I, the limits

$$\lim_{x\to c^-} f(x) \qquad \text{and} \qquad \lim_{x\to c^+} f(x)$$

exist and are finite at every interior point of I, and the appropriate one-sided limits exist and are finite at the endpoints of I. All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1 - x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.31) over $[-1, 3]$ is

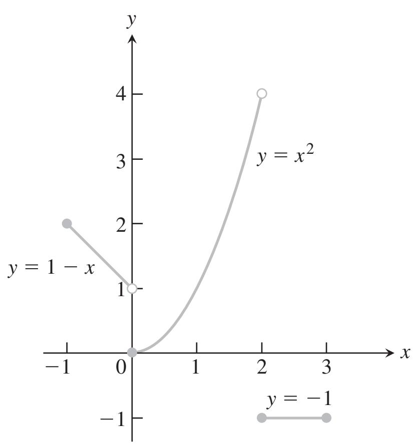


FIGURE 5.31 Piecewise continuous functions like this are integrated piece by piece.

$$\begin{aligned} \int_{-1}^{3} f(x)\,dx &= \int_{-1}^{0} (1 - x)\,dx + \int_{0}^{2} x^2\,dx + \int_{2}^{3} (-1)\,dx \\ &= \left[x - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{2} + \left[-x\right]_{2}^{3} \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx)\int_a^x f(t)\,dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule (see Exercises 27–29).

Graph the functions in Exercises 11–16 and integrate them over their domains.

11. $f(x) = \begin{cases} x^{2/3}, & -8 \leq x < 0 \\ -4, & 0 \leq x \leq 3 \end{cases}$

12. $f(x) = \begin{cases} \sqrt{-x}, & -4 \leq x < 0 \\ x^2 - 4, & 0 \leq x \leq 3 \end{cases}$

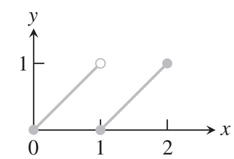
13. $g(t) = \begin{cases} t, & 0 \leq t < 1 \\ \sin \pi t, & 1 \leq t \leq 2 \end{cases}$

14. $h(z) = \begin{cases} \sqrt{1 - z}, & 0 \leq z < 1 \\ (7z - 6)^{-1/3}, & 1 \leq z \leq 2 \end{cases}$

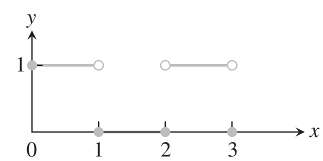
15. $f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1 - x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$

16. $h(r) = \begin{cases} r, & -1 \leq r < 0 \\ 1 - r^2, & 0 \leq r < 1 \\ 1, & 1 \leq r \leq 2 \end{cases}$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



Approximating Finite Sums with Integrals

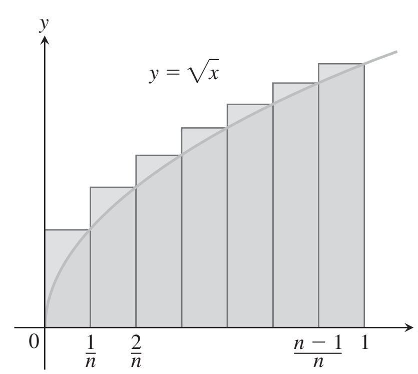
In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg]_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$\begin{aligned} S_n &= \sqrt{\frac{1}{n}} \cdot \frac{1}{n} + \sqrt{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \sqrt{\frac{n}{n}} \cdot \frac{1}{n} \\ &= \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}. \end{aligned}$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3}n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

19. Evaluate

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5\, dx$$

and evaluating the integral.

20. See Exercise 19. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3).$$

21. Let $f(x)$ be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

as a definite integral.

22. Use the result of Exercise 21 to evaluate

a. $\lim_{n\to\infty} \frac{1}{n^2}(2 + 4 + 6 + \cdots + 2n)$,

b. $\lim_{n\to\infty} \frac{1}{n^{16}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$,

c. $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$.

What can be said about the following limits?

d. $\lim_{n\to\infty} \frac{1}{n^{17}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

e. $\lim_{n\to\infty} \frac{1}{n^{15}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

23. a. Show that the area A_n of an n-sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}.$$

b. Find the limit of A_n as $n \to \infty$. Is this answer consistent with what you know about the area of a circle?

24. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n\to\infty} S_n$, show that

$$S_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right]$$

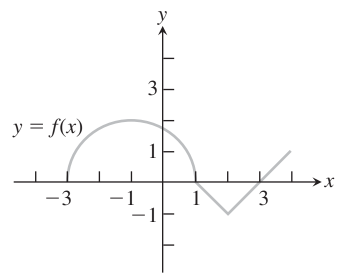
and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2\, dx.$$

(*Hint:* Partition $[0, 1]$ into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

Defining Functions Using the Fundamental Theorem

25. A function defined by an integral The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t)\, dt$.



a. Find $g(1)$. **b.** Find $g(3)$. **c.** Find $g(-1)$.

d. Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

e. Write an equation for the line tangent to the graph of g at $x = -1$.

f. Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

g. Find the range of g.

26. A differential equation Show that both of the following conditions are satisfied by $y = \sin x + \int_x^\pi \cos 2t\, dt + 1$:

i) $y'' = -\sin x + 2\sin 2x$

ii) $y = 1$ and $y' = -2$ when $x = \pi$.

Use Leibniz's Rule to find the derivatives of the functions in Exercises 27–29.

27. $f(x) = \int_{1/x}^{x} \frac{1}{t}\, dt$

28. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1 - t^2}\, dt$

29. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\, dt$

30. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

$$\int_x^{x+3} t(5 - t)\, dt.$$

Evaluating Integrals

Evaluate the integrals in Exercises 31–36.

31. $\int (\sin^{-1} x)^2\, dx$

32. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

33. $\int x \sin^{-1} x\, dx$

34. $\int \sin^{-1}\sqrt{y}\, dy$

35. $\displaystyle\int \frac{dt}{t - \sqrt{1 - t^2}}$

36. $\displaystyle\int \frac{dx}{x^4 + 4}$

Evaluate the limits in Exercises 37 and 38.

37. $\displaystyle\lim_{x\to\infty} \int_{-x}^{x} \sin t \, dt$

38. $\displaystyle\lim_{x\to 0^+} x \int_{x}^{1} \frac{\cos t}{t^2} \, dt$

Evaluate the limits in Exercises 39 and 40 by identifying them with definite integrals and evaluating the integrals.

39. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}}$

40. $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

41. For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx$$

converge? Evaluate the corresponding integral(s).

42. For each $x > 0$, let $G(x) = \int_0^{\infty} e^{-xt} \, dt$. Prove that $xG(x) = 1$ for each $x > 0$.

43. Integrating the square of the derivative If f is continuously differentiable on $[0, 1]$ and $f(1) = f(0) = -1/6$, prove that

$$\int_0^1 (f'(x))^2 \, dx \geq 2 \int_0^1 f(x) \, dx + \frac{1}{4}.$$

Hint: Consider the inequality $0 \leq \displaystyle\int_0^1 \left(f'(x) + x - \frac{1}{2} \right)^2 dx.$

Source: Mathematics Magazine, vol. 84, no. 4, Oct. 2011.

44. (*Continuation of Exercise 43.*) If f is continuously differentiable on $[0, a]$ for $a > 0$, and $f(a) = f(0) = b$, prove that

$$\int_0^a (f'(x))^2 \, dx \geq 2 \int_0^a f(x) \, dx - \left(2ab + \frac{a^3}{12} \right).$$

Hint: Consider the inequality $0 \leq \displaystyle\int_0^a \left(f'(x) + x - \frac{a}{2} \right)^2 dx.$

Source: Mathematics Magazine, vol. 84, no. 4, Oct. 2011.

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)g(x) \, dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x \, dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} and its derivatives		$\cos x$ and its integrals
e^{2x}	$(+)$	$\cos x$
$2e^{2x}$	$(-)$	$\sin x$
$4e^{2x}$	$(+)$	$-\cos x$

← *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, -1 on the right).

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x} \cos x \, dx = +(e^{2x} \sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x) \, dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x$$

or

$$\int e^{2x} \cos x \, dx = \frac{e^{2x} \sin x + 2e^{2x} \cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 45–52.

45. $\displaystyle\int e^{2x} \cos 3x \, dx$

46. $\displaystyle\int e^{3x} \sin 4x \, dx$

47. $\displaystyle\int \sin 3x \sin x \, dx$

48. $\displaystyle\int \cos 5x \sin 4x \, dx$

49. $\displaystyle\int e^{ax} \sin bx \, dx$

50. $\displaystyle\int e^{ax} \cos bx \, dx$

51. $\displaystyle\int \ln (ax) \, dx$

52. $\displaystyle\int x^2 \ln (ax) \, dx$

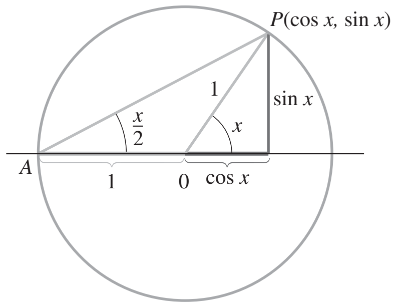
The Substitution $z = \tan (x/2)$

The substitution

$$z = \tan \frac{x}{2} \qquad (1)$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a. $$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int dz = z + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

b. $$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$

$$= \int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 53–60. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

53. $\displaystyle\int \frac{dx}{1 - \sin x}$

54. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

55. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

56. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

57. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

58. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

59. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

60. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 61 and 62.

61. $\displaystyle\int \sec\theta\,d\theta$

62. $\displaystyle\int \csc\theta\,d\theta$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt, \quad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 5.32 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 23 in Chapter 15.

63. If n is a nonnegative integer, $\Gamma(n + 1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x + 1)$ to show that $\Gamma(x + 1) = x\Gamma(x)$. This gives

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\Gamma(4) = 3\Gamma(3) = 6$$

$$\vdots$$

$$\Gamma(n + 1) = n\,\Gamma(n) = n! \tag{5}$$

c. Use mathematical induction to verify Equation (5) for every nonnegative integer n.

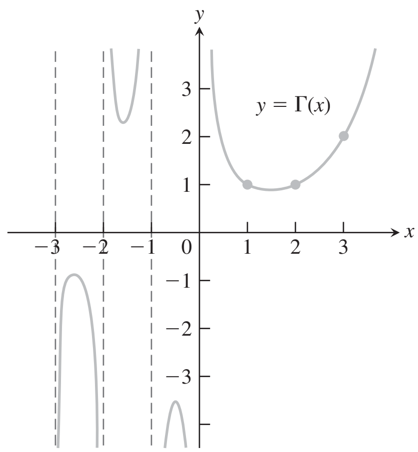


FIGURE 5.32 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x + 1))/x$, which is the subject of Exercise 51.

64. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x\sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so, for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}\,(1 + \epsilon(x)), \qquad \epsilon(x)\to 0 \text{ as } x\to\infty. \tag{6}$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}} \qquad \textbf{(Stirling's formula)}. \tag{7}$$

a. Stirling's approximation for *n*! Use Equation (7) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2n\pi} \qquad \textbf{(Stirling's approximation)}. \tag{8}$$

Equation (8) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \tag{9}$$

T **b.** Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots$, as far as your calculator can go.

6

First-Order Differential Equations and Area Under Curve

OVERVIEW Many real-world problems, when formulated mathematically, lead to differential equations. We encountered a number of these equations in previous chapters when studying phenomena such as the motion of an object moving along a straight line, the decay of a radioactive material, the growth of a population, and the cooling of a heated object placed within a medium of lower temperature.

We earlier introduced differential equations of the form $dy/dx = f(x)$, where f is given and y is an unknown function of x. When f is continuous over some interval, we learned that the general solution $y(x)$ was found directly by integration, $y = \int f(x)\,dx$. In this chapter we further extend our study to include other commonly occurring *first-order* differential equations. They involve only first derivatives of the unknown function $y(x)$, and model phenomena such as simple electrical circuits, or the resulting concentration of a chemical being added and mixed with some other fluid in a container.

6.1 Solutions, Slope Fields, and Euler's Method

We begin this section by defining general differential equations involving first derivatives. We then look at slope fields, which give a geometric picture of the solutions to such equations. Many differential equations cannot be solved by obtaining an explicit formula for the solution. However, we can often find numerical approximations to solutions. We present one such method here, called *Euler's method*, which is the basis for many other numerical methods as well.

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

We need to show that the differential equation is satisfied when we substitute into it the expressions $(C/x) + 2$ for y, and $-C/x^2$ for dy/dx. That is, we need to verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right-hand side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left side of the equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right side of the equation:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 6.1. ■

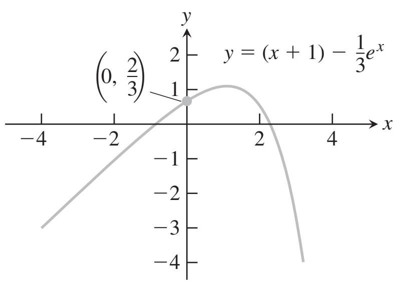


FIGURE 6.1 Graph of the solution to the initial value problem in Example 2.

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 6.2a shows a slope field, with a particular solution sketched into it in Figure 6.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

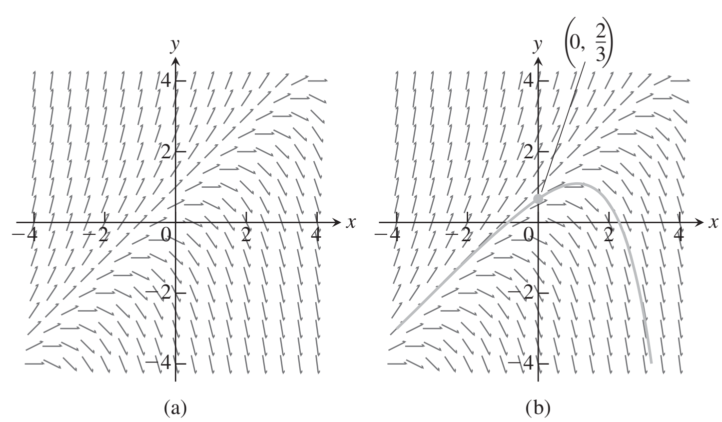


FIGURE 6.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

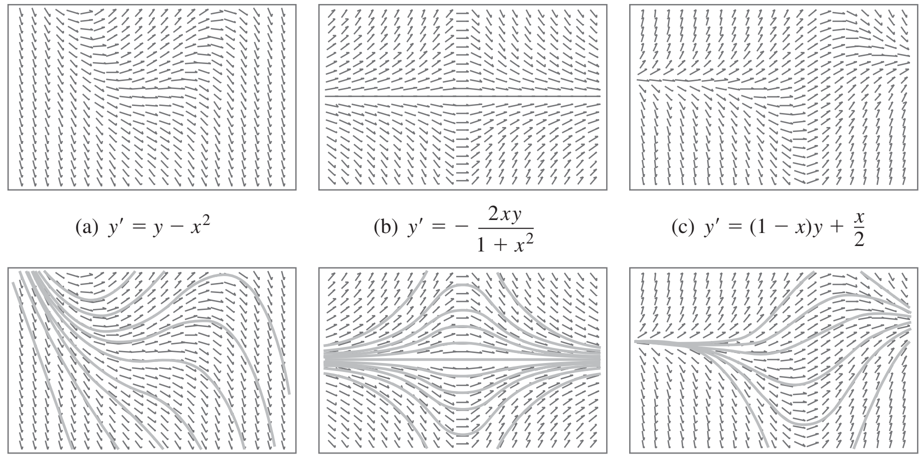


FIGURE 6.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here, but they should be considered as just tangent line segments.

Figure 6.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields. Slope fields are useful because they display the overall behavior of the family of solution curves for a given differential equation. For instance, the slope field in Figure 6.3b reveals that every solution $y(x)$ to the differential equation specified in the figure satisfies $\lim_{x\to\pm\infty} y(x) = 0$. We will see that knowing the overall behavior of the solution curves is often critical to understanding and predicting outcomes in a real-world system modeled by a differential equation.

Exponential Change and Separable Differential Equations

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in many natural and human-made situations. The variety of models based on these functions partly accounts for their importance. We now investigate the basic proportionality assumption that leads to such *exponential change*.

Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential change**.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

$$\text{Differential equation:} \qquad \frac{dy}{dt} = ky \tag{1a}$$

$$\text{Initial condition:} \qquad y = y_0 \quad \text{when} \quad t = 0. \tag{1b}$$

If y is positive and increasing, then k is positive, and we use Equation (1a) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1a) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Equation (1a) if $y_0 = 0$. To find the nonzero solutions, we divide Equation (1a) by y:

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dt} &= k && y \neq 0 \\
\int \frac{1}{y}\frac{dy}{dt}dt &= \int k\, dt && \text{Integrate with respect to } t; \ \textstyle\int (1/u)\, du = \ln|u| + C. \\
\ln|y| &= kt + C \\
|y| &= e^{kt+C} && \text{Exponentiate.} \\
|y| &= e^{C} \cdot e^{kt} && e^{a+b} = e^{a} \cdot e^{b} \\
y &= \pm e^{C} e^{kt} && \text{If } |y| = r, \text{ then } y = \pm r. \\
y &= Ae^{kt}. && A \text{ is a shorter name for } \pm e^{C}.
\end{aligned}$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}. \tag{2}$$

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$. The number k is called the **rate constant** of the change. (See Figure 6.4.)

The derivation of Equation (2) shows also that the only functions that are their own derivatives (so $k = 1$) are constant multiples of the exponential function.

Before presenting several examples of exponential change, let's consider the process we used to derive it.

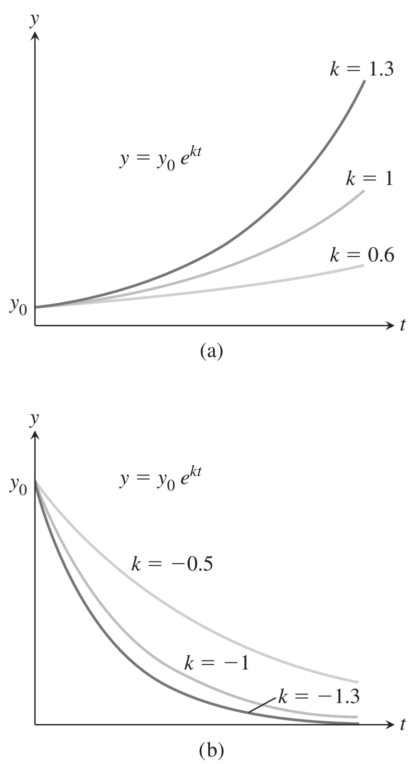


FIGURE 6.4 Graphs of (a) exponential growth and (b) exponential decay. As $|k|$ increases, the growth ($k > 0$) or decay ($k < 0$) intensifies.

Separable Differential Equations

Exponential change is modeled by a differential equation of the form $dy/dx = ky$ for some nonzero constant k. More generally, suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

where f is a function of *both* the independent and dependent variables. A **solution** of the equation is a differentiable function $y = y(x)$ defined on an interval of x-values (perhaps infinite) such that

$$\frac{d}{dx} y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into the differential equation, the resulting equation is true for all x in the solution interval. The **general**

solution is a solution $y(x)$ that contains all possible solutions and it always contains an arbitrary constant.

Equation (3) is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y). \qquad \text{g is a function of x; H is a function of y.}$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad H(y) = \frac{1}{h(y)}$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{4}$$

After completing the integrations, we obtain the solution y defined implicitly as a function of x.

The justification that we can simply integrate both sides in Equation (4) is based on the Substitution Rule (Section 5.5):

$$\begin{aligned}\int h(y)\,dy &= \int h(y(x))\frac{dy}{dx}dx \\ &= \int h(y(x))\frac{g(x)}{h(y(x))}dx && \frac{dy}{dx} = \frac{g(x)}{h(y)} \\ &= \int g(x)\,dx.\end{aligned}$$

EXAMPLE 3 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\begin{aligned}\frac{dy}{dx} &= (1 + y)e^x \\ dy &= (1 + y)e^x\,dx && \text{Treat dy/dx as a quotient of differentials and multiply both sides by dx.} \\ \frac{dy}{1 + y} &= e^x\,dx && \text{Divide by $(1 + y)$.} \\ \int \frac{dy}{1 + y} &= \int e^x\,dx && \text{Integrate both sides.} \\ \ln(1 + y) &= e^x + C && \text{C represents the combined constants of integration.}\end{aligned}$$

The last equation gives y as an implicit function of x. ■

EXAMPLE 4 Solve the equation $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

Solution We change to differential form, separate the variables, and integrate:

$$y(x+1)\,dy = x(y^2+1)\,dx$$

$$\frac{y\,dy}{y^2+1} = \frac{x\,dx}{x+1} \qquad x \neq -1$$

$$\int \frac{y\,dy}{1+y^2} = \int \left(1 - \frac{1}{x+1}\right)dx \qquad \text{Divide } x \text{ by } x+1.$$

$$\frac{1}{2}\ln(1+y^2) = x - \ln|x+1| + C.$$

The last equation gives the solution y as an implicit function of x. ■

The initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0e^{kt}$ expresses exponential change. We now present several examples of such change.

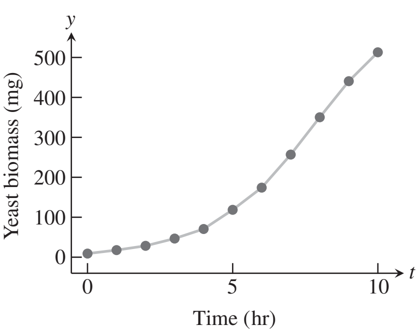


FIGURE 6.5 Graph of the growth of a yeast population over a 10-hour period, based on the data in Example 5.

Time (hr)	Yeast biomass (mg)
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441.0
10	513.3

Unlimited Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, animals, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, when the number of individuals becomes large enough, the population can be approximated by a continuous function. Differentiability of the approximating function is another reasonable hypothesis in many settings, allowing for the use of calculus to model and predict population sizes.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. Let's assume, too, that the death rate of the population is stable and proportional to $y(t)$. If, further, we neglect departures and arrivals, the growth rate dy/dt is the birth rate minus the death rate, which is the difference of the two proportionalities under our assumptions. In other words, $dy/dt = ky$ so that $y = y_0e^{kt}$, where y_0 is the size of the population at time $t = 0$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here. (We treat one model imposing such limitations in Section 9.4.) When k is positive, the proportionality $dy/dt = ky$ models *unlimited population growth.* (See Figure 6.5.)

EXAMPLE 5 The biomass of a yeast culture in an experiment is initially 29 grams. After 30 minutes the mass is 37 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will its take for the mass to double from its initial value?

Solution Let $y(t)$ be the yeast biomass after t minutes. We use the exponential growth model $dy/dt = ky$ for unlimited population growth, with solution $y = y_0e^{kt}$.

We have $y_0 = y(0) = 29$. We are also told that

$$y(30) = 29e^{k(30)} = 37.$$

Solving this equation for k, we find

$$\begin{aligned}
e^{k(30)} &= \frac{37}{29} \\
30k &= \ln\left(\frac{37}{29}\right) \\
k &= \frac{1}{30}\ln\left(\frac{37}{29}\right) \approx 0.008118.
\end{aligned}$$

Then the mass of the yeast in grams after t minutes is given by the equation

$$y = 29e^{(0.008118)t}.$$

To solve the problem we find the time t for which $y(t) = 58$, which is twice the initial amount.

$$\begin{aligned}
29e^{(0.008118)t} &= 58 \\
(0.008118)t &= \ln\left(\frac{58}{29}\right) \\
t &= \frac{\ln 2}{0.008118} \approx 85.38
\end{aligned}$$

It takes about 85 minutes for the yeast population to double. ■

In the next example we model the number of people within a given population who are infected by a disease which is being eradicated from the population. Here the constant of proportionality k is negative, and the model describes an exponentially decaying number of infected individuals.

EXAMPLE 6 One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The number of people cured is proportional to the number y that are infected with the disease. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find: the value of y_0, the value of k, and the time t when $y = 1000$.

The value of y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000e^{kt}. \tag{5}$$

The value of k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned}
8000 &= 10{,}000e^{k(1)} && \text{Eq. (5) with } t = 1 \text{ and } y = 8000 \\
e^k &= 0.8 \\
\ln(e^k) &= \ln 0.8 && \text{Logs of both sides} \\
k &= \ln 0.8 < 0. && \ln 0.8 \approx -0.223
\end{aligned}$$

At any given time t,

$$y = 10{,}000e^{(\ln 0.8)t}. \tag{6}$$

The value of t that makes $y = 1000$. We set y equal to 1000 in Equation (6) and solve for t:

$$1000 = 10{,}000e^{(\ln 0.8)t}$$

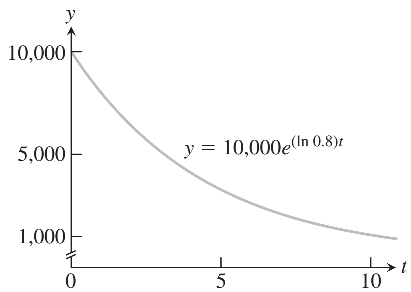


FIGURE 6.6 A graph of the number of people infected by a disease exhibits exponential decay (Example 6).

$$
\begin{aligned}
e^{(\ln 0.8)t} &= 0.1 \\
(\ln 0.8)t &= \ln 0.1 \qquad \text{Logs of both sides} \\
t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years}.
\end{aligned}
$$

It will take a little more than 10 years to reduce the number of cases to 1000. (See Figure 6.6.) ■

Radioactivity

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. It is conventional to use $-k$, with $k > 0$, to emphasize that y is decreasing. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$.

$$y = y_0 e^{-kt}, \qquad k > 0.$$

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$
\begin{aligned}
y_0 e^{-kt} &= \frac{1}{2} y_0 \\
e^{-kt} &= \frac{1}{2} \\
-kt &= \ln \frac{1}{2} = -\ln 2 \qquad \text{Reciprocal Rule for logarithms} \\
t &= \frac{\ln 2}{k}.
\end{aligned}
$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not have any effect.

$$\boxed{\text{Half-life} = \frac{\ln 2}{k}} \qquad (7)$$

The effective radioactive lifetime of polonium-210 is so short that we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3}\, t}.$$

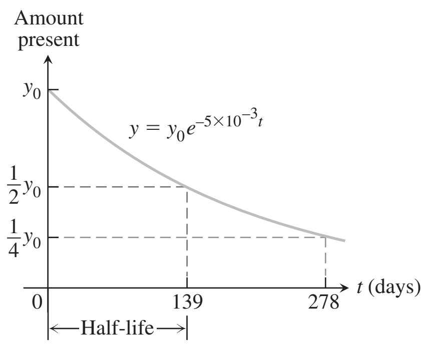


FIGURE 6.7 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present.

The element's half-life is

$$\begin{aligned} \text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (7)} \\ &= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\ &\approx 139 \text{ days.} \end{aligned}$$

This means that after 139 days, $1/2$ of y_0 radioactive atoms remain; after another 139 days (or 278 days altogether) half of those remain, or $1/4$ of y_0 radioactive atoms remain, and so on (see Figure 6.7).

EXAMPLE 7 The decay of radioactive elements can sometimes be used to date events from Earth's past. In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's atmosphere at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays.

Carbon-14 dating uses the half-life of 5730 years.

Scientists who do carbon-14 dating often use a figure of 5730 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find: the value of k and the value of t when y is $0.9y_0$ (90% of the radioactive nuclei are still present). That is, find t when $y_0 e^{-kt} = 0.9y_0$, or $e^{-kt} = 0.9$.

The value of k. We use the half-life Equation (7):

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5730} \qquad (\text{about } 1.2 \times 10^{-4}).$$

The value of t that makes $e^{-kt} = 0.9$.

$$\begin{aligned} e^{-kt} &= 0.9 \\ e^{-(\ln 2/5730)t} &= 0.9 \\ -\frac{\ln 2}{5730}t &= \ln 0.9 && \text{Logs of both sides} \\ t &= -\frac{5730 \ln 0.9}{\ln 2} \approx 871 \text{ years} \end{aligned}$$

The sample is about 871 years old. ■

Heat Transfer: Newton's Law of Cooling

Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver bar immersed in a large tub of water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's Law of Cooling*, although it applies to warming as well.

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

If we substitute y for $(H - H_S)$, then

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S) \\ &= \frac{dH}{dt} - 0 && H_S \text{ is a constant.} \\ &= \frac{dH}{dt} \\ &= -k(H - H_S) && \text{Eq. (8)} \\ &= -ky. && H - H_S = y\end{aligned}$$

Now we know that the solution of $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_S)$ for y, this says that

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$. This equation is the solution to Newton's Law of Cooling.

EXAMPLE 8 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_S = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $H = 38$ when $t = 5$:

$$\begin{aligned}38 &= 18 + 80e^{-5k} \\ e^{-5k} &= \frac{1}{4} \\ -5k &= \ln\frac{1}{4} = -\ln 4 \\ k &= \frac{1}{5}\ln 4 = 0.2\ln 4 \qquad \text{(about 0.28).}\end{aligned}$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $H = 20$:

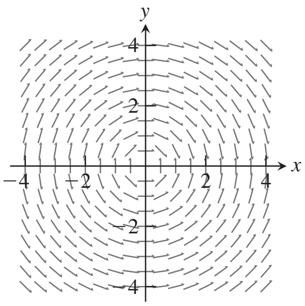
$$\begin{aligned}20 &= 18 + 80e^{-(0.2\ln 4)t} \\ 80e^{-(0.2\ln 4)t} &= 2 \\ e^{-(0.2\ln 4)t} &= \frac{1}{40} \\ -(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\ t &= \frac{\ln 40}{0.2\ln 4} \approx 13 \text{ min.}\end{aligned}$$

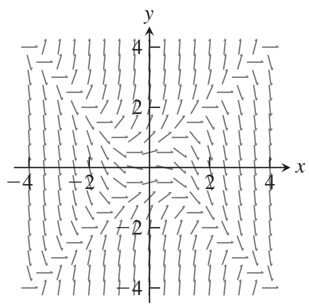
The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ■

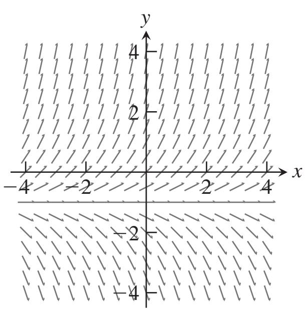
Exercises 6.1

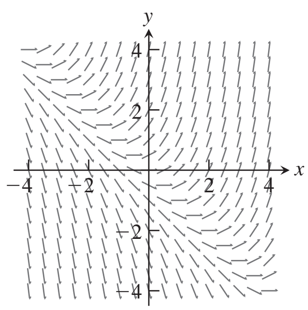
Slope Fields

In Exercises 1–4, match the differential equations with their slope fields, graphed here.


(a)


(b)


(c)


(d)

1. $y' = x + y$

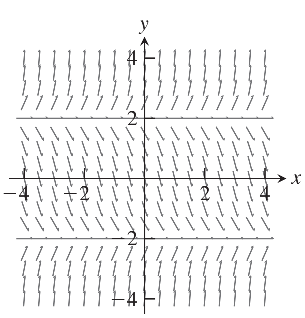
2. $y' = y + 1$

3. $y' = -\dfrac{x}{y}$

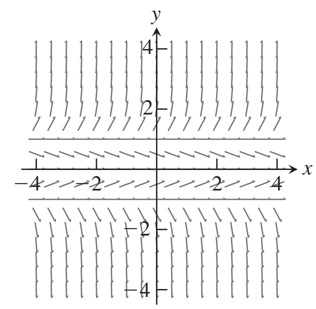
4. $y' = y^2 - x^2$

In Exercises 5 and 6, copy the slope fields and sketch in some of the solution curves.

5. $y' = (y + 2)(y - 2)$



6. $y' = y(y + 1)(y - 1)$



Integral Equations

In Exercises 7–10, write an equivalent first-order differential equation and initial condition for y.

7. $y = -1 + \displaystyle\int_1^x (t - y(t))\,dt$

8. $y = \displaystyle\int_1^x \frac{1}{t}\,dt$

9. $y = 2 - \displaystyle\int_0^x (1 + y(t)) \sin t\,dt$

10. $y = 1 + \displaystyle\int_0^x y(t)\,dt$

11. Show that the solution of the initial value problem

$$y' = x + y, \qquad y(x_0) = y_0$$

is

$$y = -1 - x + (1 + x_0 + y_0)\,e^{x - x_0}.$$

12. What integral equation is equivalent to the initial value problem $y' = f(x)$, $y(x_0) = y_0$?

13. $y' = x + y$, $y(0) = -7/10$; $-4 \le x \le 4$, $-4 \le y \le 4$; $b = 1$

14. $y' = -x/y$, $y(0) = 2$; $-3 \le x \le 3$, $-3 \le y \le 3$; $b = 2$

15. $y' = y(2 - y)$, $y(0) = 1/2$; $0 \le x \le 4$, $0 \le y \le 3$; $b = 3$

16. $y' = (\sin x)(\sin y)$, $y(0) = 2$; $-6 \le x \le 6$, $-6 \le y \le 6$; $b = 3\pi/2$

Verifying Solutions

In Exercises 17–20, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

17. $2y' + 3y = e^{-x}$

a. $y = e^{-x}$ **b.** $y = e^{-x} + e^{-(3/2)x}$

c. $y = e^{-x} + Ce^{-(3/2)x}$

18. $y' = y^2$

a. $y = -\dfrac{1}{x}$ **b.** $y = -\dfrac{1}{x + 3}$ **c.** $y = -\dfrac{1}{x + C}$

19. $y = \dfrac{1}{x}\displaystyle\int_1^x \dfrac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

20. $y = \dfrac{1}{\sqrt{1+x^4}}\displaystyle\int_1^x \sqrt{1+t^4}\,dt, \quad y' + \dfrac{2x^3}{1+x^4}y = 1$

Initial Value Problems

In Exercises 21–24, show that each function is a solution of the given initial value problem.

	Differential equation	Initial equation	Solution candidate
21.	$y' + y = \dfrac{2}{1+4e^{2x}}$	$y(-\ln 2) = \dfrac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
22.	$y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
23.	$xy' + y = -\sin x,$ $x > 0$	$y\left(\dfrac{\pi}{2}\right) = 0$	$y = \dfrac{\cos x}{x}$
24.	$x^2y' = xy - y^2,$ $x > 1$	$y(e) = e$	$y = \dfrac{x}{\ln x}$

Separable Differential Equations

Solve the differential equations in Exercises 25–38.

25. $2\sqrt{xy}\dfrac{dy}{dx} = 1, \quad x, y > 0$

26. $\dfrac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

27. $\dfrac{dy}{dx} = e^{x-y}$

28. $\dfrac{dy}{dx} = 3x^2\,e^{-y}$

29. $\dfrac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$

30. $\sqrt{2xy}\dfrac{dy}{dx} = 1$

31. $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

32. $(\sec x)\dfrac{dy}{dx} = e^{y+\sin x}$

33. $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

34. $\dfrac{dy}{dx} = \dfrac{e^{2x-y}}{e^{x+y}}$

35. $y^2\dfrac{dy}{dx} = 3x^2y^3 - 6x^2$

36. $\dfrac{dy}{dx} = xy + 3x - 2y - 6$

37. $\dfrac{1}{x}\dfrac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\,e^{x^2}$

38. $\dfrac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

Applications and Examples

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

39. Human evolution continues The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago, as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

a. If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0e^{kt}$. Then use this value of k to answer the following questions.

b. In about how many years will human teeth be 90% of their present size?

c. What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

40. Atmospheric pressure The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

a. Solve the initial value problem

Differential equation: $dp/dh = kp \quad (k \text{ a constant})$

Initial condition: $p = p_0 \quad \text{when} \quad h = 0$

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

b. What is the atmospheric pressure at $h = 50$ km?

c. At what altitude does the pressure equal 900 millibars?

41. First-order chemical reactions In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

42. The inversion of sugar The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

43. Working underwater The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

44. Voltage in a discharging capacitor Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

45. Cholera bacteria Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

46. Growth of bacteria A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

47. The incidence of a disease (*Continuation of Example 4.*) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

a. How long will it take to reduce the number of cases to 1000?

b. How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

48. Drug concentration An antibiotic is administered intravenously into the bloodstream at a constant rate r. As the drug flows through the patient's system and acts on the infection that is present, it is removed from the bloodstream at a rate proportional to the amount in the bloodstream at that time. Since the amount of blood in the patient is constant, this means that the concentration $y = y(t)$ of the antibiotic in the bloodstream can be modeled by the differential equation

$$\frac{dy}{dt} = r - ky, \quad k > 0 \text{ and constant.}$$

a. If $y(0) = y_0$, find the concentration $y(t)$ at any time t.

b. Assume that $y_0 < (r/k)$ and find $\lim_{y\to\infty} y(t)$. Sketch the solution curve for the concentration.

49. Endangered species Biologists consider a species of animal or plant to be endangered if it is expected to become extinct within 20 years. If a certain species of wildlife is counted to have 1147 members at the present time, and the population has been steadily declining exponentially at an annual rate averaging 39% over the past 7 years, do you think the species is endangered? Explain your answer.

50. The U.S. population The U.S. Census Bureau keeps a running clock totaling the U.S. population. On September 20, 2012, the total was increasing at the rate of 1 person every 12 sec. The population figure for 8:11 P.M. EST on that day was 314,419,198.

a. Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).

b. At this rate, what will the U.S. population be at 8:11 P.M. EST on September 20, 2019?

51. Oil depletion Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

52. Continuous price discounting To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

a. Find $p(x)$ by solving the following initial value problem:

Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$

Initial condition: $p(100) = 20.09.$

b. Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.

c. The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.

d. Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

53. Plutonium-239 The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

54. Polonium-210 The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

55. The mean life of a radioactive nucleus Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

56. Californium-252 What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the Western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per minute.

a. What is the value of k in the decay equation for this isotope?

b. What is the isotope's mean life? (See Exercise 39.)

c. How long will it take 95% of a sample's radioactive nuclei to disintegrate?

57. Cooling soup Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

a. How much longer would it take the soup to cool to 35°C?

b. Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

58. A beam of unknown temperature An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min it was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

59. Surrounding medium of unknown temperature A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

60. Silver cooling in air The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

a. 15 min from now?

b. 2 hours from now?

c. When will the silver be 10°C above room temperature?

61. The age of Crater Lake The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

62. The sensitivity of carbon-14 dating to measurement To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

a. A bone fragment found in central Illinois in the year 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

b. Repeat part (a), assuming 18% instead of 17%.

c. Repeat part (a), assuming 16% instead of 17%.

63. Carbon-14 The oldest known frozen human mummy, discovered in the Schnalstal glacier of the Italian Alps in 1991 and called *Otzi*, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

64. Art forgery A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

65. Lascaux Cave paintings Prehistoric cave paintings of animals were found in the Lascaux Cave in France in 1940. Scientific analysis revealed that only 15% of the original carbon-14 in the paintings remained. What is an estimate of the age of the paintings?

66. Incan mummy The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995.

a. How much of the original carbon-14 was present if the estimated age of the "Ice Maiden" was 500 years?

b. If a 1% error can occur in the carbon-14 measurement, what is the oldest possible age for the Ice Maiden?

6.2 First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 7.2) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y, e^y, \text{ or } \sqrt{dy/dx}\right)$.

EXAMPLE 1 Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by positive } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y).$$

$$v(x) \cdot y = \int v(x)Q(x)\,dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \tag{2}$$

Equation (2) expresses the solution of Equation (1) in terms of the functions $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (1) because its presence makes the equation integrable.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Derivative Product Rule}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated, } v > 0$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0, \text{ we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{3}$$

Thus a formula for the general solution to Equation (1) is given by Equation (2), where $v(x)$ is given by Equation (3). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified. Any antiderivative of P works for Equation (3).

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

When you integrate the product on the left-hand side in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

HISTORICAL BIOGRAPHY
Adrien Marie Legendre
(1752–1833)

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} \\ &= e^{-3\ln|x|} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left-hand side is } \frac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int\frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

■

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left-hand side is } vy.$$

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4. \qquad \blacksquare$$

In solving the linear equation in Example 2, we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example 3, by remembering that the left-hand side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation (2) this means that

$$v(x)y = \int v(x)Q(x)\,dx. \tag{4}$$

We need only integrate the product of the integrating factor $v(x)$ with $Q(x)$ on the right-hand side of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example 2.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable and can be solved by the method of Section 6.1:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) = 0$$

$$\frac{dy}{y} = -P(x)\,dx \qquad \text{Separating the variables} \qquad \blacksquare$$

Exercises 6.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

1. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

2. $e^x\dfrac{dy}{dx} + 2e^x y = 1$

3. $xy' + 3y = \dfrac{\sin x}{x^2}, \quad x > 0$

4. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

5. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

6. $(1 + x)\, y' + y = \sqrt{x}$

7. $2y' = e^{x/2} + y$

8. $e^{2x} y' + 2e^{2x} y = 2x$

9. $xy' - y = 2x \ln x$

10. $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y, \quad x > 0$

11. $(t - 1)^3\dfrac{ds}{dt} + 4(t - 1)^2 s = t + 1, \quad t > 1$

12. $(t + 1)\dfrac{ds}{dt} + 2s = 3(t + 1) + \dfrac{1}{(t + 1)^2}, \quad t > -1$

13. $\sin\theta\dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

14. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

17. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

18. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3 \sec\theta \tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

19. $(x + 1)\dfrac{dy}{dx} - 2(x^2 + x)y = \dfrac{e^{x^2}}{x + 1}, \quad x > -1, \quad y(0) = 5$

20. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

21. Solve the exponential growth/decay initial value problem for y as a function of t by thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

22. Solve the following initial value problem for u as a function of t:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

a. as a first-order linear equation.

b. as a separable equation.

Theory and Examples

23. Is either of the following equations correct? Give reasons for your answers.

a. $x\displaystyle\int \frac{1}{x}\,dx = x \ln|x| + C$

b. $x\displaystyle\int \frac{1}{x}\,dx = x \ln|x| + Cx$

HISTORICAL BIOGRAPHY
James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x} y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\, dy/dx$. Then $dy/dx = -y^2\, du/dx = -u^{-2}\, du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x} u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the Bernoulli equations in Exercises 24–27.

24. $y' - y = -y^2$

25. $y' - y = xy^2$

26. $xy' + y = y^{-2}$

27. $x^2y' + 2xy = y^3$

6.3 Applications

We now look at four applications of first-order differential equations. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth. The third application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles). The final application analyzes chemical concentrations entering and leaving a container. The various models involve separable or linear first-order equations.

Motion with Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. Picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 6.1)

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that an object is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k}e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k}e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t}). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: If m is large, the body will coast a long way.

In the English system, in which weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

EXAMPLE 1 For a 192-lb ice skater, the k in Equation (1) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Equation (1) for t:

$$\begin{aligned} 11e^{-t/18} &= 1 && \text{Eq. (1) with } k = 1/3, \\ e^{-t/18} &= 1/11 && m = 6, v_0 = 11, v = 1 \\ -t/18 &= \ln(1/11) = -\ln 11 \\ t &= 18 \ln 11 \approx 43 \text{ sec.}\end{aligned}$$

We answer the second question with Equation (3):

$$\begin{aligned}\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3} \\ &= 198 \text{ ft.}\end{aligned}$$

■

Inaccuracy of the Exponential Population Growth Model

In Section 6.1 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 6.1 we found the solution $P = P_0 e^{kt}$ to this model.

To assess the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

is constant. This rate is called the **relative growth rate**. Now, Table 6.1 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by the initial value problem,

$$\frac{dP}{dt} = 0.017P, \qquad P(0) = 4454.$$

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 2008 (so $t = 28$), the solution predicts the world population in midyear to be about 7169 million, or 7.2 billion (Figure 6.8), which is more than the actual population of

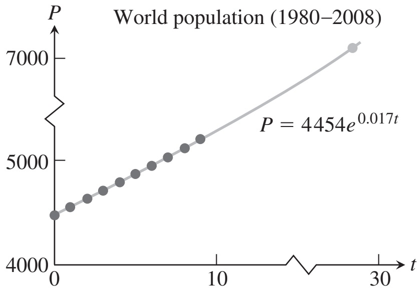


FIGURE 6.8 Notice that the value of the solution $P = 4454e^{0.017t}$ is 7169 when $t = 28$, which is nearly 7% more than the actual population in 2008.

TABLE 6.1 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 2007): www.census .gov/ipc/www/idb.

6707 million from the U.S. Bureau of the Census. A more realistic model would consider environmental and other factors affecting the growth rate, which has been steadily declining to about 0.012 since 1987. We consider one such model in Section 9.4.

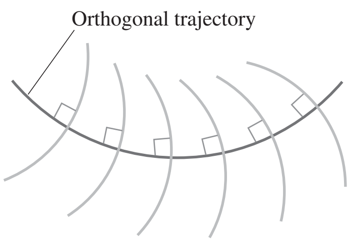


FIGURE 6.9 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 6.9). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 6.10). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to strength of an electric field and those in the other family correspond to constant electric potential. They also occur in hydrodynamics and heat-flow problems.

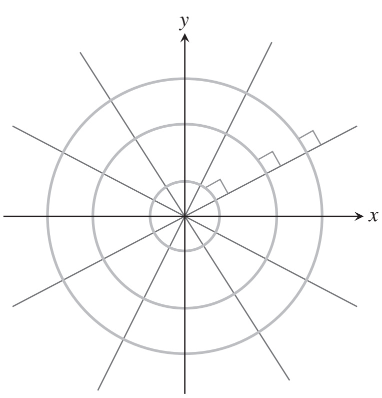


FIGURE 6.10 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 2 Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas having the coordinate axes as asymptotes. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

This differential equation is separable and we solve it as in Section 6.1:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

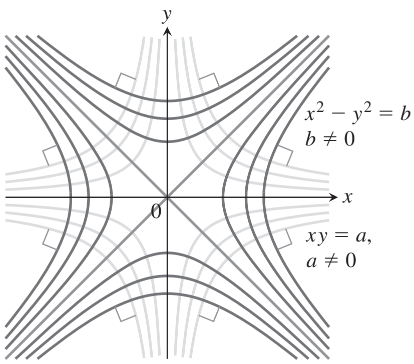


FIGURE 6.11 Each curve is orthogonal to every curve it meets in the other family (Example 2).

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{5}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (5) and sketched in Figure 6.11. ■

Mixture Problems

Suppose a chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix}. \tag{6}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate})\\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate}).\end{aligned} \tag{7}$$

Accordingly, Equation (6) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{8}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Equation (8) are

$$\frac{\text{pounds}}{\text{minutes}} = \frac{\text{pounds}}{\text{minutes}} - \frac{\text{pounds}}{\text{gallons}} \cdot \frac{\text{gallons}}{\text{minutes}}.$$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 6.12)?

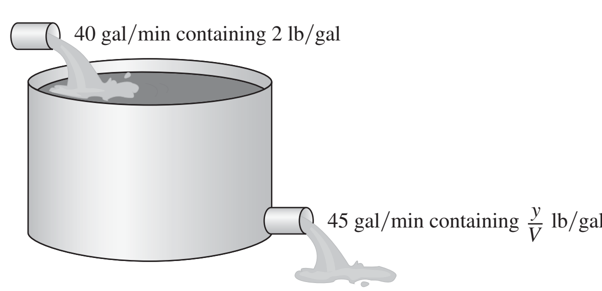


FIGURE 6.12 The storage tank in Example 3 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (7)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min and } V = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t} \qquad \text{Eq. (8)}$$

in pounds per minute.

To solve this differential equation, we first write it in standard linear form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t} y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) = e^{\int P\,dt} &= e^{\int \frac{45}{2000-5t} dt} \\ &= e^{-9 \ln (2000-5t)} && 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$\begin{aligned} (2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t} y\right) &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9} \frac{dy}{dt} + 45(2000 - 5t)^{-10} y &= 80(2000 - 5t)^{-9} \\ \frac{d}{dt}\left[(2000 - 5t)^{-9} y\right] &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9} y &= \int 80(2000 - 5t)^{-9}\, dt \\ (2000 - 5t)^{-9} y &= 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C. \end{aligned}$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^9.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$100 = 2(2000 - 0) + C(2000 - 0)^9$$
$$C = -\frac{3900}{(2000)^9}.$$

The particular solution of the initial value problem is

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive in the tank 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb.}$$ ■

Exercises 6.3

Motion Along a Line

1. Coasting bicycle A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.

a. About how far will the cyclist coast before reaching a complete stop?

b. How long will it take the cyclist's speed to drop to 1 m/sec?

2. Coasting battleship Suppose that an Iowa class battleship has mass around 51,000 metric tons (51,000,000 kg) and a k value in Equation (1) of about 59,000 kg/sec. Assume that the ship loses power when it is moving at a speed of 9 m/sec.

a. About how far will the ship coast before it is dead in the water?

b. About how long will it take the ship's speed to drop to 1 m/sec?

Orthogonal Trajectories

In Exercises 3–8, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

3. $y = mx$ **4.** $y = cx^2$

5. $kx^2 + y^2 = 1$ **6.** $2x^2 + y^2 = c^2$

7. $y = ce^{-x}$ **8.** $y = e^{kx}$

9. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.

10. Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.

a. $x\,dx + y\,dy = 0$ **b.** $x\,dy - 2y\,dx = 0$

Mixture Problems

11. Salt mixture A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

a. At what rate (pounds per minute) does salt enter the tank at time t?

b. What is the volume of brine in the tank at time t?

c. At what rate (pounds per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of salt in the tank 25 min after the process starts.

12. Mixture problem A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a. At what time will the tank be full?

b. At the time the tank is full, how many pounds of concentrate will it contain?

13. Fertilizer mixture A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

14. Carbon monoxide pollution An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

6.4 Area Under the Graph of a Nonnegative Function

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

For the first time we have a rigorous definition for the area of a region whose boundary is the graph of any continuous function. We now apply this to a simple example, the area under a straight line, where we can verify that our new definition agrees with our previous notion of area.

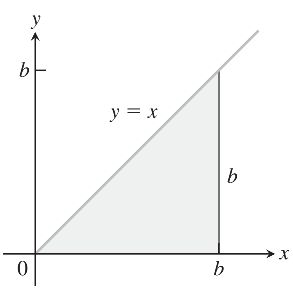


FIGURE 6.13 The region in Example 1 is a triangle.

EXAMPLE 1 Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Solution The region of interest is a triangle (Figure 6.13). We compute the area in two ways.

(a) To compute the definite integral as the limit of Riemann sums, we calculate $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$ for partitions whose norms go to zero. Theorem 1 tells us that it does not matter how we choose the partitions or the points c_k as long as the norms approach zero. All choices give the exact same limit. So we consider the partition P that subdivides the interval $[0, b]$ into n subintervals of equal width $\Delta x = (b - 0)/n = b/n$, and we choose c_k to be the right endpoint in each subinterval. The partition is $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \cdots, \frac{nb}{n}\right\}$ and $c_k = \frac{kb}{n}$. So

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\,\Delta x &= \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} && f(c_k) = c_k \\
&= \sum_{k=1}^{n} \frac{kb^2}{n^2} \\
&= \frac{b^2}{n^2} \sum_{k=1}^{n} k && \text{Constant Multiple Rule} \\
&= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} && \text{Sum of First } n \text{ Integers} \\
&= \frac{b^2}{2}\left(1 + \frac{1}{n}\right).
\end{aligned}$$

As $n \to \infty$ and $\|P\| \to 0$, this last expression on the right has the limit $b^2/2$. Therefore,

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

(b) Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base length b and height $y = b$. The area is $A = (1/2)\,b \cdot b = b^2/2$. Again we conclude that $\int_0^b x\,dx = b^2/2$. ■

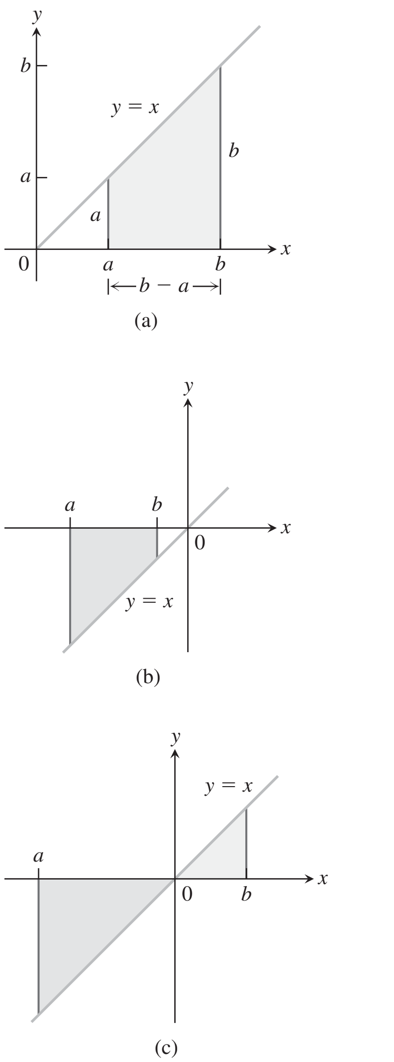


FIGURE 6.14 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

Example 1 can be generalized to integrate $f(x) = x$ over any closed interval $[a, b]$, $0 < a < b$.

$$\begin{aligned} \int_a^b x\,dx &= \int_a^0 x\,dx + \int_0^b x\,dx && \text{Rule 5} \\ &= -\int_0^a x\,dx + \int_0^b x\,dx && \text{Rule 1} \\ &= -\frac{a^2}{2} + \frac{b^2}{2}. && \text{Example 1} \end{aligned}$$

In conclusion, we have the following rule for integrating $f(x) = x$:

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \tag{2}$$

This computation gives the area of the trapezoid in Figure 6.14a. Equation (2) remains valid when a and b are negative, but the interpretation of the definite integral changes. When $a < b < 0$, the definite integral value $(b^2 - a^2)/2$ is a negative number, the negative of the area of a trapezoid dropping down to the line $y = x$ below the x-axis (Figure 6.14b). When $a < 0$ and $b > 0$, Equation (2) is still valid and the definite integral gives the difference between two areas, the area under the graph and above $[0, b]$ minus the area below $[a, 0]$ and over the graph (Figure 6.14c).

The following results can also be established using a Riemann sum calculation similar to that in Example 1 (Exercises 63 and 65).

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \tag{3}$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \tag{4}$$

Average Value of a Continuous Function Revisited

In Section 5.1 we introduced informally the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$. In integral notation we write this as

$$\text{Average} = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

We can use this formula to give a precise definition of the average value of any continuous (or integrable) function, whether positive, negative, or both.

Alternatively, we can use the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n. A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 6.15). The average of the n sampled values is

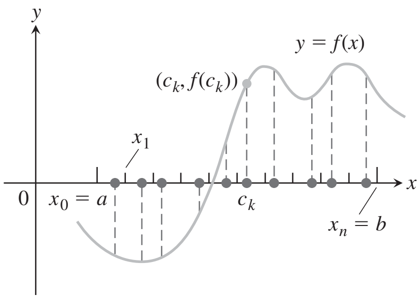


FIGURE 6.15 A sample of values of a function on an interval $[a, b]$.

$$\begin{aligned}\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n}\sum_{k=1}^{n} f(c_k) \\ &= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) && \Delta x = \frac{b-a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b-a} \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\,\Delta x. && \text{Constant Multiple Rule}\end{aligned}$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the size of the sample and let the norm of the partition approach zero, the average approaches $(1/(b-a))\int_a^b f(x)\,dx$. Both points of view lead us to the following definition.

> **DEFINITION** If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is
>
> $$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 2 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

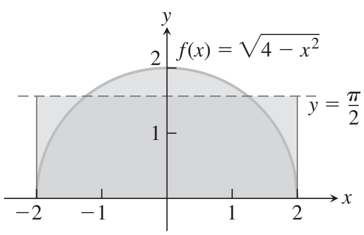


FIGURE 6.16 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 2). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 6.16).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2}\cdot \pi r^2 = \frac{1}{2}\cdot \pi(2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4 - x^2}\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 6.16). ■

Total Area

Area is always a nonnegative quantity. The Riemann sum contains terms such as $f(c_k)\,\Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k)\,\Delta x_k$ is the negative of the rectangle's area. When we add up such terms for a negative function, we get the negative of the area between the curve and the x-axis. If we then take the absolute value, we obtain the correct positive area.

EXAMPLE 3 Figure 6.17 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x-axis. For each function, compute.

(a) the definite integral over the interval $[-2, 2]$, and

(b) the area between the graph and the x-axis over $[-2, 2]$.

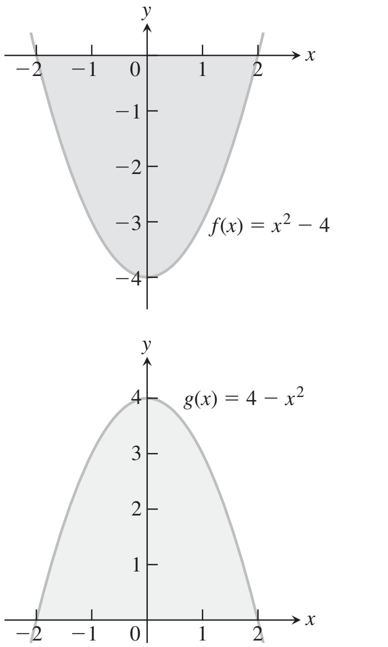


FIGURE 6.17 These graphs enclose the same amount of area with the x-axis, but the definite integrals of the two functions over $[-2, 2]$ differ in sign (Example 3).

Solution

(a) $\displaystyle\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$

and

$$\int_{-2}^{2} g(x)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$

(b) In both cases, the area between the curve and the x-axis over $[-2, 2]$ is $32/3$ square units. Although the definite integral of $f(x)$ is negative, the area is still positive. ■

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x-axis when the function takes on both positive and negative values, we must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign. Otherwise we might get cancelation between positive and negative signed areas, leading to an incorrect total. The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign. The term "area" will be taken to mean this *total area*.

EXAMPLE 4 Figure 6.18 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

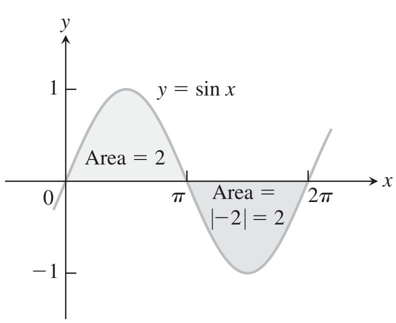


FIGURE 6.18 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 4).

Solution

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_{0}^{2\pi} \sin x\,dx = -\cos x\Big]_{0}^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

(b) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_{0}^{\pi} \sin x\,dx = -\cos x\Big]_{0}^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values,

$$\text{Area} = |2| + |-2| = 4.$$ ■

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

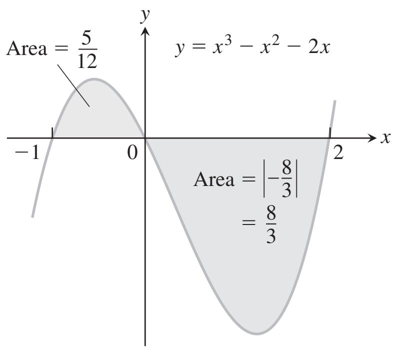


FIGURE 6.19 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 5).

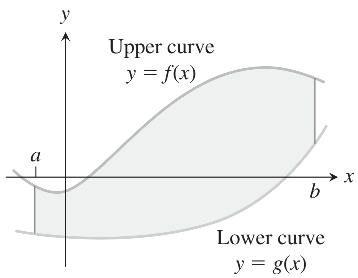


FIGURE 6.20 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

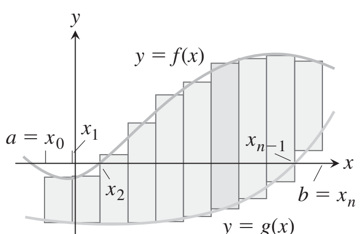


FIGURE 6.21 We approximate the region with rectangles perpendicular to the x-axis.

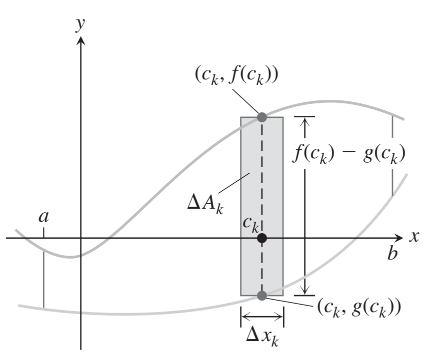


FIGURE 6.22 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

EXAMPLE 5 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x, -1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x\left(x^2 - x - 2\right) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 6.19). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$ ■

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 6.20). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area with an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ (Figure 6.21). The area of the kth rectangle (Figure 6.22) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\,\Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\|P\| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because f and g are continuous. We take the area of the region to be the value of this integral. That is,

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k = \int_a^b [f(x) - g(x)]\,dx.$$

DEFINITION If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

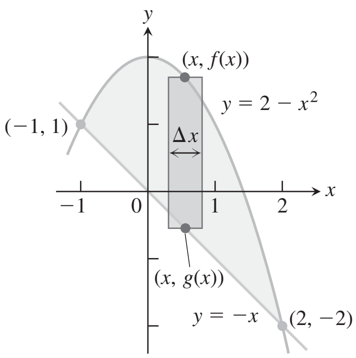


FIGURE 6.23 The region in Example 6 with a typical approximating rectangle.

When applying this definition it is helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

EXAMPLE 6 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 6.23). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \qquad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^2 [(2 - x^2) - (-x)]\,dx \\ &= \int_{-1}^2 (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}. \end{aligned}$$

■

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

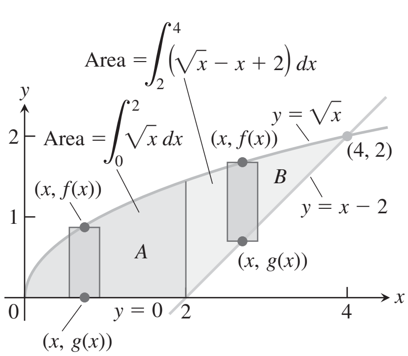


FIGURE 6.24 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 8.

EXAMPLE 7 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 6.24) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 6.24.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x - 2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x - 1)(x - 4) &= 0 && \text{Factor.} \\ x = 1, \qquad x &= 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\text{For } 0 \le x \le 2: \qquad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$$

$$\text{For } 2 \le x \le 4: \qquad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$$

We add the areas of subregions A and B to find the total area:

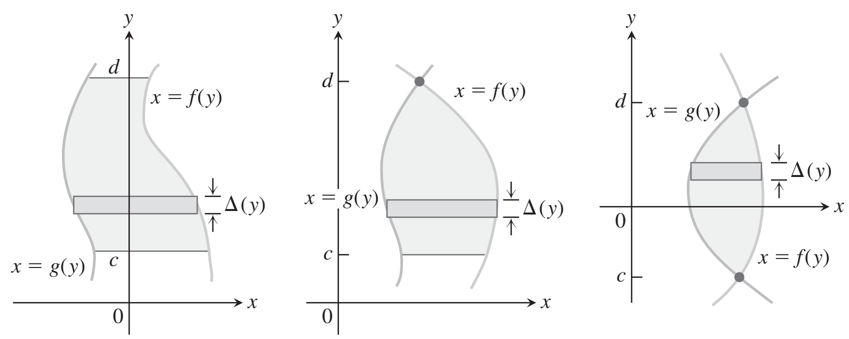
$$\begin{aligned}
\text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B} \\
&= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\
&= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\
&= \frac{2}{3}(8) - 2 = \frac{10}{3}.
\end{aligned}$$

■

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$A = \int_c^d [\,f(y) - g(y)\,]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

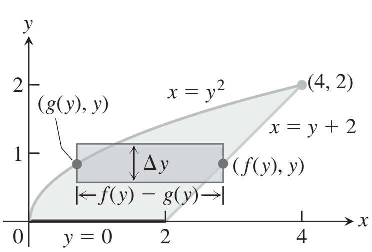


FIGURE 6.25 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 8).

EXAMPLE 8 Find the area of the region in Example 7 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 6.25). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned}
y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\
y^2 - y - 2 &= 0 && \text{Rewrite.} \\
(y + 1)(y - 2) &= 0 && \text{Factor.} \\
y = -1, \quad y &= 2 && \text{Solve.}
\end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$\begin{aligned} A = \int_c^d [f(y) - g(y)]\,dy &= \int_0^2 [y + 2 - y^2]\,dy \\ &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

This is the result of Example 8, found with less work. ■

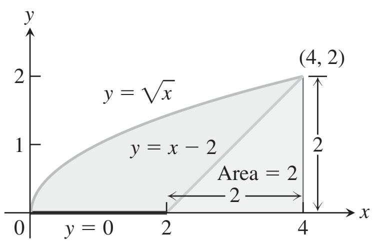


FIGURE 6.26 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 7 by integrating with respect to y rather than x (just as we did in Example 8), there is an easier way yet. Looking at Figure 6.26, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

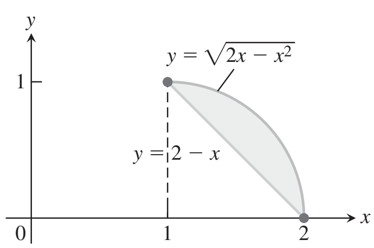


FIGURE 6.27 The region described by the curves in Example 9.

EXAMPLE 9 Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$.

Solution A sketch of the region is displayed in Figure 6.27, and we see that the line and curve intersect at the points (1, 1) and (2, 0). Using vertical rectangles, the area of the region is given by

$$A = \int_1^2 \left(\sqrt{2x - x^2} + x - 2\right) dx.$$

However, we don't know how to find an antiderivative for the term involving the radical, and no simple substitution is apparent.

To use horizontal rectangles, we first need to express each bounding curve as a function of the variable y. The line on the left is easily found to be $x = 2 - y$. For the curve $y = \sqrt{2x - x^2}$ on the right-hand side in Figure 6.27, we have

$$\begin{aligned} y^2 &= 2x - x^2 \\ &= -(x^2 - 2x + 1) + 1 && \text{Complete the square.} \\ &= -(x - 1)^2 + 1. \end{aligned}$$

Solving for x,

$$\begin{aligned} (x - 1)^2 &= 1 - y^2, \\ x &= 1 + \sqrt{1 - y^2}. && x \ge 1,\ 0 \le y \le 1 \end{aligned}$$

The area of the region is then given by

$$A = \int_0^1 \left[\left(1 + \sqrt{1 - y^2}\right) - (2 - y)\right] dy$$

$$= \int_0^1 \left(\sqrt{1 - y^2} + y - 1\right) dy.$$

Again, we don't know yet how to integrate the radical term (although we will see how to do that in Section 8.4). We conclude that neither vertical nor horizontal rectangles lead to an integral we currently can evaluate.

Nevertheless, as we found with Example 8, sometimes a little observation proves to be helpful. If we look again at the algebra for expressing the right-hand side curve $y = \sqrt{2x - x^2}$ as a function of y, we see that $(x - 1)^2 + y^2 = 1$, which is the equation of the unit circle with center shifted to the point (1, 0). From Figure 6.27, we can then see that the area of the region we want is the area of the upper right quarter of the unit circle minus the area of the triangle with vertices (1, 1), (1, 0), and (2, 0). That is, the area is given by

$$A = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4} \approx 0.285. \qquad \blacksquare$$

Exercises 6.4

Finding Area by Definite Integrals

In Exercises 1–4, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

1. $y = 3x^2$

2. $y = \pi x^2$

3. $y = 2x$

4. $y = \dfrac{x}{2} + 1$

Finding Average Value

In Exercises 5–12, graph the function and find its average value over the given interval.

5. $f(x) = x^2 - 1$ on $\left[0, \sqrt{3}\right]$

6. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

7. $f(x) = -3x^2 - 1$ on $[0, 1]$

8. $f(x) = 3x^2 - 3$ on $[0, 1]$

9. $f(t) = (t - 1)^2$ on $[0, 3]$

10. $f(t) = t^2 - t$ on $[-2, 1]$

11. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

12. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Area

In Exercises 13–16, find the total area between the region and the x-axis.

13. $y = -x^2 - 2x, \quad -3 \le x \le 2$

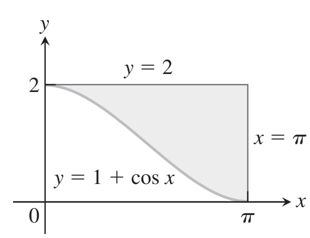
14. $y = 3x^2 - 3, \quad -2 \le x \le 2$

15. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

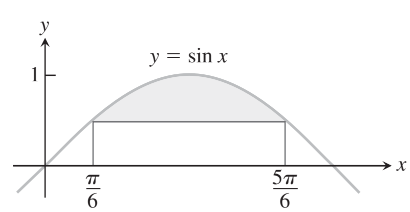
16. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 17–20.

17.



18.



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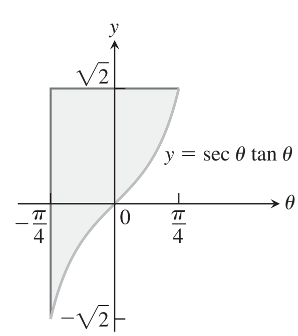

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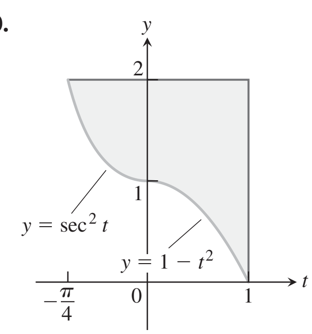

Area

Find the total areas of the shaded regions in Exercises 21–36.

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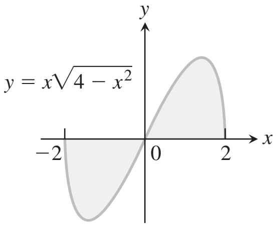

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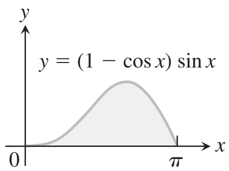

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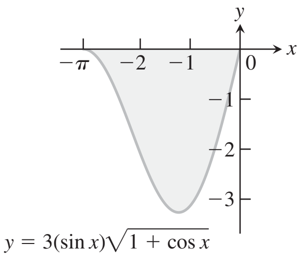

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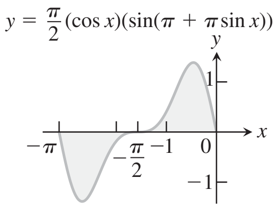

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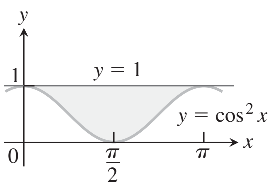

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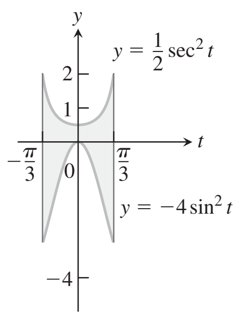

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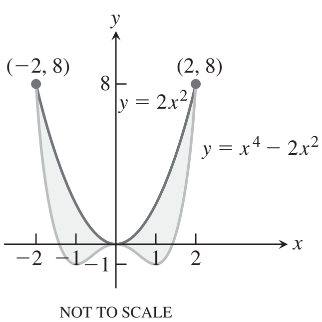

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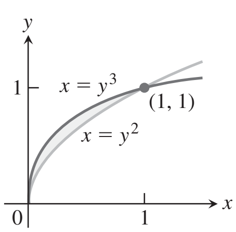

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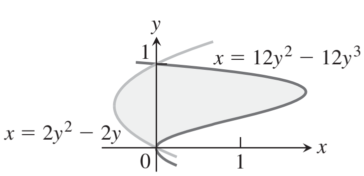

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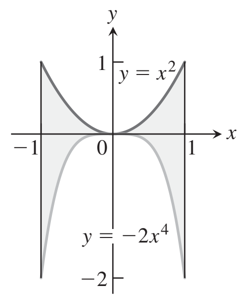

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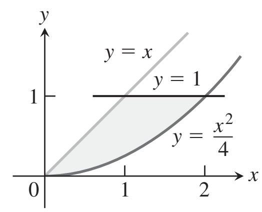

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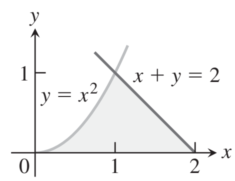

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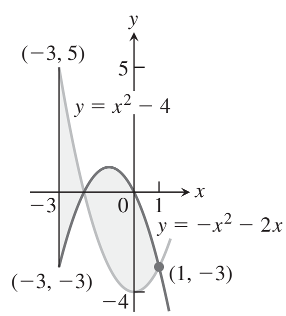

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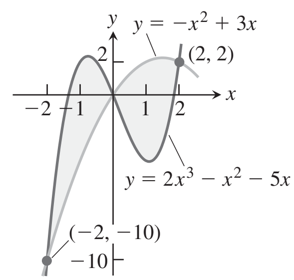

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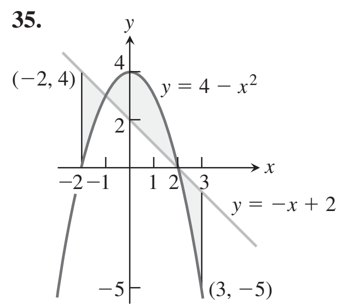

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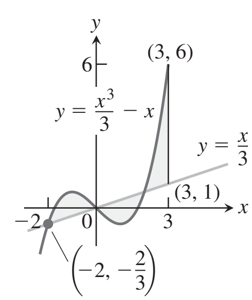


Find the areas of the regions enclosed by the lines and curves in Exercises 37–46.

37. $y = x^2 - 2$ and $y = 2$

38. $y = 2x - x^2$ and $y = -3$

39. $y = x^4$ and $y = 8x$

40. $y = x^2 - 2x$ and $y = x$

41. $y = x^2$ and $y = -x^2 + 4x$

42. $y = 7 - 2x^2$ and $y = x^2 + 4$

43. $y = x^4 - 4x^2 + 4$ and $y = x^2$

44. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

45. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

46. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 47–54.

47. $x = 2y^2$, $x = 0$, and $y = 3$

48. $x = y^2$ and $x = y + 2$

49. $y^2 - 4x = 4$ and $4x - y = 16$

50. $x - y^2 = 0$ and $x + 2y^2 = 3$

51. $x + y^2 = 0$ and $x + 3y^2 = 2$

52. $x - y^{2/3} = 0$ and $x + y^4 = 2$

53. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

54. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 55–58.

55. $4x^2 + y = 4$ and $x^4 - y = 1$

56. $x^3 - y = 0$ and $3x^2 - y = 4$

57. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

58. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 59–66.

59. $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

60. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

61. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

62. $y = \sin(\pi x/2)$ and $y = x$

63. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

64. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

65. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

66. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

67. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

68. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

69. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

70. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

71. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

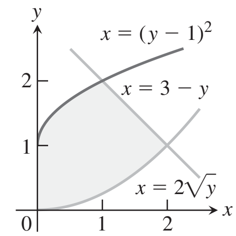
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

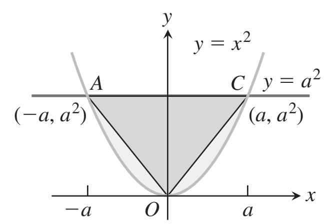
72. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

73. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

74. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



75. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

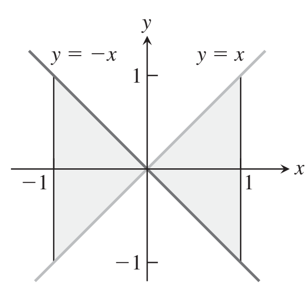


76. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

77. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\displaystyle\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

b. $\displaystyle\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



78. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

Chapter 6 Questions to Guide Your Review

1. What is a first-order differential equation? When is a function a solution of such an equation?
2. What is a general solution? A particular solution?
3. What is the slope field of a differential equation $y' = f(x, y)$? What can we learn from such fields?
4. How do you solve linear first-order differential equations?
5. What is an orthogonal trajectory of a family of curves? Describe how one is found for a given family of curves.
6. How do you solve separable first-order differential equations?
7. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
8. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
9. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
10. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

Chapter 6 Practice Exercises

In Exercises 1–16 solve the differential equation.

1. $y' = xe^y\sqrt{x-2}$
2. $y' = xye^{x^2}$
3. $\sec x\, dy + x\cos^2 y\, dx = 0$
4. $2x^2\, dx - 3\sqrt{y}\csc x\, dy = 0$
5. $y' = \dfrac{e^y}{xy}$
6. $y' = xe^{x-y}\csc y$
7. $x(x-1)\, dy - y\, dx = 0$
8. $y' = (y^2 - 1)x^{-1}$
9. $2y' - y = xe^{x/2}$
10. $\dfrac{y'}{2} + y = e^{-x}\sin x$
11. $xy' + 2y = 1 - x^{-1}$
12. $xy' - y = 2x\ln x$
13. $(1 + e^x)\, dy + (ye^x + e^{-x})\, dx = 0$
14. $e^{-x}\, dy + (e^{-x}y - 4x)\, dx = 0$
15. $(x + 3y^2)\, dy + y\, dx = 0$ (*Hint:* $d(xy) = y\, dx + x\, dy$)
16. $x\, dy + (3y - x^{-2}\cos x)\, dx = 0, \quad x > 0$

Initial Value Problems

In Exercises 17–22 solve the initial value problem.

17. $(x+1)\dfrac{dy}{dx} + 2y = x, \quad x > -1, \quad y(0) = 1$
18. $x\dfrac{dy}{dx} + 2y = x^2 + 1, \quad x > 0, \quad y(1) = 1$
19. $\dfrac{dy}{dx} + 3x^2y = x^2, \quad y(0) = -1$
20. $x\, dy + (y - \cos x)\, dx = 0, \quad y\left(\dfrac{\pi}{2}\right) = 0$
21. $xy' + (x-2)y = 3x^3e^{-x}, \quad y(1) = 0$
22. $y\, dx + (3x - xy + 2)\, dy = 0, \quad y(2) = -1, \quad y < 0$

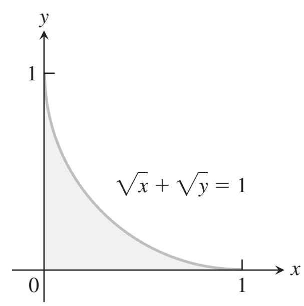
Area

In Exercises 23–26, find the total area of the region between the graph of f and the x-axis.

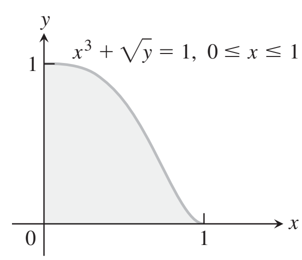
23. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$
24. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$
25. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$
26. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 27–37.

27. $y = x, \quad y = 1/x^2, \quad x = 2$
28. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$
29. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



30. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



31. $x = 2y^2, \quad x = 0, \quad y = 3$
32. $x = 4 - y^2, \quad x = 0$
33. $y^2 = 4x, \quad y = 4x - 2$
34. $y^2 = 4x + 4, \quad y = 4x - 16$
35. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$
36. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$
37. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

38. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

39. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

40. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

41. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

42. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

43. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

44. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

Average Values

45. Find the average value of $f(x) = mx + b$

a. over $[-1, 1]$ **b.** over $[-k, k]$

46. Find the average value of

a. $y = \sqrt{3x}$ over $[0, 3]$ **b.** $y = \sqrt{ax}$ over $[0, a]$

47. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

48. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

Chapter 6 Single Choice Questions

1. The order and degree of the differential equation $\sqrt[3]{\frac{dy}{dy}} - 4\frac{d^2y}{dx^2} - 7x = 0$ are a and b then $(a + b)$ is

a. 3 **b.** 4 **c.** 5 **d.** 6

2. The differential equation of the family of curves represented by $y = a + bx + ce^{-x}$ (where a, b, c are arbitrary constants) is

a. $y''' = y'$ **b.** $y''' + y'' = 0$

c. $y''' - y'' + y' = 0$ **d.** $y''' + y'' - y' = 0$

3. The general solution of the differential equation $dy/dx = (x^3 - 2x\tan^{-1}y)(1 + y^2)$ is

a. $2\tan^{-1}x = y^2 - 1 + 2ce^{-x^2}$

b. $2\tan^{-1}y = x^2 - 1 + 2ce^{-x^2}$

c. $2\tan^{-1}y = y^2 - 1 + 2ce^{-x^2}$

d. $2\tan^{-1}x = x^2 - 1 + 2ce^{-y^2}$

4. Let $(x^2 + 1)\frac{d^2y}{dx^2} = 2x\frac{dy}{dx}$, where $y'(0) = 3$ and $y(0) = 1$, then $y(1)$ is equal to

a. 2 **b.** 3 **c.** 4 **d.** 5

5. The solutions of $y'y''' = 3(y'')^2$ is

a. $x = ay^2 + by + c$ **b.** $y = ax^2 + bx + c$

c. $y = ae^x + bx^{-x} + c$ **d.** $x = ae^y + be^{-y} + c$

6. Solutions of the differential equation is/are

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}(\cos x + \sec x) + 1 = 0$$

a. $y = -\ln|\sec x + \tan x| + c$ **b.** $y = \ln|\sec x + \tan x| + c$

c. $y = \sin x + c$ **d.** $y = \cos x + c$

7. If $y = x\sin(\ln 3x)$ then $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + 2y^2$ is equal to

a. $2x^2$ **b.** $\frac{x^2}{2}$ **c.** x^2 **d.** $2\ln|x|$

8. The solution of differential equation $\left(x\tan\left(\frac{y}{x}\right) - y\sec^2\left(\frac{y}{x}\right)\right)dx + x\sec^2\left(\frac{y}{x}\right)dy = 0$ satisfying the initial conditions $y(1) = \frac{\pi}{4}$ is

a. $x\sec\frac{y}{x} = \sqrt{2}$ **b.** $x\tan\frac{y}{x} = 1$

c. $x\tan^2\frac{y}{x} = 1$ **d.** $x\sec^2\frac{y}{x} = 2$

9. The differential equation of family of hyperbolas with asymptotes as the lines $x + y = 1$ and $x - y = 1$ is

a. $yy' + x = 0$ **b.** $yy' = x - 1$

c. $yy' + y' = 0$ **d.** $y' + xy = 0$

10. Solutions of the differential equation $(2x - y + 2)dx + (4x - 2y - 1)dy = 0$ is

a. $2x - y = ce^{-(x + 2y)}$ **b.** $2x + y = ce^{(2x - y)}$

c. $x - 2y = ce^{-(x + 2y)}$ **d.** $2x + y = ce^{(x + 2y)}$

11. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where $c > 0$, is a parameter, is of order and degree as follows

a. order 1, degree 2 **b.** order1, degree 1

c. order 1, degree 3 **d.** order 2, degree2

12. Let C be a curve such that the tangent at any point P on it meets x-axis and y-axis at A and B, respectively. IF BP:PA = 3:1 and the curve passes through the point (1, 1), then

a. The curve passes through $\left(3, \frac{1}{8}\right)$

b. Equation of normal to the curve at (1, 1) is $3y - x = 2$

c. The differential equation for the curve is $3y' + x^2y = 0$

d. The differential equation for the curve is $xy' + 3xy = 0$

13. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim\limits_{t\to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

a. $\dfrac{1}{3x} + \dfrac{2x^2}{3}$ **b.** $\dfrac{-1}{3x} + \dfrac{4x^2}{3}$

c. $\dfrac{-1}{x} + \dfrac{2}{x^2}$ **d.** $\dfrac{1}{x}$

14. The differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ determine a family of circles with

a. variable radii and a fixed center at (0, 1)

b. variable radii and a fixed center at (0, –1)

c. fixed radius 1 and variable centers along the x-axis

d. fixed radius 1 and variable centers along the y-axis.

15. Let a solution $y = y(x)$ of the differential equation, $x\sqrt{x^2-1}dy - y\sqrt{y^2-1}dx = 0$ satisfy $y(2) = \dfrac{2}{\sqrt{3}}$.

Statement-1: $y(x) = \sec\left(\sec^{-1} x - \dfrac{\pi}{6}\right)$

and

Statement-2: $y(x)$ is given by $\dfrac{1}{y} = \dfrac{2\sqrt{3}}{x} - \sqrt{1 - \dfrac{1}{x^2}}$

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is false, statement-2 is true.

d. Statement-1 is true, statement-2 is false

16. If $\left(\dfrac{2+\sin x}{1+y}\right)\dfrac{dy}{dx} = -\cos x$, $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right)$ is equal to

a. 1 **b.** 1/2

c. 1/3 **d.** 1/4

17. The solution of primitive integral equation $(x^2 + y^2)dy = xy\,dx$, is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is

a. $\sqrt{2(e^2-1)}$ **b.** $\sqrt{2(e^2+1)}$

c. $\sqrt{3}e$ **d.** $\sqrt{\dfrac{e^2+1}{2}}$

18. For the primitive integral equation $ydx + y^2dy = xdy$; $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then $y(-3)$ is

a. 3 **b.** 2 **c.** 1 **d.** 5

19. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then

a. $y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{8\sqrt{2}}$ **b.** $y'\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{18}$

c. $y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{9}$ **d.** $y'\left(\dfrac{\pi}{3}\right) = \dfrac{4\pi}{9} + \dfrac{2\pi^2}{3\sqrt{3}}$

20. Let $f: \left[\dfrac{1}{2}, 1\right] \to$ R (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\dfrac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

a. (2e – 1, 2e) **b.** (e – 1, 2e – 1)

c. $\left(\dfrac{e-1}{2}, e-1\right)$ **d.** $\left(0, \dfrac{e-1}{2}\right)$

21. A curve passes through the point $\left(1, \dfrac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\dfrac{y}{x} + \sec\left(\dfrac{y}{x}\right)$, $x > 0$. Then the equation of the curve is

a. $\sin\left(\dfrac{y}{x}\right) = \log x + \dfrac{1}{2}$ **b.** $\operatorname{cosec}\left(\dfrac{y}{x}\right) = \log x + 2$

c. $\sec\left(\dfrac{2y}{x}\right) = \log x + 2$ **d.** $\cos\left(\dfrac{2y}{x}\right) = \log x + \dfrac{1}{2}$

22. Let $y'(x) + \dfrac{g'(x)}{g(x)}.y(x) = \dfrac{g'(x)}{1+g^2(x)}$, where $f'(x)$ denotes $\dfrac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function an R. If $g(1) = y(1) = 1$ and $g(e) = \sqrt{(2e-1)}$ then $y(e)$ equals

a. $\dfrac{3}{2g(e)}$ **b.** $\dfrac{1}{2g(e)}$ **c.** $\dfrac{2}{3g(e)}$ **d.** $\dfrac{1}{3g(e)}$

23. If the function $y = f(x)$ satisfies $f'(x) + f(x)\cot x - 2\cos x = 0$, $f\left(\dfrac{\pi}{2}\right) = 1$, then $f\left(\dfrac{\pi}{3}\right)$ is equal to

a. 0 **b.** $\dfrac{1}{2}$ **c.** $\dfrac{\sqrt{3}}{2}$ **d.** 2

24. If slope of the tangent at the point (x, y) on the curve is $\dfrac{y-1}{x^2+x}$, then the equation of the curve passing through M(1, 0) is

a. $(y-1)(x+1) + 2x = 0$ **b.** $(y-1)(x-1) = 0$

c. $(y+1)(x+1) - 2x = 0$ **d.** $(y+1)(x-1) = 0$

25. Let $f(x)$ be a differentiable non-decreasing function such that $\int_0^x (f(t))^3 dt = \dfrac{1}{x^2}\left(\int_0^x f(t)dt\right)^3$ $\forall x \in$ R– {0} and $f(1) = 1$. If $\int_0^x f(t)dt = g(x)$ then $\dfrac{x.g'(x)}{g(x)}$ is

a. Always equal to 1

b. Always equal to –2

c. May be 1 or –2

d. Not independent of x

26. The family of curve whose differential equation is $(xy + 1)(3x^2ydx - x^3dy) = (x^6 + y^2)(xdy + ydx)$, is

a. $\tan^{-1}\left(\dfrac{x^3}{y}\right) = \ln(|1+xy|) + c$

b. $\sin^{-1}\left(\dfrac{x^3}{y}\right) = \ln(|1-xy|) + c$

c. $\sin^{-1}x^3 = \ln(1+xy) + c$

d. None of these

27. Solutions of differential equation $x^5y + x = y\dfrac{dx}{dy}$ is (where C is an integration constant)

a. $\dfrac{x^5}{5} + \dfrac{y^4}{4x^4} = C$ **b.** $\dfrac{x^5}{5} + \dfrac{y^4}{4y^4} = C$

c. $\dfrac{y^5}{5} + \dfrac{y^4}{4x^4} = C$ **d.** $\dfrac{y^5}{5} + \dfrac{x^4}{4y^4} = C$

28. Statement-1: Order and degree of differential equation of curve $y = 2\lambda^2 x + 3\lambda^{5/2}$ are 1 and 5, respectively.

Statement-2: Order of a differential equation is number of independent arbitrary constants involved in it and degree of differential equation is the degree of highest order differential coefficient involved in it.

a. Statement-1 is true, Statement-2 is True, Statement-2 is a correct explanation for statement-1

b. Statement-1 is true, Statement-2 is True, Statement-2 is not a correct explanation for statement-1

c. Statement-1 is true, statement-2 is false

d. Statement-1 is false, statement-2 is true.

29. If a curve passing through (1, 1) satisfies the differential equation $x^2dy = y^2dx + xyd(xy)$, then $\ln|xy|$ is equal to

a. $\dfrac{1}{x^2} - \dfrac{1}{y^2}$ **b.** $\dfrac{1}{x} - \dfrac{1}{y}$

c. $\dfrac{1}{y} - \dfrac{1}{x}$ **d.** $\dfrac{1}{y^2} - \dfrac{1}{x^2}$

30. The equation of a curve whose slope at any point is twice its abscissa and which passes through (1, 2) is

a. $x^2 + y^2 = 5$ **b.** $y = 3 - x^2$

c. $y = x^3 + 1$ **d.** $y = x^2 + 1$

31. Solutions of differential equation $xdy = (y - x^2 - y^2)\,dx$ is (where c is an arbitrary constant)

a. $y = x\cos(c + x)$ **b.** $y = x\tan(c - x)$

c. $y = x\operatorname{cosec}(x - c)$ **d.** $y = x\sec(c + x)$

32. If a curve represented by differential equation $(x^3 - x + 2xy)dx + x^2dy = 0$ passes through the point (2, 2), then the curve passes through the point

a. $\left(1, \dfrac{1}{4}\right)$ **b.** $\left(1, \dfrac{41}{4}\right)$

c. (3, 1) **d.** $\left(3, \dfrac{-23}{36}\right)$

33. The area bounded by the curve $y = f(x)$, the ordinates $x = 1$ and $x = e^a$ $(a > 0)$ and the x-axis is given by ae^a, then $f(x)$ equals (where $f(x) > 0$)

a. $x + \ln x$ **b.** $e^x + xe^x$ **c.** $1 + \ln x$ **d.** $x + e^x$

34. For the curve $f(x) = x^2e^x$ which of the following is true?

a. $f(x)$ has local minima at $x = -2$.

b. $f(x)$ has two points of inflection.

c. Area bounded by the curve $\dfrac{f(x)}{x^2}$, coordinate axes and the line $x = 1$ is 2e.

d. The local maximum value of $f(x)$ is $\dfrac{4}{e^2}$.

35. Area bounded by curve $y = (x-2)^2$ and lines $y = x$ and $y = 0$, is

a. $\dfrac{5}{6}$ **b.** $\dfrac{6}{5}$ **c.** $\dfrac{2}{3}$ **d.** 1

36. Area bounded by $y = 2\sqrt{x}$ and $x = 3\sqrt{y}$ is equal to (in sq. units)

a. 12 **b.** 8 **c.** 10 **d.** 6

37. The area bounded between $x^2 - y = 1$ and $y = |x - 1| + x$ is

a. $\dfrac{7+4\sqrt{2}}{3}$ **b.** $\dfrac{7-4\sqrt{2}}{3}$

c. $\dfrac{5+4\sqrt{2}}{3}$ **d.** $\dfrac{8+4\sqrt{2}}{3}$

38. Let $g(x)$ be the inverse of $f(x)$ and $f(x) = 11x^5 + 7x^3 + x + 1$, then area bounded by $g(x)$, x-axis and $x = 20$, is

a. $\dfrac{59}{12}$ **b.** $\dfrac{179}{12}$ **c.** $\dfrac{189}{12}$ **d.** $\dfrac{199}{12}$

39. Area bounded by $y = \dfrac{x-5}{x^2 - 10x + 26}$, x-axis and $x = 0$ to $x = 10$.

a. $4\log_e 26$ **b.** $3\log_e 26$

c. $2\log_e 26$ **d.** $\log_e 26$

40. Area bounded by $f(x) = \min(\{x - 2\}, \{3 - x\})$, $x = 4$, $x = -4$ and x-axis is equal to (where $\{\cdot\}$ denotes fractional part)

a. 1 **b.** 3/2 **c.** 2 **d.** 4

41. Area bounded by $(x-1)^2 + y^2 \geq 1$, $y^2 \leq 4x$ and $0 \leq x \leq 2$ is equal to

a. $(8\sqrt{2} + \pi)$ **b.** $8\sqrt{2} - \pi$

c. $\dfrac{8\sqrt{2}}{3} - \pi$ **d.** $\dfrac{16\sqrt{2}}{3} - \pi$

42. The area of region in xy-plane consisting of all points (a, b) such that quadratic equation $ax^2 + 2(a + b - 7)x + 2b = 0$ has fewer than two real solutions is

a. 4π **b.** 9π **c.** 25π **d.** 49π

43. The area bounded by the parabola $x^2 = 8y$ and the line $x - 2y + 8 = 0$ is

a. 36 **b.** 72 **c.** 18 **d.** 9

44. If the area bounded by $y = ax^2$ and $x = ay^2$, $a > 0$, is 1, then $a =$

a. 1 **b.** $\dfrac{1}{\sqrt{3}}$ **c.** $\dfrac{1}{3}$ **d.** $-\dfrac{1}{\sqrt{3}}$

45. The area bounded by the parabolas $y = (x+1)^2$ and $y = (x-1)^2$ and the line $y = 1/4$ is

a. 4 sq. units

b. 1/6 sq. units

c. 4/3 sq. units

d. 1/3 sq. units

46. The area of the region between the curves $y = \sqrt{\dfrac{1+\sin x}{\cos x}}$ and $y = \sqrt{\dfrac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \dfrac{\pi}{4}$ is

a. $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\mathrm{d}t$ **b.** $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\mathrm{d}t$

c. $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\mathrm{d}t$ **d.** $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\mathrm{d}t$

47. Area of the region bounded by the curve $y = \mathrm{e}^x$ and lines $x = 0$ and $y = \mathrm{e}$ is

a. $\mathrm{e} - 1$ **b.** $\displaystyle\int_1^e \ln(e+2-y)\,\mathrm{d}y$

c. $\mathrm{e} - \displaystyle\int_0^1 e^{3x}\,\mathrm{d}x$ **d.** $\displaystyle\int_1^e \ln y\,\mathrm{d}y$

48. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts R_1 $(0 \le x \le b)d$ and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \dfrac{1}{4}$. Then b equals

a. $\dfrac{3}{4}$ **b.** $\dfrac{1}{2}$ **c.** $\dfrac{1}{3}$ **d.** $\dfrac{1}{4}$

49. Let f: $[-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \displaystyle\int_{-1}^{2} x\,f(x)\mathrm{d}x$ and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then.

a. $R_1 = 2R_2$ **b.** $2R_1 = 3R_2$

c. $2R_1 = R_2$ **d.** $3R_1 = R_2$

50. The area enclosed by the curve $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \dfrac{\pi}{2}\right]$ is

a. $4\left(\sqrt{2}-1\right)$ **b.** $2\sqrt{2}\left(\sqrt{2}-1\right)$

c. $2\left(\sqrt{2}+1\right)$ **d.** $2\sqrt{2}\left(\sqrt{2}+1\right)$

51. The positive value of the parameter a for which the area of the figure bounded by the curve $y = \cos ax$, $y = 0$, $x = \dfrac{\pi}{6a}$ and $x = \dfrac{5\pi}{6a}$ is greater than 3 are

a. ϕ **b.** $\left(0, \dfrac{1}{3}\right)$

c. $(3, \infty)$ **d.** $\left(\dfrac{1}{3}, 3\right)$

52. Tangent is drawn from (1, 0) to $y = e^x$, then the area bounded between the coordinate axes and the tangent is equal to

a. $\dfrac{e}{2}$ **b.** e

c. $\dfrac{e^2}{2}$ **d.** e^2

53. The positive value of the parameter k for which the area of the figure bounded by the curve $y = \sin(kx)$, $x = \dfrac{2\pi}{3k}$, $x = \dfrac{5\pi}{3k}$ and x-axis is less than 2 can be

a. $\dfrac{1}{8} < k < \dfrac{3}{8}$ **b.** $0 < k < \dfrac{1}{8}$

c. $1 < k < 2$ **d.** $\dfrac{3}{8} < k < \dfrac{5}{8}$

54. Area enclosed by parabola $ay = 3(a^2 - x^2)$ and x-axis is 64, then value of a is

a. 4 and –4 **b.** 2 and –2

c. 3 and –3 **d.** 5 and –5

55. The value of positive real parameter a such that area of region bounded by parabolas $y = x - ax^2$, $ay = x^2$ attains its maximum value is equal to

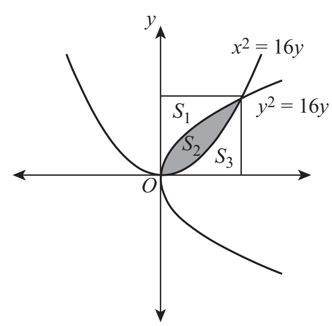
a. $\dfrac{1}{2}$ **b.** 2 **c.** $\dfrac{1}{3}$ **d.** 1

56. Area of region bounded by $x = 0$, $y = 0$, $x = 2$, $y = 2$, $y \le \mathrm{e}^x$ and $y \ge \ln x$ is

a. $6 - 4 \ln 2$ **b.** $4 \ln 2 - 2$

c. $2 \ln 2 - 4$ **d.** $6 - 2 \ln 2$

57. The parabolas $y^2 = 16x$ and $x^2 = 16y$ divide the square region bounded by lines $x = 16$, $y = 16$ and coordinate axes. If S_1, S_2, S_3 are, respectively, the areas of three parts numbered from top to bottom as shown in figure, then



a. S_3, S_1, S_2 form a G.P.

b. S_1, S_2, S_3 form an A.P.

c. S_2, S_1, S_3 form a H.P.

d. $S_1 + S_3 = \dfrac{215}{3}$

58. If p and q are the degree and order, respectively, for the differential equation obtained on eliminating arbitrary constants a, b, c, d from $y + a\cos^2 x + b\sin^2 x + c\cos 2x + d\sin 2x = 0$, then

a. $p + q = 5$

b. $p^2 + q^2 = 20$

c. $p + q = 3$

d. $p^2 + q^2 = 5$

Chapter 6 Multiple Choice Questions

1. Let $y = f(x)$ defined on R satisfies the differential equation $(1 + x^2)\dfrac{dy}{dx} + 2xy = 2x$ and $f(0) = 2$, the which of the following statements is(are) correct?
   - **a.** $f(x)$ is neither even nor odd function
   - **b.** $f(x)$ increases on $(-\infty, 0)$ and decreases on $(0, \infty)$
   - **c.** The x-intercept of normal on the graph of $y = f(x)$ at $x = 1$ equals $\dfrac{1}{4}$.
   - **d.** The area bounded by $y = f(x)$ with x-axis between ordinates at $x = 0$ and $x = 1$ equals $\left(\dfrac{\pi + 4}{4}\right)$

2. If $S(K)$ denotes the area bounded by $y = x^2 - 3$ and $y = kx + 2$ then
   - **a.** $S(K) = \dfrac{1}{6}(K^2 + 20)^{3/2}$
   - **b.** $S(K) = \dfrac{1}{12}(K^2 + 20)^{3/2}$
   - **c.** Least value of $S(K) = \dfrac{20\sqrt{5}}{3}$
   - **d.** Least value of $S(K) = \dfrac{20\sqrt{5}}{6}$

3. Let $f(x)$ be a real-valued function satisfying $f(x/y) = f(x) - f(x)$ $\forall$ x and $y > 0$ and $\lim\limits_{x \to 0} \dfrac{f(1+x)}{x} = 3$, then
   - **a.** Area bounded by $y = f(x)$; y-axis and the line $y = 3$ is $3e$
   - **b.** $f(x)$ is a transcendental function
   - **c.** $f(x)$ is an odd function
   - **d.** $f(e^2) = 6$

4. The area of the region bounded by the graph of the function $f(x) = x^4 - 8x^3 + 22x^2 - 23x + 10$ and the tangent line which touches the graph at two distinct points is:
   - **a.** $\displaystyle\int_1^3 (x-1)^2(x-3)^2 dx$
   - **b.** $\displaystyle\int_{-1}^1 (t^2 - 1)dt$
   - **c.** $\dfrac{16}{15}$
   - **d.** $\dfrac{8}{15}$

5. The area bounded by the graph of $y = \sqrt[3]{x + \sqrt{x^2+1}} + \sqrt[3]{x - \sqrt{x^2+1}}$, the line $x - y - 1 = 0$ and the x-axis is
   - **a.** $\displaystyle\int_0^1 \left[(y+1) - \left(\frac{y^3 + 3y}{2}\right)\right] dy$
   - **b.** $\displaystyle\int_0^1 (y - y^3 + 2)dy$
   - **c.** $\dfrac{5}{8}$
   - **d.** $\dfrac{9}{4}$

6. If $y = f(x)$ is non-negative function such that $f(1) = 2$ and satisfying $(x^2 - y^2)dx + xydy = 0$, then select the correct option(s)
   - **a.** $f(e) = \sqrt{2}e$
   - **b.** $f(e) = 2e$
   - **c.** Area bounded by the curve $g(x) = (f(x))^2$, the x-axis, ordinates $x = 1$ and $x = e$ is $\dfrac{8e^3 - 14}{9}$
   - **d.** Area bounded by the curve $g(x) = (f(x))^2$, the x-axis, ordinates $x = 1$ and $x = e$ is $\dfrac{8e^3 - 14}{3}$

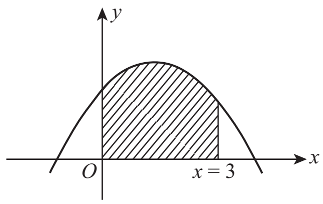
7. Let S be the area bounded by the curve $y = \sin x$ $(0 \le x \le \pi)$ and the x-axis and T be the area bounded by the curves $y = \sin x$ $\left(0 \le x \le \dfrac{\pi}{2}\right)$, $y = a \cos x$ $\left(0 \le x \le \dfrac{\pi}{2}\right)$ and the x-axis (where $a \in R^+$).
   - **a.** If $S{:}T = 3{:}1$ then $3a = 4$
   - **b.** If $S{:}T = 3{:}1$ then $3a = 5$
   - **c.** If $S{:}T = 6{:}1$ then $6a = 4$
   - **d.** If $S{:}T = 6{:}1$ then $12a = 5$

8. Let f: $R_0 \to R$ is such that $f(x) = 2x - \displaystyle\int_1^x \frac{f(t)}{t} dt$, then which of the following is true?
   - **a.** max value of $f(x)$ is 2 and min value is –2
   - **b.** $f(x)$ has two asymptotes
   - **c.** $f(x)$ is one-one function
   - **d.** Area bounded by $y = f(x)$: x-axis and the lines $x = 1$ and $x = 2$ is

9. If $f'(x)$ is a non-decreasing function for which $f'(0) = 0$, $f(0) = 5$ and $f''(x) - f'(x) \le 0$ $x \in R$, then the area bounded by the $y = f(x)$ and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{23}{2}$ lies in the interval
   - **a.** [50, 55]
   - **b.** [50, 60]
   - **c.** [51, 56]
   - **d.** [40, 75]

10. The area bounded by the circle $x^2 + y^2 = 4$, the curve $y = \left[\sin^2 \dfrac{x}{4} + \cos \dfrac{x}{4}\right]$ (where $[\cdot]$ denotes greatest integer function) and the x-axis is
    - **a.** $\dfrac{4\pi}{3} - \sqrt{3}$
    - **b.** $\dfrac{2\pi}{3} + \sqrt{3}$
    - **c.** $\dfrac{\pi}{3}\displaystyle\int_0^{\pi} \sin x dx + \tan\frac{\pi}{3}$
    - **d.** $\dfrac{2\pi}{3}\displaystyle\int_0^{\pi} \sin x dx + \tan\left(\frac{2\pi}{3}\right)$

11. The shaded area enclosed by $f(x) = 12 + ax - x^2$ coordinate axes and the ordinate at $x = 3$ is 45 square units. If m and n are the x-intercepts of the graph of $y = f(x)$, then



a. $m^2 + n^2 = 40$

b. $a = 4$

c. $a = 8$

d. $m + n + a = 8$

12. Given function $f: R \to R$ is defined by $f(x) = \begin{cases} -x+2, & x \le 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}$. Which of the following holds good?

a. $f(x)$ is continuous and differentiable at $x = 1$

b. Area bounded by $y = f(x)$, y-axis and $x = e$ and $x = 0$ is $\dfrac{5}{2}$

c. $f'(x)$ is continuous everywhere

d. Range of $f(x)$ is $(0, \infty)$

13. If $f: (0, 1) \to R$ is defined by $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{2^n}}{1 - x^{2^{n+1}}}$, then

a. $f^{-1}(x) = \dfrac{x}{x+1}$

b. Area bounded by $y = f(x)$, x-axis and the lines $x = 0$ and $x = 1/2$ is $\dfrac{(\ln 4) - 1}{2}$

c. Area bounded by $y = f(x)$, x-axis and the lines $x = 0$ and $x = 1$ is $(\ln 2) - 1$

d. Area bounded by $y = f(x)$, x-axis and the lines $x = 0$ and $x = 1$ does not converge.

14. Let f be a differentiable function satisfying the relation $f(xy) = x f(x) + y f(x) - 2xy$ (where $x, y > 0$) and $f'(1) = 3$, then

a. $f(x) = x \ln x + 3x - \dfrac{x^2}{2}$

b. Area bounded by $y = f(x)$, x-axis and the line $x = 1$ and $x = 2$ is $\dfrac{4\ln 4 + 9}{4}$

c. $x = e^{-3}$ is the abscissa of the point of inflection of $f(x)$

d. Graph of $y = f(x)$ has no asymptote

15. Let $f(x) = \begin{bmatrix} (x-2)^2 + 3 & x \ge 2 \\ k + x^2 & x < 2 \end{bmatrix}$, then which of the following statements is/are true?

a. If $f(x)$ is continuous at $x = 2$, then $k = -1$

b. There exists at least one value of k for which $f(x)$ is derivable $\forall\, x \in R$

c. If $f(x)$ has a local minima at $x = 2$, then $k > -1$

d. Area bounded by the graph of $y = f(x)$, x-axis and the lines $x = 2$ and $x = 3$ is 10/3

16. Let $f(x) = \begin{cases} \sin^{-1}(\sin x) & x > 0 \\ \dfrac{\pi}{2} & x = 0 \\ \cos^{-1}(\cos x) & x < 0 \end{cases}$; then

a. $x = 0$ is point of maxima.

b. $f(x)$ is continuous for $x \in R$

c. Area bounded by $y = f(x)$; x-axis and the lines $x = -2\pi$ and $x = \pi$ is $\dfrac{5\pi^2}{4}$

d. Area bounded by $y = f(x)$; x-axis and the lines $x = -2\pi$ and $x = \pi$ is $\dfrac{3\pi^2}{4}$

17. Which of the following is equal to $\pi/2$?

a. Area bounded by $f(x) = \dfrac{1}{1 + (\sin x)^{\cos x}}$, x-axis and the lines $x = 0$ and $x = \pi$

b. Area bounded by $f(x) = \dfrac{1}{1 + (\tan x)^5}$, x-axis and then line $x = 0$ and $x = \pi/2$

c. Area bounded by $f(x) = \dfrac{x^2 + 1}{x^4 - x^2 + 1}$ and co-ordinate axes

d. Area bounded by $f(x) = \ln(\sec x)\,(e^{\ln \ln 2})^{-1}$ x-axis and $x = 0$ and $x = \pi/2$

18. In a question, a student was given to find the derivative of product of two functions f and g. The student by mistake used $(fg)' = f'g'$ for his questions and got the correct answer. If $f(x) = x^3$ and $g(4) = 1$, then which of the following is/are correct possibilities (where $g(x) > 0$)?

a. Area bounded by $y = g(x)$ x-axis and $x = 4$ and $x = 5$ is 3/5

b. Area bounded by $y = g(x)$ x-axis and $x = 4$ and $x = 5$ is 3/8

c. $g(x)$ is decreasing in $(3, \infty)$

d. $g(x)$ is increasing in $(-\infty, 3)$

19. Area of the region bounded by the curve $y = \tan x$ and lies $y = 0$ and $x = 1$ is

a. $\int_0^1 \tan(1-x)\,dx$

b. $\tan 1 - \int_0^{\tan 1} \tan^{-1} x\,dx$

c. $\int_0^{\tan 1} \tan^{-1} y\,dx$

d. $\int_0^1 \tan^{-1} x\,dx$

20. If $y = ax^p + bx^{-q}$, where $p, q \in N$ and a, b are constants such that $x^2y'' + 2xy' = 12y$, then

a. $p + q = 7$

b. $q - p = 1$

c. $p = q$

d. $q = p + 1$

21. A solution of the differential equation $(x^2y^2 - 1)dy + 2xy^3dx = 0$ is

a. $1 + x^2y^2 = cx$

b. $1 + x^2y^2 = cy$

c. $y = 0$

d. $y = -\dfrac{1}{x^2}$

22. A student made a mistake in writing quotient rule of differentiation and writes $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f(x).g'(x) - g(x)f'(x)}{(g(x))^2}$ and still gets correct answer.

Let $F(n) = \frac{f(n)}{g(n)}$, $n \in N$

a. $F(n) = F(n + 1)$

b. $F(n) = F(n + 2)$

c. $\sum_{n=1}^{20} F(n) = \sum_{n=1}^{20} F(2n-1)$

d. $F(1), F(2), F(3)\ldots$ from a G.P.

23. The solutions of $\left(\frac{dy}{dx}\right)^2 - 2\left(x + \frac{1}{4x}\right)\frac{dy}{dx} + 1 = 0$

a. $y = x^2 + c$ **b.** $y = \frac{1}{2}$ ln (x)+ c, $x > 0$

c. $y = \frac{x}{2} + c$ **d.** $y = \frac{x^2}{2} + c$

24. Given $y = f(x)$ be the solutions of differential equation e^{-x} dy – x^2dx = 2xdx with $y(0) = 0$, then which of the following is true?

a. $f(x)$ has local minimum at $x = -2$

b. $f(x)$ has two points of inflection

c. Area bounded by the curve $\frac{f(x)}{x^2}$, coordinate axes and the line $x = 1$ is $2e$.

d. The local maximum value of $f(x)$ is $\frac{4}{e^2}$ (where $e = \lim_{x\to 0}(1+x)^{1/x}$)

25. If $y(x)$ is the solutions of $(2 + y)\frac{dy}{dx}$ $(x - 1) = 0$ and $y(0) = 0$ and the area bounded by the curve $y(x)$, x-axis and y-axis in first quadrant is A, then

a. $y(x)$ is circle of radius 5.

b. $y(x)$ is a circle of radius $\sqrt{5}$

c. $\frac{5\pi}{8} - 2 < A < \frac{5\pi}{8} - 2$

d. $\frac{5\pi}{6} - 2 < A < \frac{5\pi}{4} - 2$

26. Equation of curve represented by differential equation $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ and passing through (0, 1) is

a. $\sin^{-1}x = \sin^{-1}y$ **b.** $\sin^{-1}x + \cos^{-1}y = 0$

c. $\sin^{-1}x = \cos^{-1}y$ **d.** $\cos^{-1}x + \sin^{-1}y = \pi$

27. Given the equation of curve is $y = Ae^x \cos x + Be^x \sin x$, a differential equation of the given curve is formed, then which of the following is/are true?

a. Order of differential equation is two

b. Degree of differential equation is one

c. Differential equation is linear

d. Differential equation is $\frac{d^2y}{dx^2} = 2\left(y + \frac{dy}{dx}\right)$

28. Solution of the differential equation xdx + ydx = $(x^4 + 2x^2y^2 + y^4)$ $(x$d$x - y$d$y)$ is

a. $x^2 + y^2 + \frac{1}{x^2 - y^2} = C$ **b.** $x^4 - y^4 + 1 = C(x^2 - y^2)$

c. $x^2 - y^2 + \frac{1}{x^2 + y^2} = C$ **d.** $x^4 - y^4 + 1 = C(x^2 + y^2)$

Chapter 6 Passage Type Questions

Passage 1

Consider the curve C$_1$: $y = \frac{8}{27}x^3$ and C$_2$: $y = (x + a)^2$

1. The range of a for which there exists two common tangents to the curves C$_1$ and C$_2$ there than x-axis is

a. $(0, \infty)$ **b.** $\left(-\frac{1}{2}, \infty\right) - \{0\}$

c. $\left(-\frac{3}{2}, \infty\right) - \{0\}$ **d.** $[1, \infty)$

2. If $a = 4$, then the area bounded by the two tangent lines mentioned in the previous questions and the given parabola C$_2$ is

a. 100 **b.** 144 **c.** 180 **d.** 200

Passage 2

Suppose the curve C: $y = x\sqrt{9 - x^2}$ $(x \ge 0)$

On the basis of above information, answer the following questions:

1. The maximum value of y is

a. $\frac{9}{4}$ **b.** $\frac{9}{8}$ **c.** $\frac{9}{2}$ **d.** $\frac{81}{4}$

2. The area of the figure bounded by the curve C and this x-axis is

a. 6 **b.** 9 **c.** 18 **d.** 27

3. The x-intercept of the tangent drawn to the curve C at $x = \frac{9}{4}$ is

a. $\frac{81}{8}$ **b.** $\frac{81}{4}$ **c.** 9 **d.** $\frac{9}{8}$

Passage 3

Given two points P and Q on the parabola C: $y = x^2 - x - 2$ in the xy-plane. Also the x coordinate of point P is less than that of Q.

On the basis of above information, answer the following questions:

1. If the origin O is the midpoint of line segment PQ, then equation of line PQ is

   a. $2x + 3y = 0$ **b.** $x + y = 0$

   c. $x + 2y = 0$ **d.** $3x + y = 0$

2. If a variable circle intersects the curve C at two points on x-axis, then the length of tangent from the point (–3, 0) to the variable circle is

   a. 10 **b.** 8 **c.** $2\sqrt{2}$ **d.** $\sqrt{10}$

3. If the area bounded by the line $2tx - y - t = 0$, $(t \in R)$ and the curve C is minimum, then the value of t is

   a. 0 **b.** $\frac{1}{2}$ **c.** $\frac{2}{3}$ **d.** 1

passage 4

Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$ satisfying $g(0) = 0$

1. If $f(-10)\ \sqrt{2}\ = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

   a. $\frac{4\sqrt{2}}{7^3 3^2}$ **b.** $-\frac{4\sqrt{2}}{7^3 3^2}$

   c. $\frac{4\sqrt{2}}{7^3 3}$ **d.** $-\frac{4\sqrt{2}}{7^3 3}$

2. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

   a. $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

   b. $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

   c. $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$

   d. $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$

3. $\int_{-1}^{1} g'(x) dx =$

   a. $2g(-1)$ **b.** 0

   c. $-2g(1)$ **d.** $2g(1)$

Passage 5 (Q.12 to Q.14)

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$

1. The real number s lies in the interval

   a. $\left(-\frac{1}{4}, 0\right)$ **b.** $\left(-11, -\frac{3}{4}\right)$

   c. $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ **d.** $\left(0, \frac{1}{4}\right)$

2. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

   a. $\left(\frac{3}{4}, 3\right)$ **b.** $\left(\frac{21}{64}, \frac{11}{16}\right)$

   c. (9, 10) **d.** $\left(0, \frac{21}{64}\right)$

3. The function $f'(x)$ is

   a. increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

   b. decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

   c. increasing in $(-t, t)$

   d. decreasing in $(-t, t)$

Passage 6

For $j = 0, 1, 2, \ldots n$. Let S_i be the area of region bounded by the x-axis and the curve $ye^x = \sin x$ for $j\pi \le x \le (j + 1)\pi$

1. The value of S_0 is

   a. $\frac{1}{2}(1 + e^{\pi})$ **b.** $\frac{1}{2}(1 + e^{-\pi})$

   c. $\frac{1}{2}(1 - e^{-\pi})$ **d.** $\frac{1}{2}(e^{\pi} - 1)$

2. The ratio $\frac{S_{2009}}{S_{2010}}$ equals

   a. $e^{-\pi}$ **b.** e^{π} **c.** $\frac{1}{2}e^{\pi}$ **d.** $2e^{\pi}$

3. The value of $\sum_{j=0}^{\infty} S_j$ equal to

   a. $\frac{e^{\pi}(1 + e^{\pi})}{2(e^{\pi} - 1)}$ **b.** $\frac{1 + e^{\pi}}{2(e^{\pi} - 1)}$

   c. $\frac{1 + e^{\pi}}{e^{\pi} - 1}$ **d.** $\frac{e^{\pi}(1 + e^{\pi})}{(e^{\pi} - 1)}$

Passage 7

Consider the function $f(x)$ and $g(x)$, both defined from $R \to R$

$f(x) = \frac{x^3}{2} + 1 - x\int_0^x g(t)dt$ and $g(x) = x - \int_0^1 f(t)dt$, then

1. Minimum value of $f(x)$

   a. 0 **b.** 1 **c.** 3/2 **d.** does not exist

2. The number of points of intersection of $f(x)$ and $g(x)$ is/are

   a. 0 **b.** 1 **c.** 2 **d.** 3

3. The area bounded by $g(x)$ with co-ordinate axes is (in square units.)

   a. $\frac{9}{4}$ **b.** $\frac{9}{2}$ **c.** $\frac{9}{8}$ **d.** None of these

Chapter 6 Matrix Match Type Questions

1.

Column-I	**Column-II**
(a) The cosine of the angle between the curves $y = 3^{x-1}\ln x$ and $y = x^{x-1}$ at their point of intersection on the line $y = 0$, is	(p) 0
(b) The area bounded by the curves $x = -4y^2$ and $(x-1) = -5y^2$ is	(q) 1
(c) The value of the integral $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \ln(\sin x)^{\sin x})dx$, is	(r) 4/3
(d) A continuous function f: $[1, 6] \to [0, \infty]$ is such that $f'(x) = \dfrac{2}{x + f(x)}$ and $f(1) = 0$, then the maximum value of f cannot exceed	(s) 2 ln 6

2.

Column-I	**Column-II**
(a) The area of triangle formed by tangents from (–1, 0) to $y^2 = 4x$ and chord of contact is	(p) 6
(b) The area of triangle formed by tangent at any point to $xy = 3$ and its asymptotes is	(q) 4
(c) The area enclosed between the parabola $4y = 3x^2$ ad the line $2y = 3x + 12$ is	(r) 5/6
(d) The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$.	(s) 27
If the curve passes through (1, 2), then area of region bounded by curve, the x-axis and line $x = 1$, is	(t) 7

3. Match the statement/expression is column-I with the open intervals in column-II

Column-I	**Column-II**
(a) Interval contained in the domain of definition of non-zero solution of the differential equation $(x-3)^2y' + y = 0$	(p) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
(b) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q) $\left(0, \dfrac{\pi}{2}\right)$
(c) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\dfrac{\pi}{8}, \dfrac{5\pi}{4}\right)$
(d) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \dfrac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

4.

Column-I	**Column-II**
(a) Solution of differential equation $y^2(x^2 + 1 + ce^{x^2}) = 1$	(p) $[3x^2y + 2xy - e^x(1 + x)]dx + (x^3 + x^2)dy = 0$
(b) Solution of differential equation $ydx - xdx - 3xy^2 e^{x^2} dx = 0$	(q) $(x^2 + x^3)y - xe^x = c$
(c) Solution of differential equation $\dfrac{dy}{dx} = xy\,(x^2y^2 - 1)$	(r) $\dfrac{x}{y} - \dfrac{3}{2}e^{x^2} = c$
(d) Solution of differential equation $\dfrac{dy}{dx}(x^2y^3 + xy) = 1$	(s) $\dfrac{1}{x} = 2 - y^2 + ce^{-y^2/2}$
	(t) $= 1 - y^2 + ce^{-y/2}$

5.

Column-I	**Column-II**
(a) If $y(x)$ is the solution of the equation $(x + 1)\dfrac{dy}{dx} - xy = 1$, $y(0) = -1$ then $y\left(-\dfrac{3}{2}\right) =$	(p) 0
(b) If $(x^2 + y^2)dy = xydx$ and $y(1) = 1$, $y(\alpha) = e$, then $\alpha^4 = ke^4$, where $k =$	(q) 1
(c) $xdy = y(dx + ydy)$, $y > 0$, $y(1) = 1$, $y(\alpha) = 2$, then $\alpha =$	(r) 2
(d) $\dfrac{dy}{dx} + \dfrac{2y}{x} = 0$, $y(1) = 1$, then $y\left(\dfrac{1}{2}\right) =$	(s) 4
	(t) 9

Chapter 6 Integer Type Questions

1. The area of the region enclosed between the curves $y + 4 = x^2$ and $y = |x|\sqrt{4 - x^2}$ is $2k$, then the value of k is

2. Consider a curve $y = \dfrac{1}{x}$ in first quadrant. If the area bounded by the tangent to the curve $y = \dfrac{1}{x}$ at (α, β), $x = 1$, $x = 2$ and x-axis is maximum such that $(\alpha + \beta) = \dfrac{p}{q}$ (where p and q are relative prime), then $(p + q)$ is (where $1 < \alpha < 2$)

3. Area bounded by the curves $y = e^{x^2}$, the x-axis and the lines $x = 1$, $x = 2$ is given to be a square units. If the area bounded by the curve $y = \sqrt{\ln x}$, the x-axis and the lines $x = \text{e}$ and $x = \text{e}^4$ is expressed as $(pe^4 - qe - \alpha)$, (where p and q are positive integers), then $(p + q)$ is

4. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

5. Let $f(x)$ be a quadratic polynomial and a, b, c be distinct real numbers such that

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

Let V be the point of maximum of the curve $y = f(x)$. If A and B are the points on the curve such that the curve meets the positive x-axis at A and the chord AB subtends a right angle at V, then find the area enclosed by the curve and the chord AB.

6. Area bounded by the straight lines $x^2y - y^3 - x^2 + 5y^2 - 8y + 4 = 0$ is equal to (in sq. units)

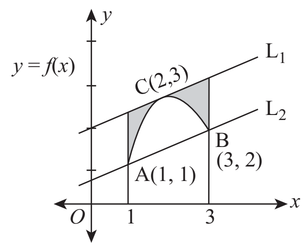
7. Let $g(x) = x^3 - 3k^2x + 16$. If $f(|x|)$ has exactly two distinct real roots, then the area bounded by $y = g(x)$ and x-axis in (in square units).

8. The positive value of parameter a for which the area bounded by parabolas $y = x - ax^2$ and $ay = x^2$ attains its maximum value is

9. Let the function f: $[-4, 4] \to [-1, 1]$ be defined implicitly by the equation $x + 5y - y^5 = 0$. Let A be the area of triangle formed by tangent and normal to $f(x)$ at $x = 0$ and the line $y = 5$, then $\left[\dfrac{A}{10}\right] =$ (where $[\cdot]$ denote greatest integer function.)

10. Let $f(x)$ be a function which satisfy the equation $f(xy) = f(x) + f(x)$ for all $x > 0$, $y > 0$ such that $f'(1) = 2$. Let A be the area of the region bounded by the curves $y = f(x)$, $y = |x^3 - 6x^2 + 11x - 6|$ and $x = 0$, then find value of $\dfrac{28}{17}A$.

11. The following figure shows the graph of a continuous function $y = f(x)$ on the interval $[1, 3]$. The points A, B, C have co-ordinates $(1, 1)$ $(3, 2)$, $(2, 3)$, respectively, and the lines L_1 and L_2 are parallel with L_1 being tangent to the curve at C. If the area under the graph of $y = f(x)$ from $x = 1$ to $x = 3$ is 4 square units, then find the area (in square units) of shaded region.



12. Let $y = f(x)$ be a curve which satisfies the differential equation $\text{e}^x\text{d}y + (x - 1)\,\text{d}x = 0$ and $f(0) = 0$. If area enclosed by the curve, the x-axis and a line $x = \text{c}$, where $(c, f(c))$ is the point of inflection on the curve is $\left(1 - \dfrac{p}{e^q}\right)$, $p, q \in N$, then find the value of $(p + q)$.

13. Let $f(x)$ be a polynomial of degree 3. If the curve $y = f(x)$ has relative extrema at $x = \dfrac{+2}{\sqrt{3}}$ and passes through $(0, 0)$ and $(1, -2)$ dividing the circle $x^2 + y^2 = 4$ in two parts, then the area bounded by $x^2 + y^2 = 4$ and $y \le f(x)$ is $\dfrac{k\pi}{2}$. Find the value of k.

14. Let $P(x)$ be a polynomial function of degree n satisfying $(P(x))^2 P'''(x) = (P''(x))^3P'(x)$ $\forall$ $x \in R$. Let $f(x)$ a polynomial whose degree is same as of $P(x)$. If the area bounded by $y = f(x)$, the x-axis and the ordinates of two local minima is $\dfrac{p}{q}$, where p and q are co-prime then find the value of $(p - q)$. Given $f'(0) = f'(-1) = f'(1) = 0$, $f(0) = 4$ and $f(1) = f(-1) = 3$.

15. Let $y = \operatorname{cosec} x - \cot x$ such that $\dfrac{\frac{\text{d}^2y}{\text{d}x^2}}{y} = \lambda \sec^2 \dfrac{x}{2}$, then 1000λ is equal to

16. If the solution of $x^2y'^2 + 2y^2 = x^2 + 2xyy'$ is $y = x\sin(\ln kx^n)$, then n is equal to $(n \in N)$

17. If $\dfrac{\text{d}y}{\text{d}x} = \dfrac{e^x(\sin x + \cos x)}{(\tan y + y\sec^2 y)}$; and if the curve passes through origin and (x_1, π), where $6 < x_1 < 7$, then $\dfrac{x_1}{\pi}$ is equal to

18. If $(2xy - y^2 - y)\text{d}x = (2xy + x - x^2)\text{d}y$ and $y(1) = 1$, then the value of $12|y(-1)|$ is

19. Let the function $y = f(x)$ passes through point $(0, 1)$ and satisfies the differential equation $2xy^2\text{d}x = \text{e}^x(\text{d}y - y\text{d}x)$. If $f(x) = \dfrac{e^x}{g(x)}$, then $|g(0)|$ is

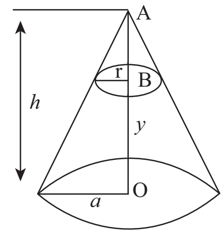
20. If the solutions of the differential $\dfrac{\text{d}y}{\text{d}x} = \dfrac{3yx^2}{x^3 + 2y^4}$ can be expressed as $Px^3y^{-1} = Qy^3 + c$ (where c is constant of integration and P, Q are co-prime numbers), then $(P + Q)$ is

21. Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $\text{P}(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to

22. Let $y'(x) + y(x)g'(x) = g(x)g'(x).y(0) = 0,\ x \in R$, where $f'(x)$ denotes $\dfrac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is

23. If the family of integral curves of the differential equation $\dfrac{dy}{dx} + x^3 y = x$ is cut by the line $x = 2$, the tangents at the points of intersection are concurrent at (λ, μ). Then the value of $\left[\dfrac{\lambda}{\mu}\right]$, where $[\cdot]$ denotes greatest integer function.

24. Let the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$, where g is the gravitational acceleration and y is the height of water level above the hole. If the time required to empty a tank having the shape of a right circular cone of base radius 5 units and height 16 units filled completely with water and having a hole of area $\dfrac{\pi}{4}$ sq. units in the base, is 't' then the value of $(0.09t)$ is (Take $g = 32$)



25. The solutions of $x^2\,dy - y^2dx + xy^2(x - y)dy = 0$ is $\ln\left|\dfrac{x-y}{xy}\right| = \dfrac{y^k}{2} + c$, then the value of k is

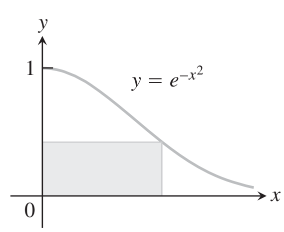
Chapter 6 Additional and Advanced Exercises

1. Area Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

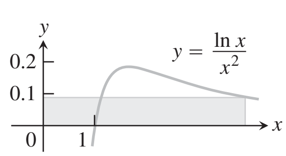
2. a. Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

3. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



4. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



In Exercises 5–8 solve the differential equation.

5. $\dfrac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$

6. $y' = \dfrac{3y(x+1)^2}{y-1}$

7. $yy' = \sec y^2 \sec^2 x$

8. $y\cos^2 x\,dy + \sin x\,dx = 0$

In Exercises 9–12 solve the initial value problem.

9. $\dfrac{dy}{dx} = e^{-x-y-2}, \quad y(0) = -2$

10. $\dfrac{dy}{dx} = \dfrac{y\ln y}{1+x^2}, \quad y(0) = e^2$

11. $x\,dy - \left(y + \sqrt{y}\right)dx = 0, \quad y(1) = 1$

12. $y^{-2}\dfrac{dx}{dy} = \dfrac{e^x}{e^{2x}+1}, \quad y(0) = 1$

13. Assume the hypotheses of Exercise 3, and assume that $y_1(x)$ and $y_2(x)$ are both solutions to the first-order linear equation satisfying the initial condition $y(x_0) = y_0$.

a. Verify that $y(x) = y_1(x) - y_2(x)$ satisfies the initial value problem

$$y' + P(x)y = 0, \quad y(x_0) = 0.$$

b. For the integrating factor $v(x) = e^{\int P(x)\,dx}$, show that

$$\frac{d}{dx}\left(v(x)\left[y_1(x) - y_2(x)\right]\right) = 0.$$

Conclude that $v(x)\left[y_1(x) - y_2(x)\right] \equiv$ constant.

c. From part (a), we have $y_1(x_0) - y_2(x_0) = 0$. Since $v(x) > 0$ for $a < x < b$, use part (b) to establish that $y_1(x) - y_2(x) \equiv 0$ on the interval (a, b). Thus $y_1(x) = y_2(x)$ for all $a < x < b$.

Homogeneous Equations

A first-order differential equation of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is called *homogeneous*. It can be transformed into an equation whose variables are separable by defining the new variable $v = y/x$. Then, $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution into the original differential equation and collecting terms with like variables then gives the separable equation

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0.$$

After solving this separable equation, the solution of the original equation is obtained when we replace v by y/x.

Solve the homogeneous equations in Exercises 14–19. First put the equation in the form of a homogeneous equation.

14. $(x^2 + y^2)\,dx + xy\,dy = 0$

15. $x^2\,dy + (y^2 - xy)\,dx = 0$

16. $(xe^{y/x} + y)\,dx - x\,dy = 0$

17. $(x + y)\,dy + (x - y)\,dx = 0$

18. $y' = \dfrac{y}{x} + \cos\dfrac{y - x}{x}$

19. $\left(x\sin\dfrac{y}{x} - y\cos\dfrac{y}{x}\right)dx + x\cos\dfrac{y}{x}\,dy = 0$

20. Find $f(\pi/2)$ from the following information.

i) f is positive and continuous.

ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a.$$

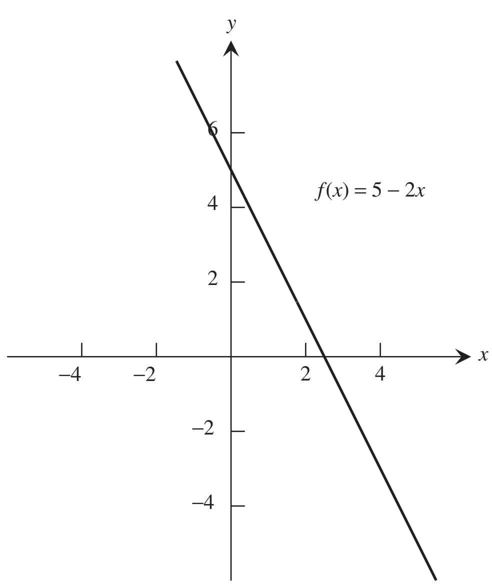
21. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

Answers to Exercises

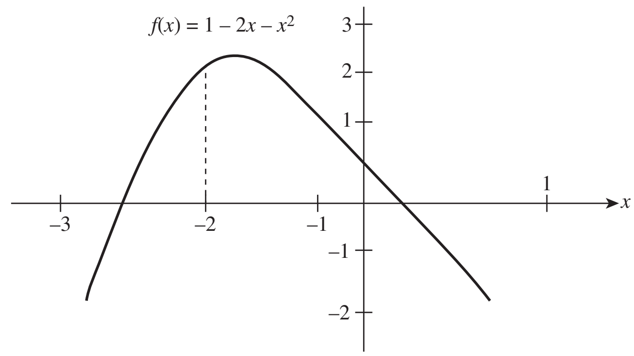
Chapter 1

Exercises 1.1

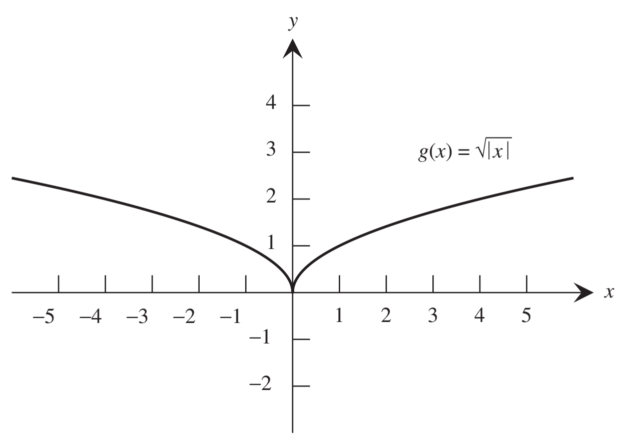
1. $D:[0,\infty); R:(-\infty,1]$
2. $D:(-\infty,0]\cup[3,\infty); R:[0,\infty)$
3. $D:(-\infty,3)\cup(3,\infty); R:(-\infty,0)\cup(0,\infty)$
4. $D: (-\infty,-4)\cup(-4,4)\cup(4,\infty);$ $R:(-\infty,\frac{-1}{8}]\cup(0,\infty)$
5. **(a)** Not a function of x because some values of x have two values of y
   (b) A function of x because for every x there is only one possible y.
6. **(a)** Not the graph of a function of x since it fails the vertical line test.
   (b) Not the graph of a function of x since it fails the vertical line test.
7. $A=\frac{\sqrt{3}}{4}x^2, p=3x$
8. s = side length
   $\Rightarrow s^2+s^2=d^2 \Rightarrow s=\frac{d}{\sqrt{2}}$; and area is $a=s^2 \Rightarrow a=\frac{1}{2}d^2$
9. $x=\frac{d}{\sqrt{3}}, A=2d^2, V=\frac{d^3}{3\sqrt{3}}$
10. The coordinates of P are $(x, \sqrt{x})$, so the slope of the line joining P to the origin is $m=\frac{\sqrt{x}}{x}=\frac{1}{\sqrt{x}}$ $(x>0)$. Thus, $(x, \sqrt{x})=\left(\frac{1}{m^2},\frac{1}{m}\right)$.
11. $L=\frac{\sqrt{20x^2-20x+25}}{4}$
12. $L=\sqrt{y^4-y^2+1}$
13. $(-\infty,\infty)$



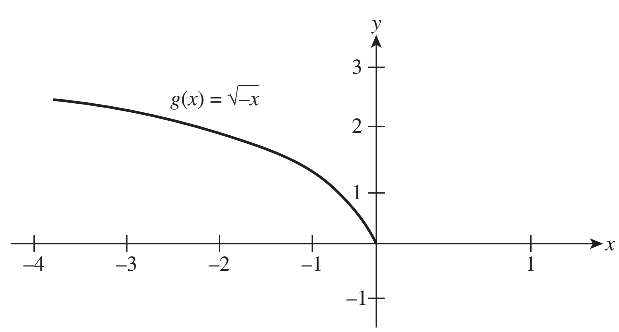
14. The domain is $(-\infty,\infty)$



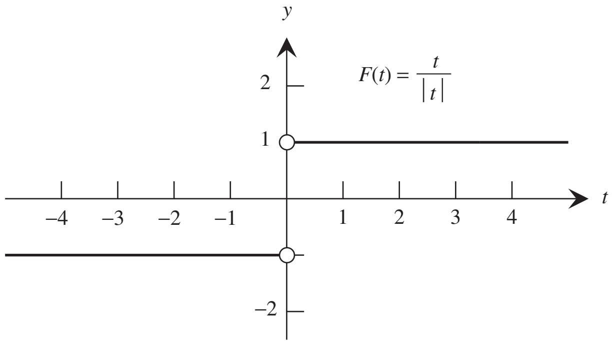
15. $(-\infty,\infty)$



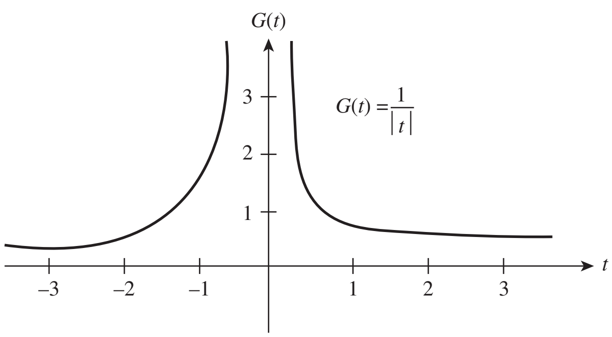
16. The domain is $(-\infty,0)$.



17. $(-\infty,0)\cup(0,\infty)$



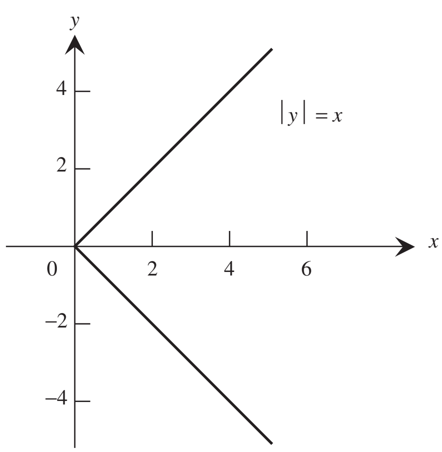
18. The domain is $(-\infty,0)\cup(0,\infty)$



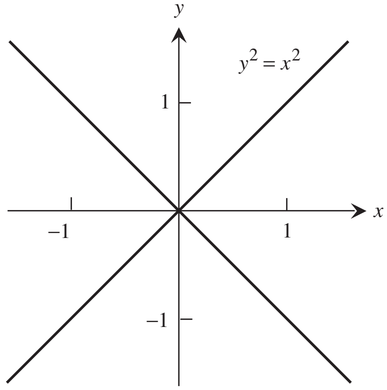
19. $(-\infty,-5)\cup(-5,-3]\cup[3,5)\cup(5,\infty)$

20. $R:[2,3)$

21. **(a)** For each positive value of x, there are two values of y.

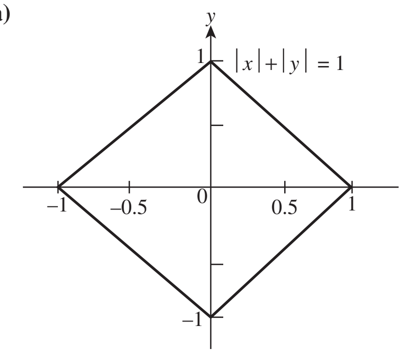


(b) For each value of $x\neq 0$, there are two values of y.

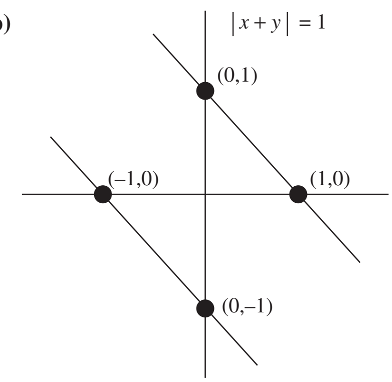


22. Neither graph passes the vertical line test.

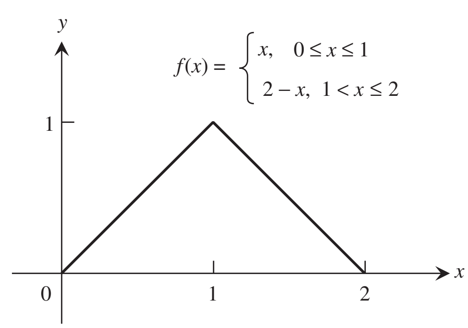
(a)



(b)

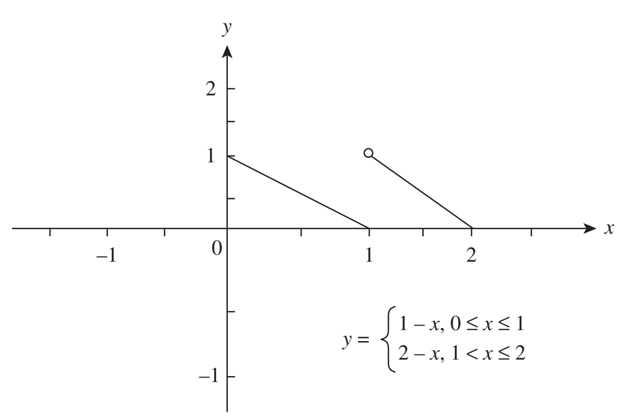


23.

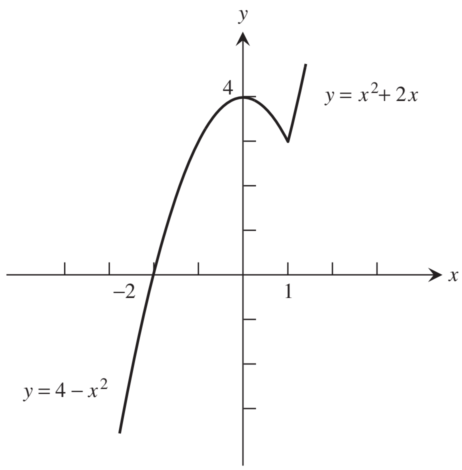


24.

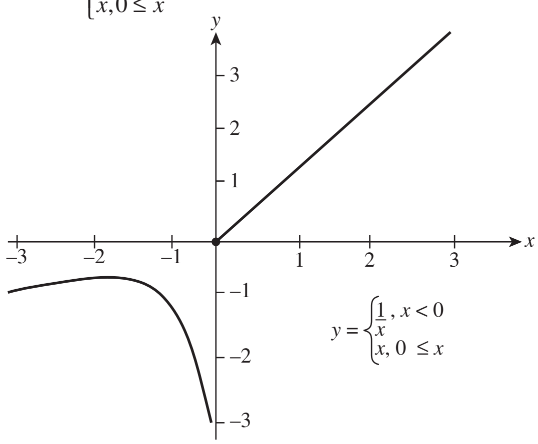
x	0	1	2
y	1	0	0



25.



26. $G(x) = \begin{cases} \dfrac{1}{x}, x < 0 \\ x, 0 \le x \end{cases}$



27. **(a)** $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x+2 & 1 < x \le 2 \end{cases}$

(b) $f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \\ 2, & 2 \le x \le 3 \\ 0, & 3 \le x \le 4 \end{cases}$

28. **(a)** $f(x) = \begin{cases} -x+2, 0 < x \le 2 \\ -\dfrac{1}{3}x + \dfrac{5}{3}, 2 < x \le 5 \end{cases}$

(b) $f(x) = \begin{cases} -3x-3, -1 < x \le 0 \\ -2x+3, 0 < x \le 2 \end{cases}$

29. **(a)** $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \\ -\dfrac{1}{2}x + \dfrac{3}{2}, & 1 < x < 3 \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{1}{2}x, & -2 \le x \le 0 \\ -2x+2, & 0 < x \le 1 \\ -1, & 1 < x \le 3 \end{cases}$

30. **(a)** $f(x) = \begin{cases} 0, & 0 \le x \le \dfrac{T}{2} \\ \dfrac{2}{T}x - 1, & \dfrac{T}{2} < x \le T \end{cases}$

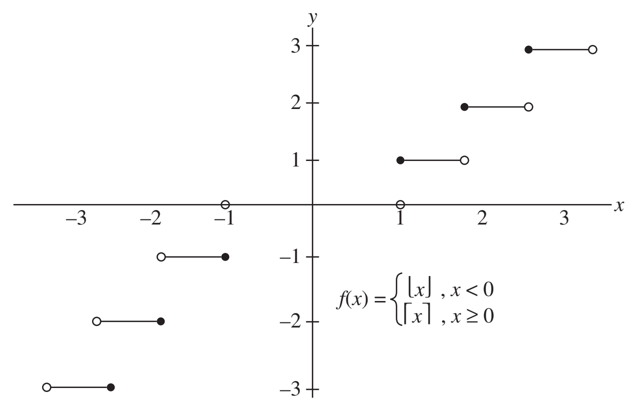
(b) $f(x) = \begin{cases} A, & 0 \le x < \dfrac{T}{2} \\ -A, & \dfrac{T}{2} \le x < T \\ A, & T \le x \le \dfrac{3T}{2} \\ -A & \dfrac{3T}{2} \le x \le 2T \end{cases}$

31. **(a)** $0 \le x < 1$
(b) $-1 < x \le 0$

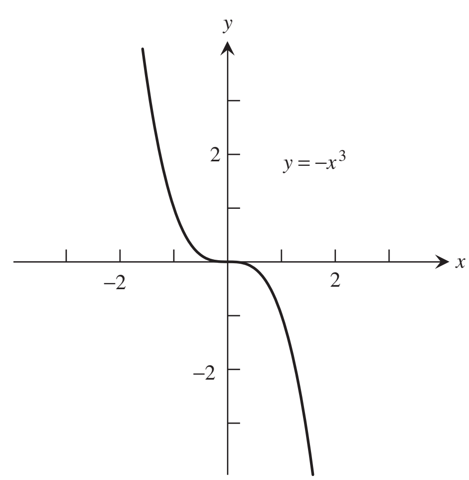
32. $[x] = [x]$ only when x is an integer.

33. Yes

34. To find $f(x)$ you delete the decimal or fractional portion of x, leaving only the integer part.



35. Symmetric about the origin

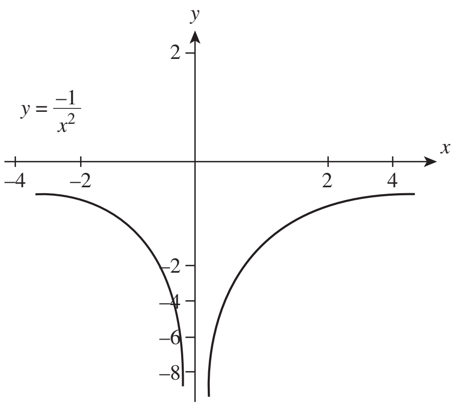


Dec: $-\infty < x < \infty$

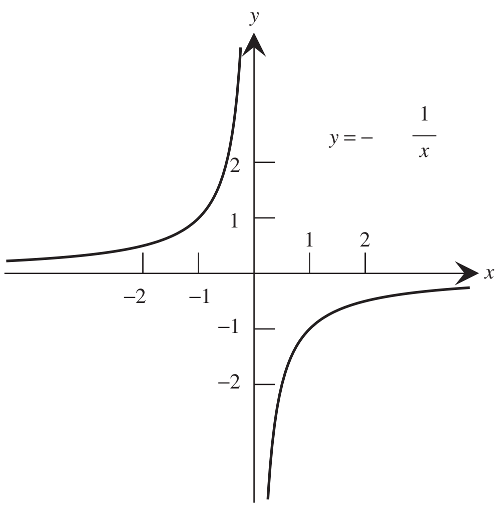
36. Symmetric about the y-axis

Dec: $-\infty < x < 0$

Inc: $0 < x < \infty$

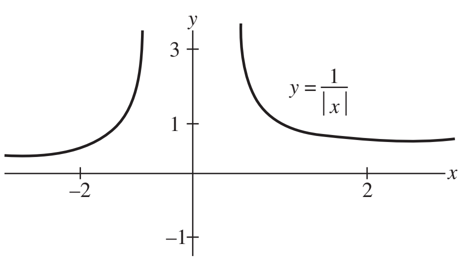


37. Symmetric about the origin

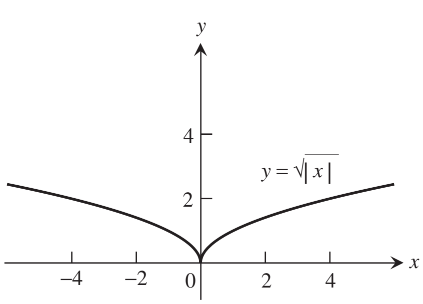


Inc. $-\infty < x < 0$ and $0 < x < \infty$

38. Symmetric about y-axis Dec: $0 < x < \infty$ Inc: $-\infty < x < 0$

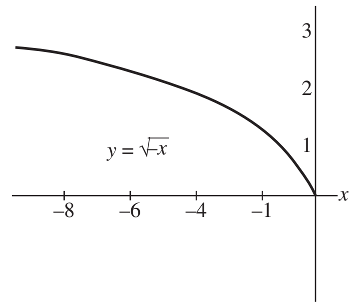


39. Symmetric about the y-axis

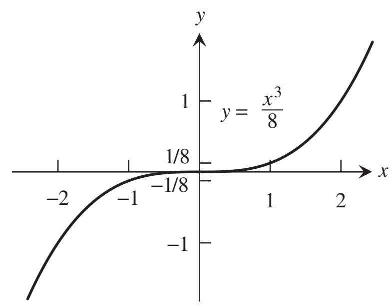


Dec. $-\infty < x \leq 0$; Inc. $0 \leq x < \infty$

40. No symmetry Dec: $-\infty < x \leq 0$ Inc: nowhere

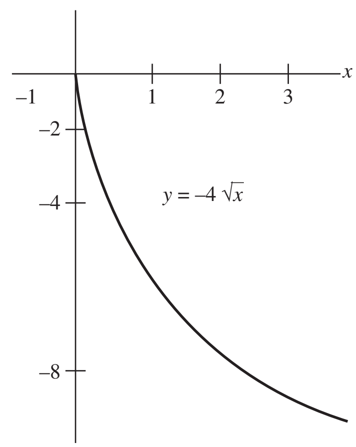


41. Symmetric about the origin

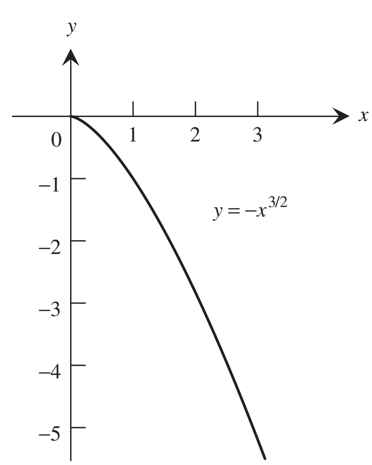


Inc: $-\infty < x < \infty$

42 No symmetry Dec: $0 \leq x < \infty$ Inc: nowhere

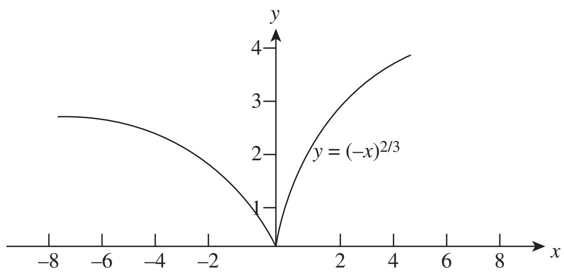


43. No symmetry



Dec. $0 \leq x < \infty$

44. Symmetric about the y-axis
Dec: $-\infty < x \le 0$
Inc: $0 \le x < \infty$



45. Even
46. Odd
47. Even
48. Neither even nor odd
49. Odd
50. Even
51. Even
52. Odd
53. Neither even nor odd
54. Even
55. Neither even nor odd
56. Even
57. $t = 180$
58. 4000 Joules
59. $s = 2.4$
60. $V \approx 628.2\text{ in}^3$
61. $V = x(14 - 2x)(22 - 2x)$
62. **(a)** $y = f(x) = -x + 1;\ x \in [0,1]$
(b) $A = 2x(1 - x);\ x \in [0,1]$
63. **(a)** h **(b)** f **(c)** g
64. **(a)** f **(b)** g **(c)** h
65. **(a)** $(-2,0) \cup (4,\infty)$
66. $x \in (-\infty,-5) \cup (-1,1)$
67. $R(x) = (40 + 5x)(300 – 25x)$
68. $c(h) = 5h\left(\sqrt{2} + 2\right)$
69. **(a)** $C(x) = 180\sqrt{800^2 + x^2} + 100(10{,}560 - x)$
(b) $C(0) = \$1{,}200{,}000$
(c) $C(500) = \$1{,}175{,}812$
(d) $C(1000) = \$1.186{,}512$
(e) $C(2500) \approx \$1{,}278{,}479$
(f) $C(3000) \approx \$1{,}314{,}870$
Values beyond this are all larger. It would appear that the least expensive location is less than 2000 feet from the point *P*.

Exercises 1.2

1. $D_f : -\infty < x < \infty, D_g : x \ge 1, R_f : -\infty < y < \infty,$
$R_g : y \ge 0, D_{f+g} = D_{f \cdot g} = D_g, R_{f+g}; y \ge 1, R_{f \cdot g} : y \ge 0$

2. $D_f : x + 1 \ge 0 \Rightarrow x \ge -1, D_g : x - 1 \ge 0 \Rightarrow x \ge 1.$
Therefore $D_{f+g} = D_g : x \ge 1$
$R_f = R_g : y \ge 0, R_{f+g} : y \ge \sqrt{2}, R_{fg} : y \ge 0$

3. $D_f : -\infty < x < \infty, D_g : -\infty < x < \infty, R_f : y = 2,$
$R_g : y \ge 1, D_{f/g} : -\infty < x < \infty, R_{f/g} : 0 < y \le 2,$
$D_{g/f} : -\infty < x < \infty, R_{g/f} : y \ge 1/2$

4. $D_f : -\infty < x < \infty, D_g : x \ge 0, D_{f/g} : x \ge 0, D_{g/f} : x \ge 0$
$R_f : y = 1, R_g : y \ge 1, R_{f/g} : 0 < y \le 1, R_{g/f} : 1 \le y < \infty$

5. **(a)** 2 **(b)** 22 **(c)** $x^2 + 2$ **(d)** $x^2 + 10x + 22$
(e) 5 **(f)** –2 **(g)** $x + 10$ **(h)** $x^4 - 6x^2 + 6$

6. **(a)** $\dfrac{-1}{3}$ **(b)** 2 (c) $\dfrac{1}{x+1} - 1 = \dfrac{-x}{x+1}$ (d) $\dfrac{1}{x}$ **(e)** 0
(f) $\dfrac{3}{4}$ **(g)** $x - 2$ **(h)** $\dfrac{1}{\dfrac{1}{x+1} + 1} = \dfrac{1}{\dfrac{x+2}{x+1}} = \dfrac{x+1}{x+2}$

7. $13 - 3x$
8. $6x^2 + 1$
9. $\sqrt{\dfrac{5x+1}{4x+1}}$
10. $\dfrac{8 - 3x}{7 - 2x}$

11. **(a)** $f(g(x))$ **(b)** $j(g(x))$ **(c)** $g(g(x))$
(d) $j(j(x))$ **(e)** $g(h(f(x)))$ **(f)** $h(j(f(x)))$

12. **(a)** $(f \circ j)(x)$ **(b)** $(g \circ h)(x)$ **(c)** $(h \circ h)(x)$
(d) $(f \circ f)(x)$ **(e)** $(j \circ g \circ f)(x)$ **(f)** $(g \circ f \circ h)(x)$

13.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	$x - 7$	$\sqrt{x}$	$\sqrt{x - 7}$
(b)	$x + 2$	$3x$	$3x + 6$
(c)	x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(d)	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	x
(e)	$\dfrac{1}{x-1}$	$1 + \dfrac{1}{x}$	x
(f)	$\dfrac{1}{x}$	$\dfrac{1}{x}$	x

14. **(a)** $\dfrac{1}{|x-1|}$ **(b)** $x + 1$ **(c)** x^2 (d) x^2

15. **(a)** 1 **(b)** 2 **(c)** –2
(d) 0 **(e)** –1 **(f)** 0

16. **(a)** –1 **(b)** –1 **(c)** 1
(d) 0 **(e)** 2 **(f)** $\dfrac{-1}{2}$

17. **(a)** $f(g(x)) = \sqrt{\dfrac{1}{x} + 1}, g(f(x)) = \dfrac{1}{\sqrt{x+1}}$
(b) $D_{f \circ g} = (-\infty,-1] \cup (0,\infty), D_{g \circ f} = (-1,\infty)$
(c) $R_{f \circ g} = [0.1) \cup (1,\infty), R_{g \circ f} = (0,\infty)$

18. **(a)** $(f \circ g)(x) = f(g(x)) = 1 - 2\sqrt{x} + x$
$(g \circ f)(x) = g(f(x)) = 1 - |x|$
(b) $D(f \circ g) : [0,\infty), D(g \circ f) : (-\infty,\infty)$
(c) $R(f \circ g) : (0,\infty), R(g \circ f) : (-\infty,1]$

19. $g(x) = \dfrac{2x}{x-1}$ **20.** $g(x) = \sqrt[3]{\dfrac{x+6}{2}}$

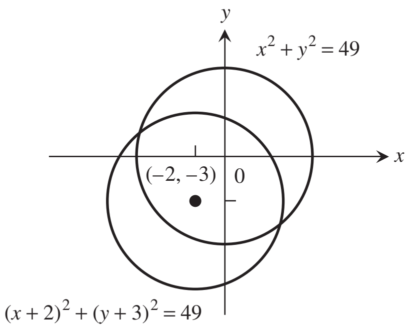
21. **(a)** $y = -(x + 7)^2$ (b) $y = -(x - 4)^2$

22. **(a)** $y = x^2 + 3$ **(b)** $y = x^2 - 5$

23. **(a)** Position 4 **(b)** Position 1
(c) Position 2 **(d)** Position 3

24. **(a)** $y = -(x - 1)^2 + 4$ **(b)** $y = -(x + 2)^2 + 3$
(c) $y = -(x + 4)^2 - 1$ **(d)** $y = -(x - 2)^2$

25. $(x+2)^2+(y+3)^2=49$



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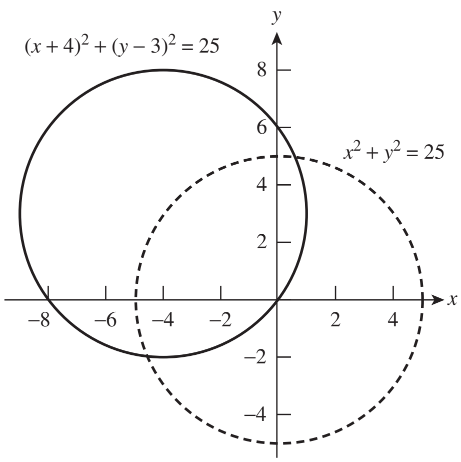
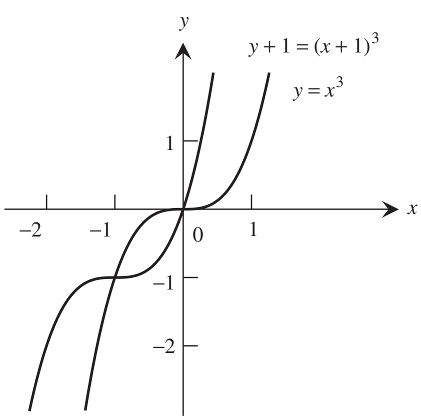


27. $y+1=(x+1)^3$



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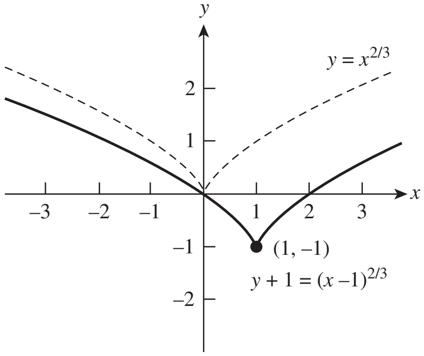
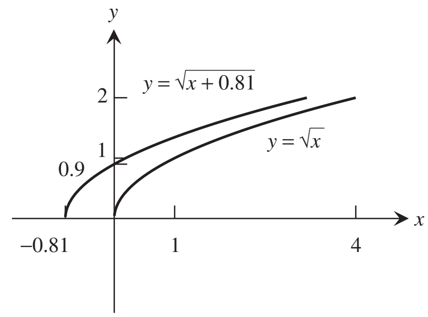
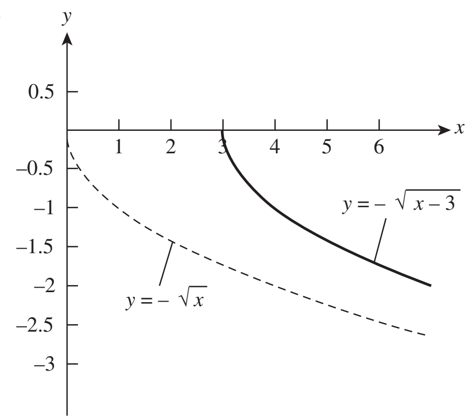


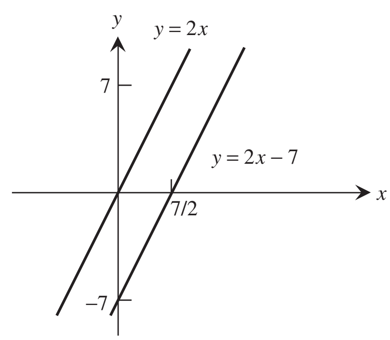
29. $y=\sqrt{x+0.81}$



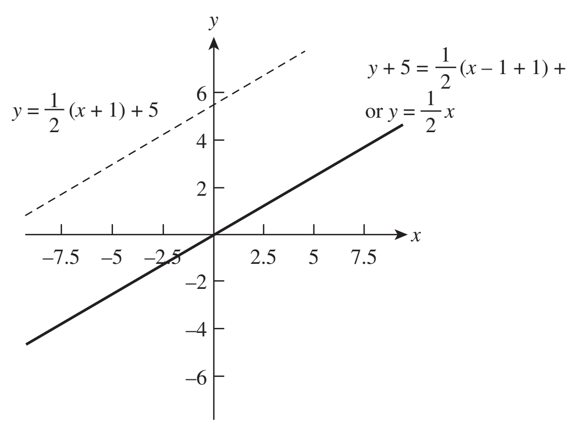
30.



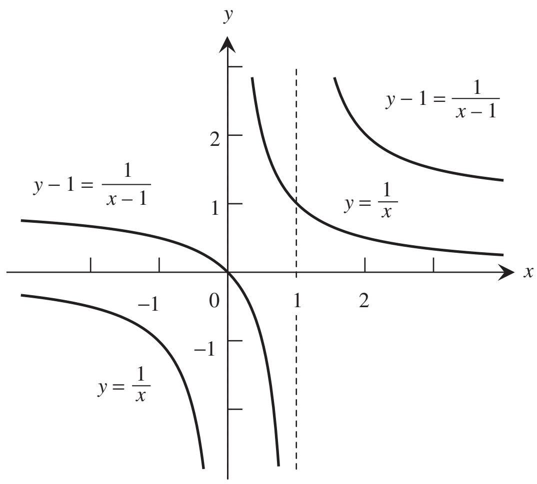
31. $y=2x$



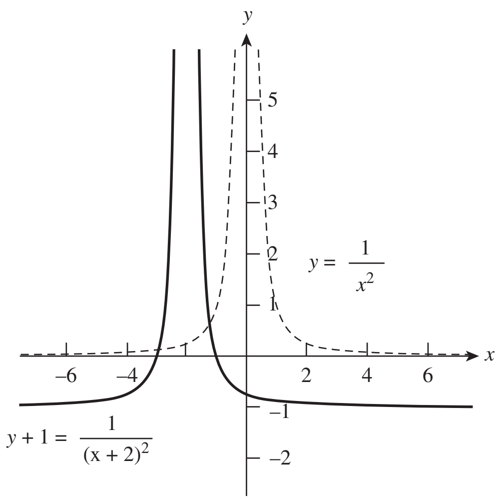
32.



33. $y-1=\dfrac{1}{x-1}$



34.



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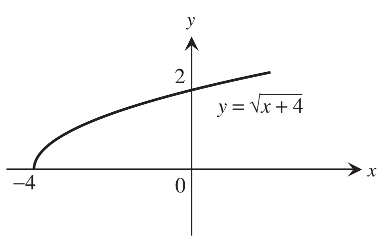

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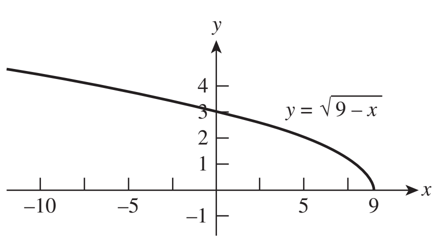

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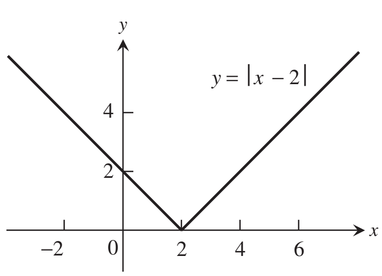

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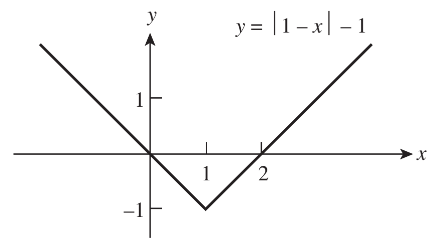
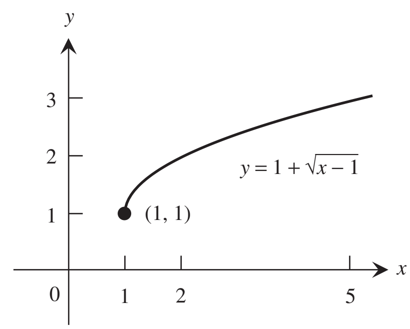


39.



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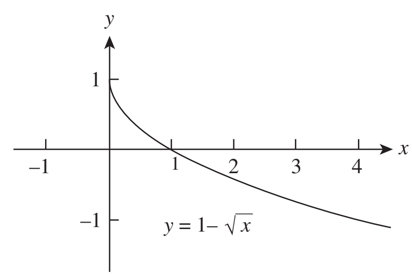

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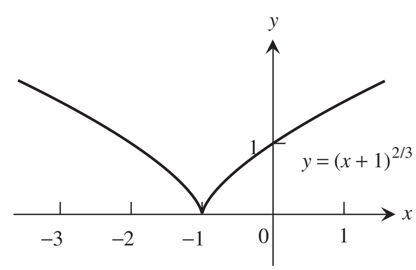

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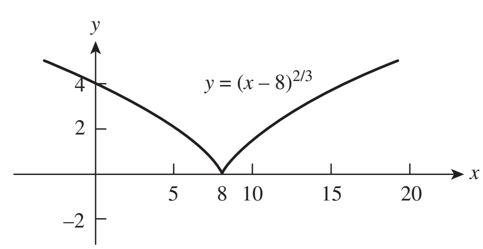

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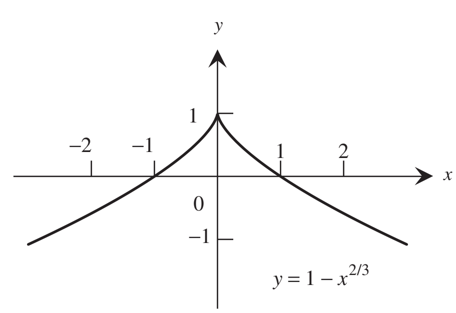

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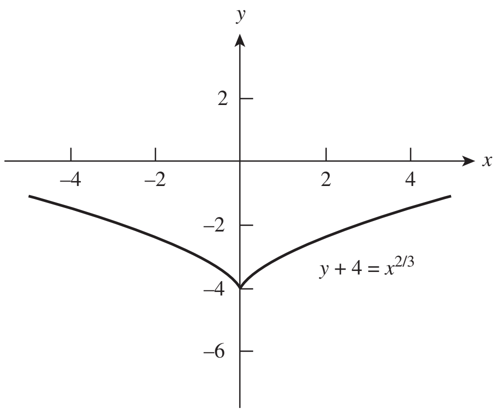

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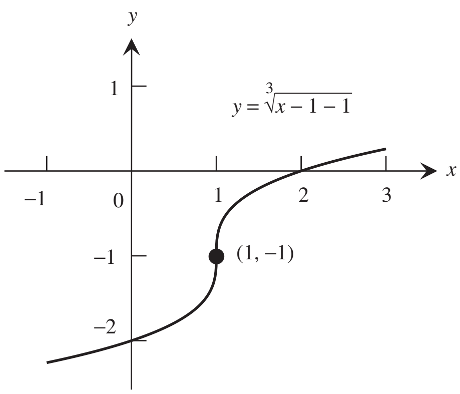

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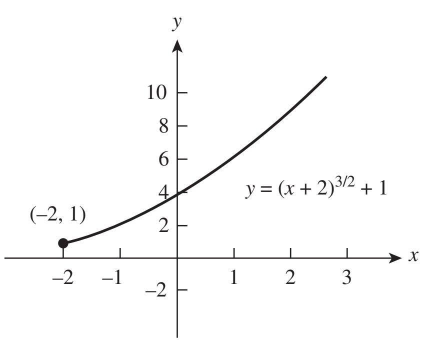

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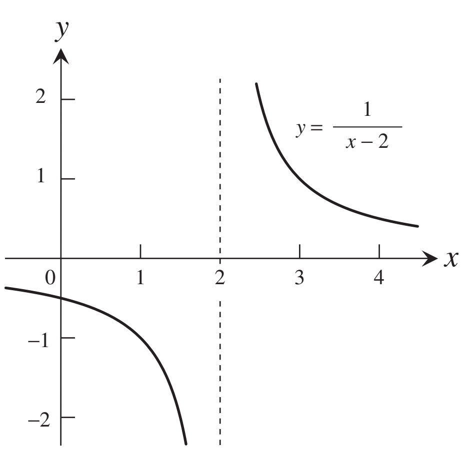

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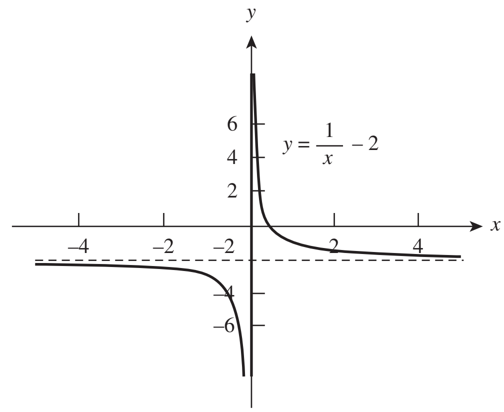

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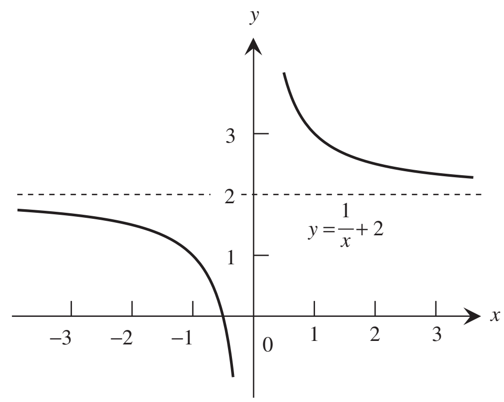

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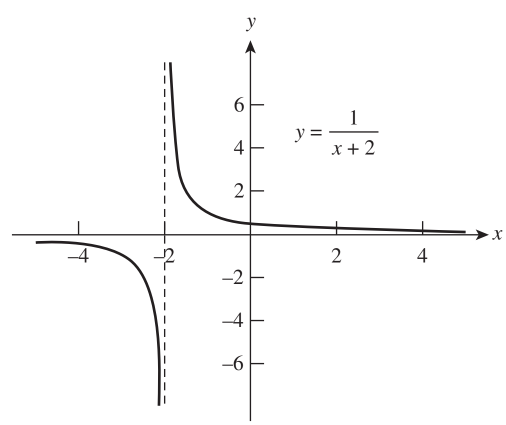

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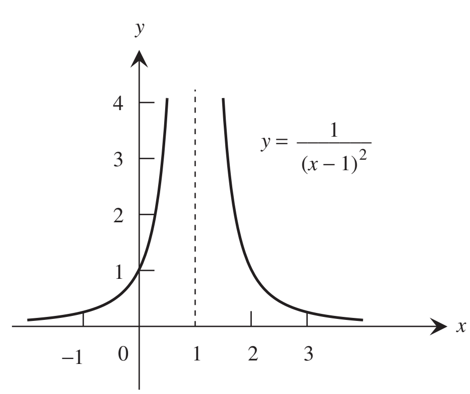
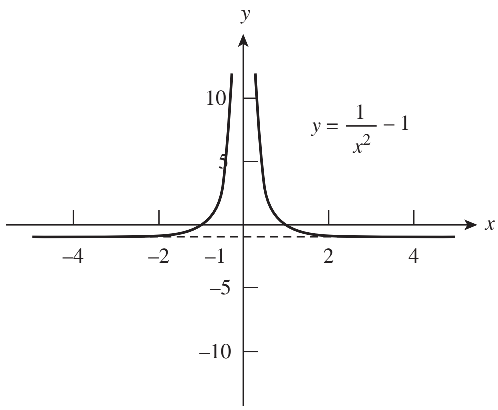
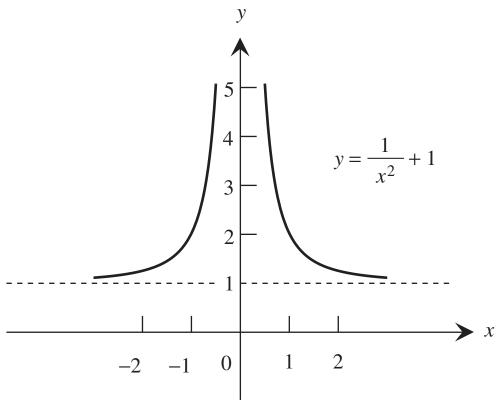


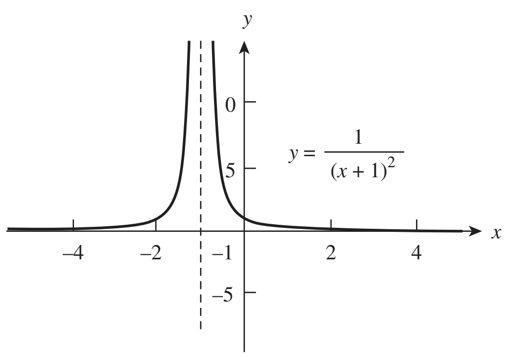
52.



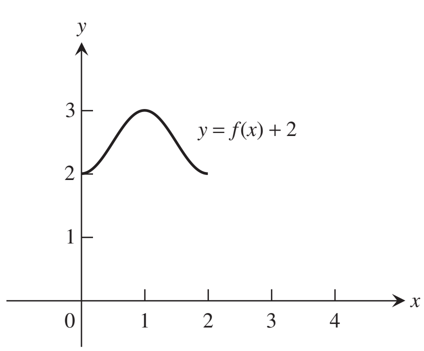
53.



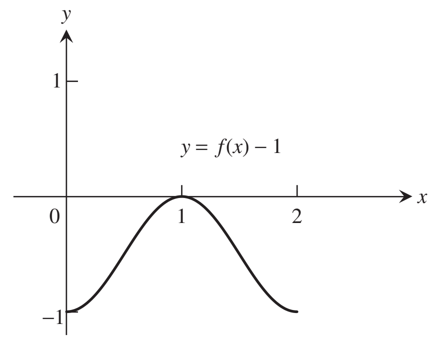
54.



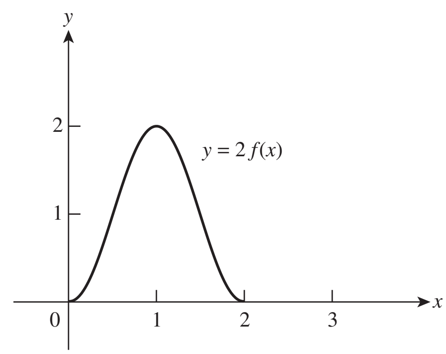
55. **(a)** $D : [0, 2], R : [2, 3]$



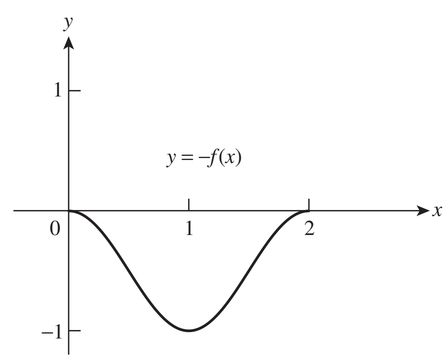
(b) D: $[0, 2]$, R : $[-1, 0]$



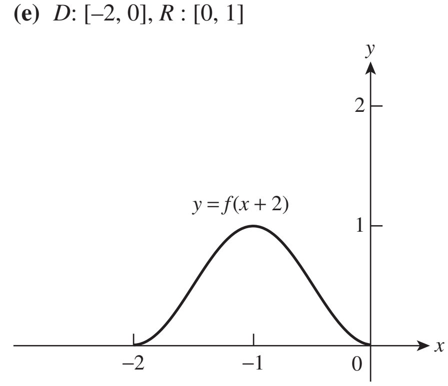
(c) D: $[0, 2]$, R : $[0, 2]$



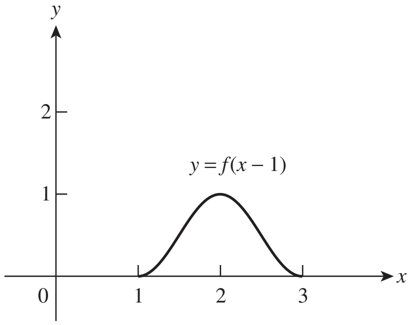
(d) D: $[0, 2]$, R : $[-1, 0]$



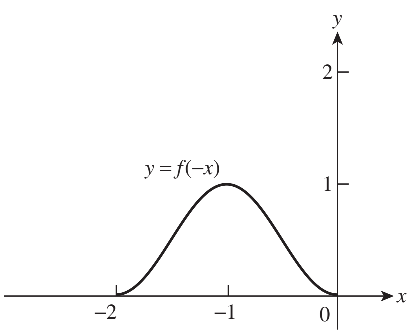
(e) D: $[-2, 0]$, R : $[0, 1]$



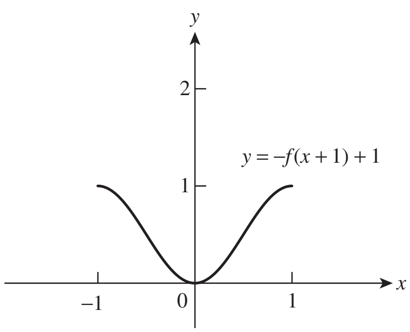
(f) D: [1, 3], R : [0, 1]



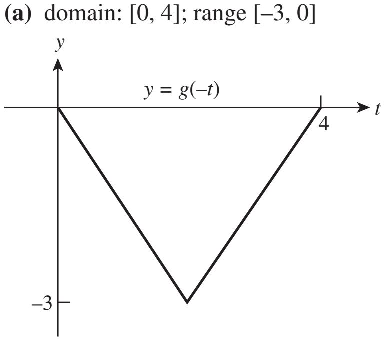
(g) D: [–2, 0], R : [0, 1]



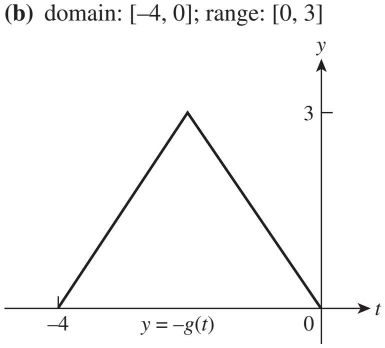
(h) D: [–1, 1], R : [0, 1]



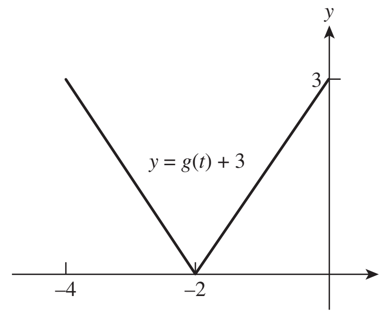
56. (a) domain: [0, 4]; range [–3, 0]



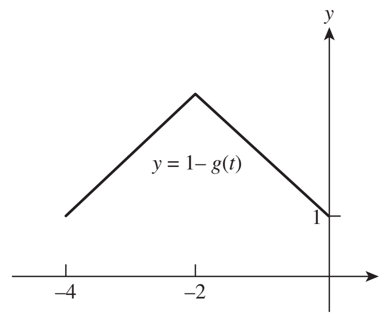
(b) domain: [–4, 0]; range: [0, 3]



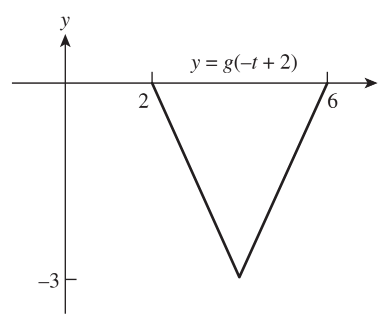
(c) domain: [–4, 0]; range: [0, 3]



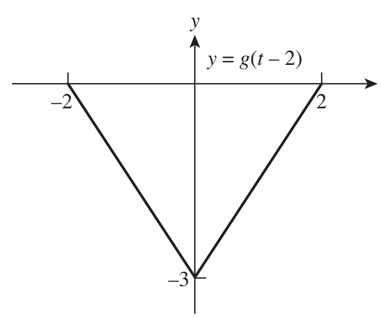
(d) domain: [–4, 0]; range: [1, 4]



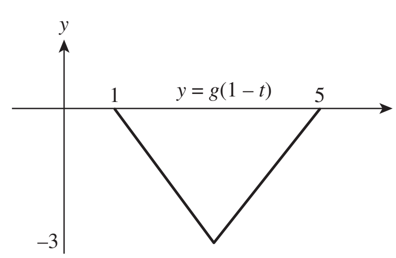
(e) domain: [2, 4]; range: [–3, 0]



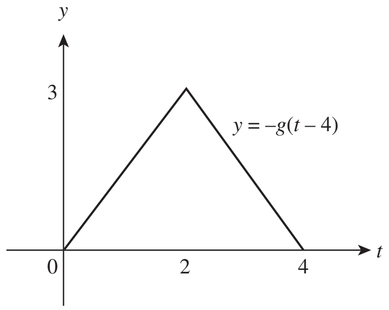
(f) domain: [–2, 2]; range: [–3, 0]



(g) domain: [1, 5]; range: [–3, 0]



(h) domain: [0, 4]; range: [0, 3]



57. $y = 3x^2 - 3$

58. $y = (2x)^2 - 1 = 4x^2 - 1$

59. $y = \frac{1}{2} + \frac{1}{2x^2}$

60. $y = 1 + \frac{1}{(x/3)^2} = 1 + \frac{9}{x^2}$

61. $y = \sqrt{4x+1}$

62. $y = 3\sqrt{x+1}$

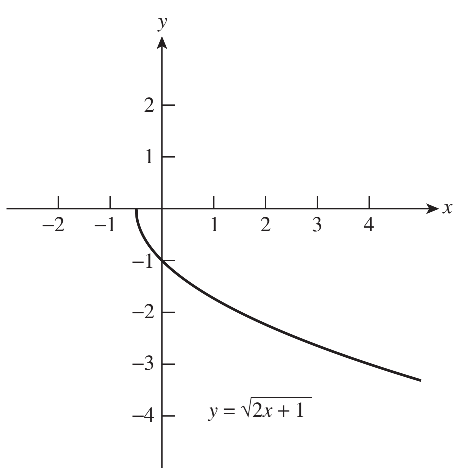
63. $y = \sqrt{4 - \frac{x^2}{4}}$

64. $y = \frac{1}{3}\sqrt{4 - x^2}$

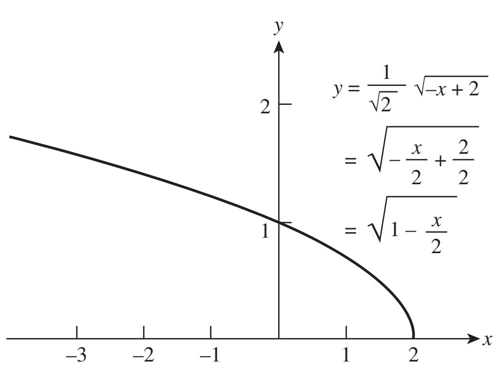
65. $y = 1 - 27x^3$

66. $y = 1 - \left(\frac{x}{2}\right)^3 = 1 - \frac{x^3}{8}$

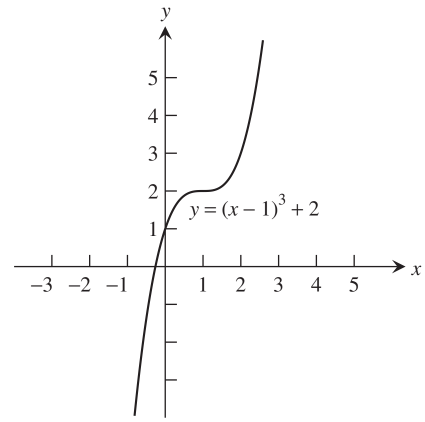
67.



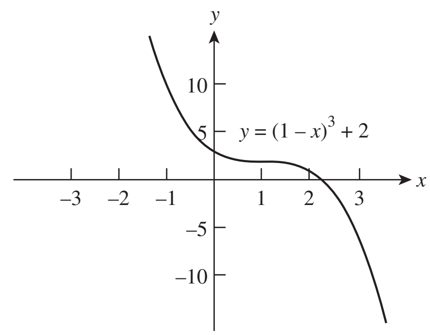
68.



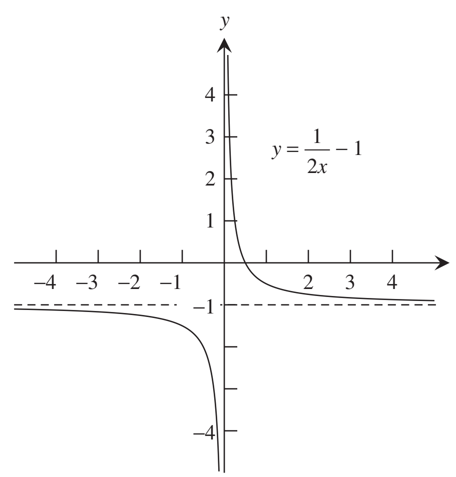
69.



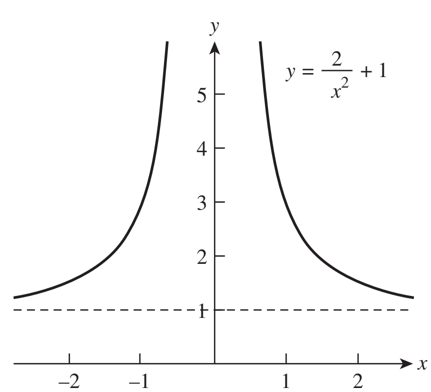
70.



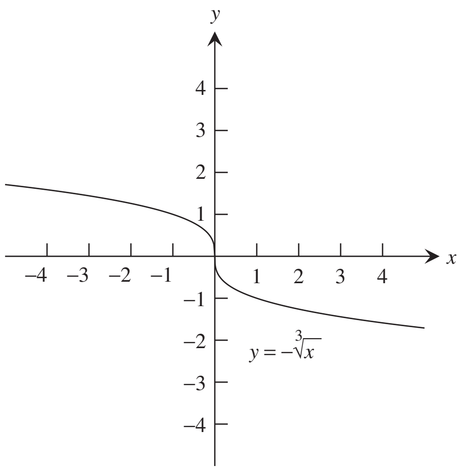
71.



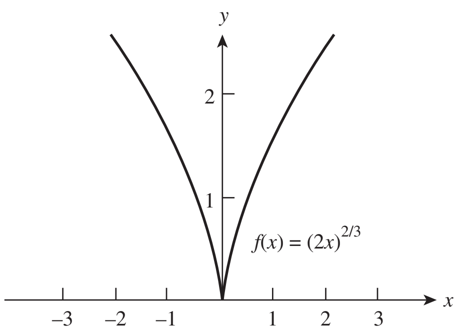
72.



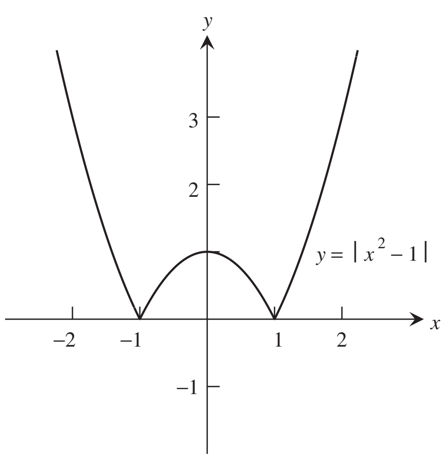
73.



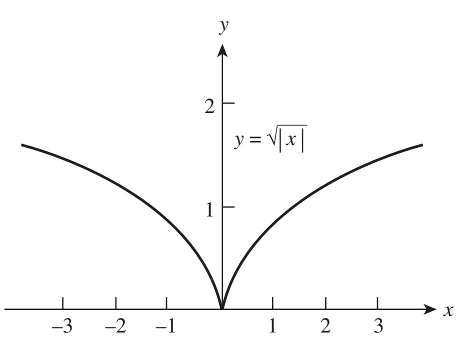
74.



75.



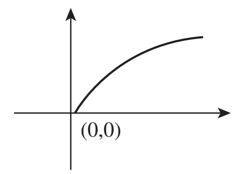
76.



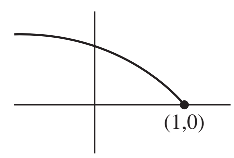
77. **(a)** Odd **(b)** Odd
(c) Odd **(d)** Even
(e) Even **(f)** Even
(g) Even **(g)** Even
(h) Even **(i)** Odd

78. Yes, $f(x) = 0$ is both even and odd since $f(-x) = 0 = f(x)$ and $f(-x) = 0 = -f(x)$.

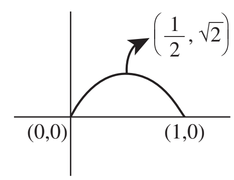
79. Graph of $f(x) = \sqrt{x}$



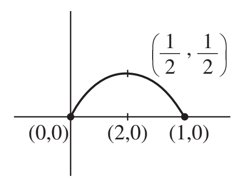
Graph of $g(x) = \sqrt{1-x}$



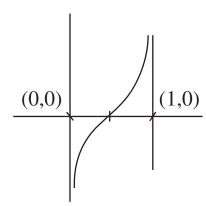
(a) Graph of $f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$



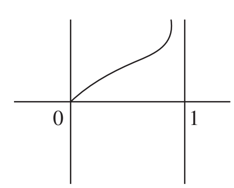
(b) Graph of $f(x) \cdot g(x) = \sqrt{x - x^2}$



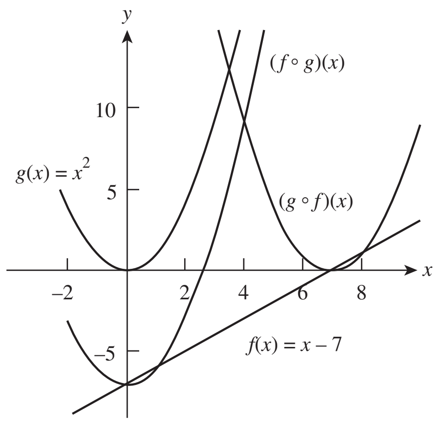
(c) Graph of $f(x) - g(x) = \sqrt{x} - \sqrt{1-x}$



(d) Graph of $\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x}}{\sqrt{1-x}}$



80.



Exercises 1.3

1. **(a)** 8π m **(b)** $\frac{55\pi}{9}$ m

2. $\frac{5\pi}{4}$ radians and 225°.

3. 8.4 in.

4. 0.6 radians or $\approx 34°$.

5.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$	0	$-\frac{\sqrt{3}}{2}$	0	1	$\frac{1}{\sqrt{2}}$
$\cos\theta$	–1	$-\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	0	$\sqrt{3}$	0	UND	–1
$\cot\theta$	UND	$\frac{1}{\sqrt{3}}$	UND	0	–1
$\sec\theta$	–1	–2	1	UND	$-\sqrt{2}$
$\mathrm{cosec}\,\theta$	UND	$-\frac{2}{\sqrt{3}}$	UND	1	$\sqrt{2}$

6.

θ	$-\frac{3\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{6}$
$\sin\theta$	1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$
$\tan\theta$	UND	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	1	$-\frac{1}{\sqrt{3}}$
$\cot\theta$	0	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	1	$-\sqrt{3}$
$\sec\theta$	UND	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	$-\frac{2}{\sqrt{3}}$
$\mathrm{cosec}\,\theta$	1	$-\frac{2}{\sqrt{3}}$	–2	$\sqrt{2}$	2

7. $\cos x = -4/5, \tan x = -3/4$

8. $\sin x = \frac{2}{\sqrt{5}}, \cos x = \frac{1}{\sqrt{5}}$

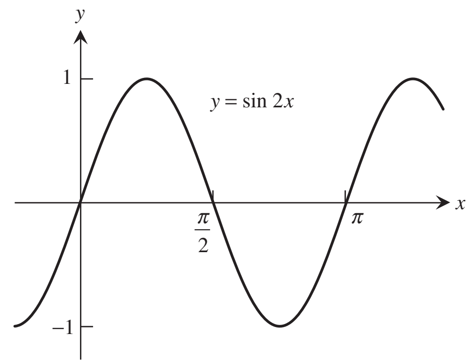
9. $\sin x = -\frac{\sqrt{8}}{3}$, $\tan x = -\sqrt{8}$

10. $\sin x = \frac{12}{13}, \tan x = -\frac{12}{5}$

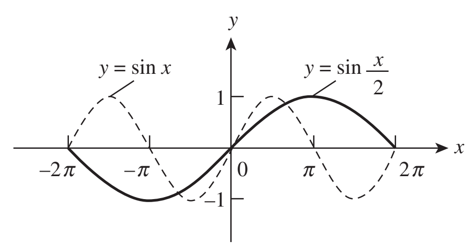
11. $\sin x = -\frac{1}{\sqrt{5}}, \cos x = -\frac{2}{\sqrt{5}}$

12. $\cos x = -\frac{\sqrt{3}}{2}, \tan x = \frac{1}{\sqrt{3}}$

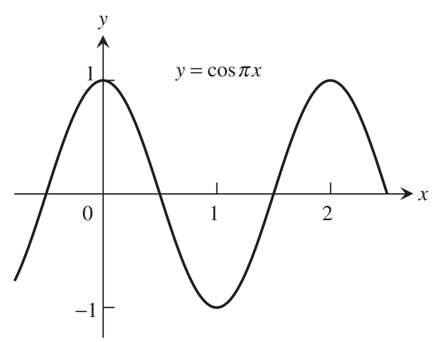
13. Period π



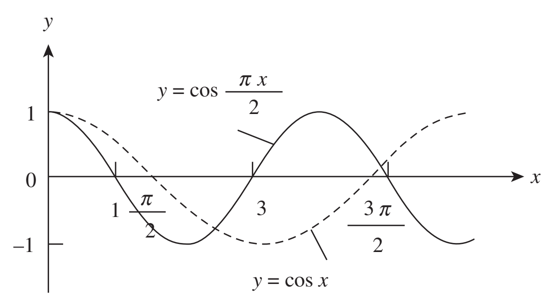
14. Period = 4π



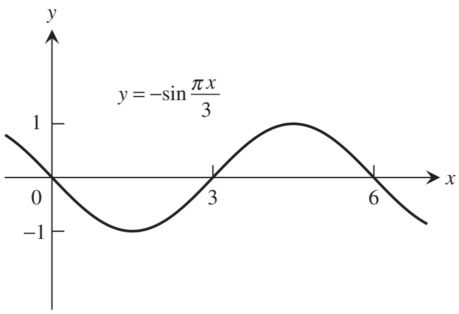
15. Period 2



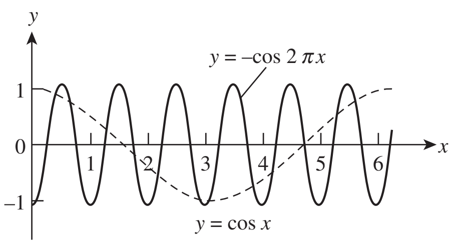
16. Period = 4



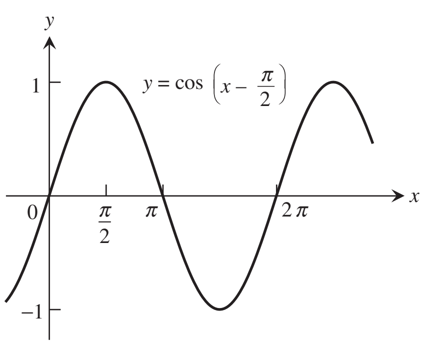
17. Period 6



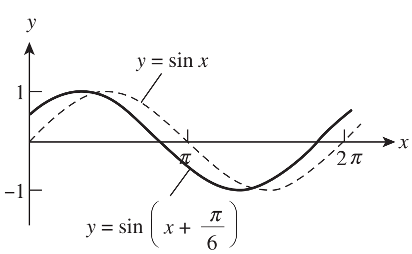
18. Period = 1



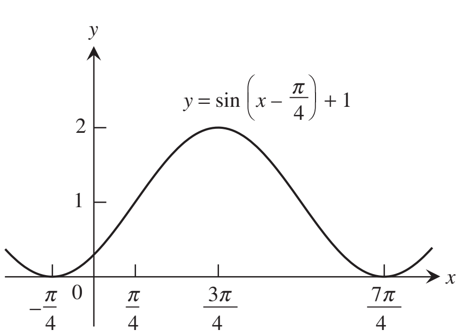
19. Period 2π



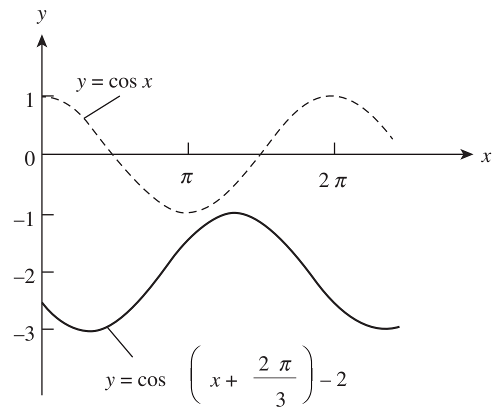
20. Period = 2π



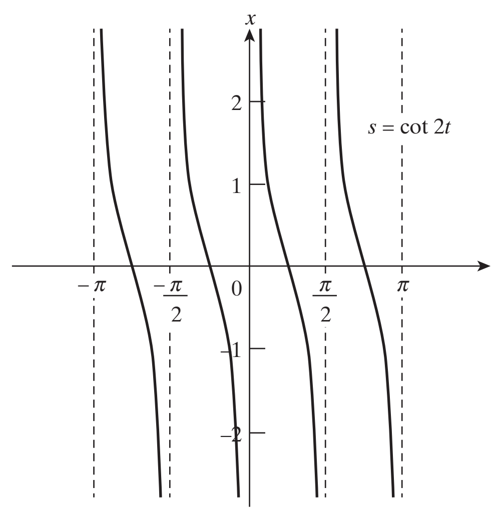
21. Period 2π



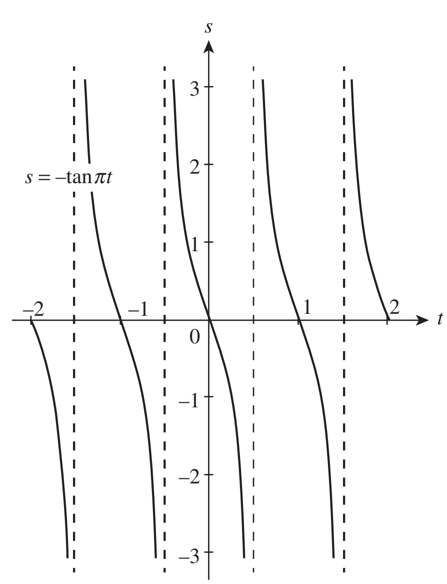
22. Period = 2π



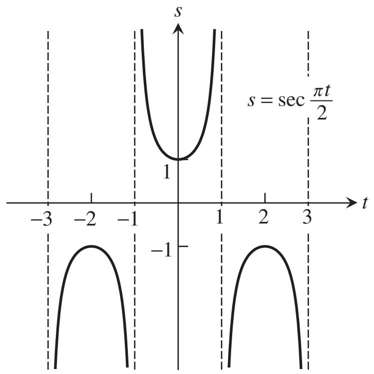
23. Period $\pi/2$, symmetric about the origin



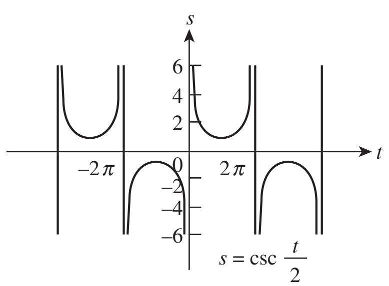
24. Period = 1, symmetric about the origin.

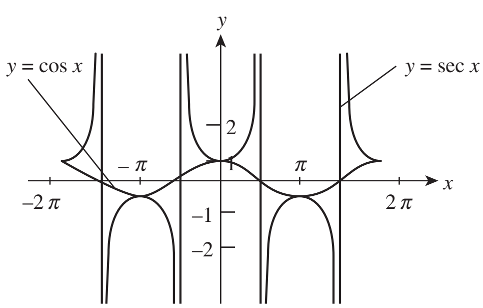


25. Period 4, symmetric about the y-axis.

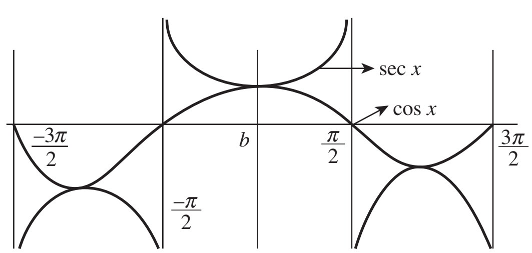


26. Period = 4π, symmetric about the origin.



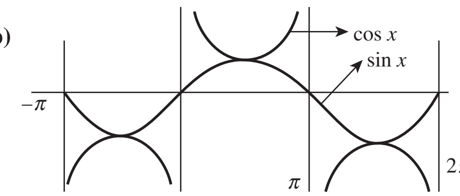


27 **(a)** Clearly $\sec x = \dfrac{1}{\cos x}$. Hence $\cos x$ and $\sec x$ will have same sign wherever defined. Also wherever defined if $\cos x$ is increasing sec x is decreasing and vice-versa.



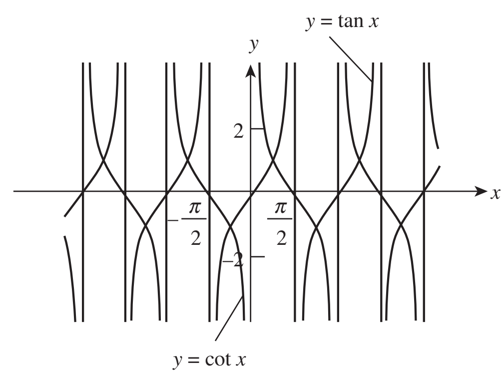
(a)

(b)

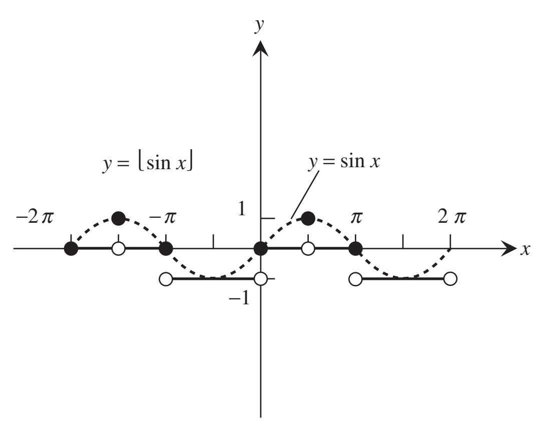


(b)

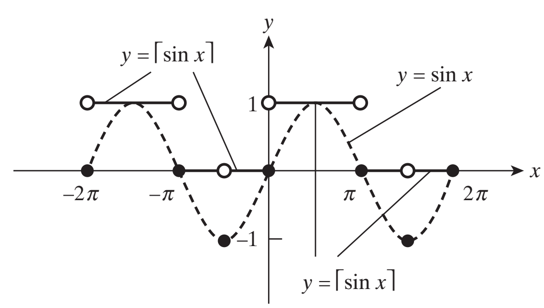
28. Since $\cot x = \dfrac{1}{\tan x}$, cot x is undefined when $\tan x = 0$ and is zero when tan x is undefined. As tan x approaches zero through positive values, cot x approaches infinity. Also, cot x approaches negative infinity as tan x approaches zero through negative values.



29. $D : (-\infty, \infty)$,
$R : y = -1, 0, 1$



30. $D : -\infty < x < \infty; R : y = -1, 0, 1$



31. $\cos\left(x-\frac{\pi}{2}\right)=\cos x\cos\left(-\frac{\pi}{2}\right)-\sin x\sin\left(-\frac{\pi}{2}\right)$
$=(\cos x)(0)-(\sin x)(-1)=\sin x$

32. $\cos\left(x+\frac{\pi}{2}\right)=\cos x\cos\left(\frac{\pi}{2}\right)-\sin x\sin\left(\frac{\pi}{2}\right)$
$=(\cos x)(0)-(\sin x)(1)=-\sin x$

33. $\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\left(\frac{\pi}{2}\right)+\cos x\sin\left(\frac{\pi}{2}\right)$
$=(\sin x)(0)+(\cos x)(1)=\cos x$

34. $\sin\left(x-\frac{\pi}{2}\right)=\sin x\cos\left(-\frac{\pi}{2}\right)+\cos x\sin\left(-\frac{\pi}{2}\right)$
$=(\sin x)(0)+(\cos x)(-1)=-\cos x$

35. $\cos(A-B)=\cos(A+(-B))=\cos A\cos(-B)-\sin A\sin(-B)$
$=\cos A\cos B-\sin A(-\sin B)$
$=\cos A\cos B+\sin A\sin B$

36. $\sin(A-B)=\sin(A+(-B))$
$=\sin A\cos(-B)\cos A\sin(-B)$
$=\sin A\cos B+\cos A(-\sin B)$
$=\sin A\cos B-\cos A\sin B$

37. If $B=A, A-B=0\Rightarrow\cos(A-B)=\cos 0=1$.
Also $\cos(A-B)=\cos(A-A)=\cos A\cos A+\sin A\sin A$
$=\cos^2 A+\sin^2 A$. Therefore, $\cos^2 A+\sin^2 A=1$.

38. If $B=2\pi$, then $\cos(A+2\pi)=\cos A\cos 2\pi-\sin A\sin 2\pi$
$=(\cos A)(1)-(\sin A)(0)=\cos A$
and
$\sin(A+2\pi)=\sin A\cos 2\pi+\cos A\sin 2\pi$
$=(\sin A)(1)+(\cos A)(0)$
$=\sin A.$

The result agrees with the fact that the cosine and sine functions have period 2π.

39. $-\cos x$ **40.** $-\sin x$

41. $-\cos x$ **42.** $\sin x$

43. $\frac{\sqrt{6}+\sqrt{2}}{4}$ **44.** $\frac{\sqrt{2}+\sqrt{6}}{4}$

45. $\frac{\sqrt{2}+\sqrt{6}}{4}$ **46.** $\frac{1+\sqrt{3}}{2\sqrt{2}}$

47. $\frac{2+\sqrt{2}}{4}$ **48.** $\frac{2-\sqrt{3}}{4}$

49. $\frac{2-\sqrt{3}}{4}$ **50.** $\frac{2+\sqrt{2}}{4}$

51. $\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$ **52.** $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

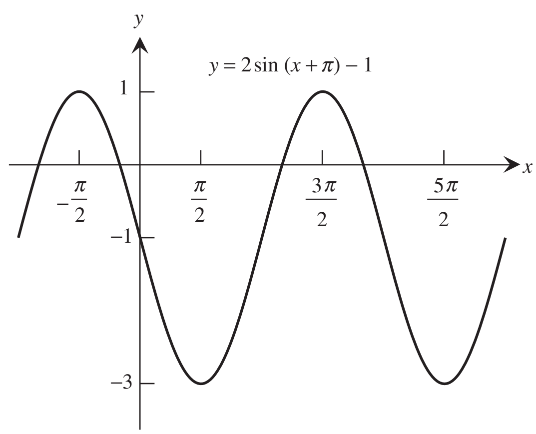
53. $\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$ **54.** $\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3}$

59. $\sqrt{7}\approx 2.65$ **60.** $c\approx 1.951$

62. $\sin B\approx 0.982$ **63.** $a=1.464$

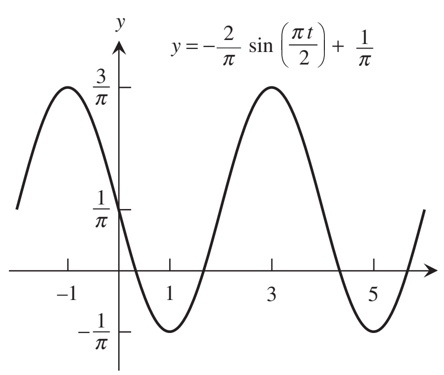
64. **(a)** The graphs of $y=\sin x$ and $y=x$ nearly coincide when x is near the origin (when the calculator is in radians mode).
(b) In degree mode, when x is near zero degrees the sine of x is much closer to zero than x itself. The curves look like intersecting straight lines near the origin when the calculator is in degree mode.

65. $A=2, B=2\pi, C=-\pi, D=-1$



66. $A=\frac{1}{2}, B=2, C=1, D=\frac{1}{2}$

67. $A=-\frac{2}{\pi}, B=4, C=0, D=\frac{1}{\pi}$



68. $A=\frac{1}{2\pi}, B=L, C=0, D=0$

Answers to Practice Exercises

1. The area is $A = \pi r^2$ and the circumference is $C = 2\pi r$. Thus,

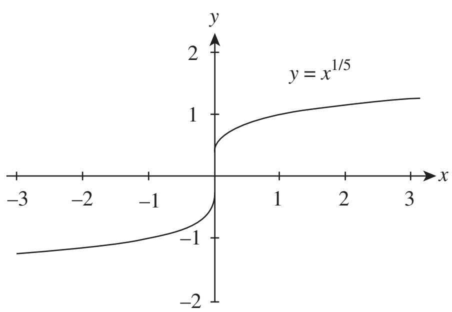
$$r = \frac{C}{2\pi} \Rightarrow A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$$

2. The surface area is $S = 4\pi r^2 \Rightarrow r = \left(\frac{S}{4\pi}\right)^{1/2}$. The volume is $V = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$ Substitution into the formula for surface area gives $S = 4\pi r^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.

3. $\left(x, x^2\right) = (\tan\theta, \tan^2\theta)$.

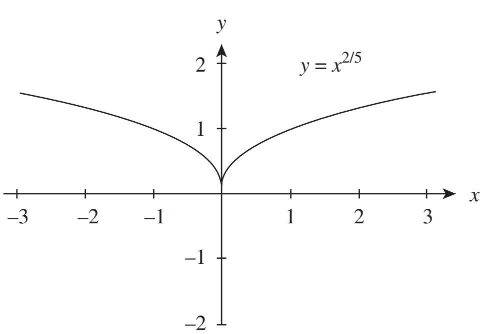
4. $h = 500\tan\theta$ ft.

5.



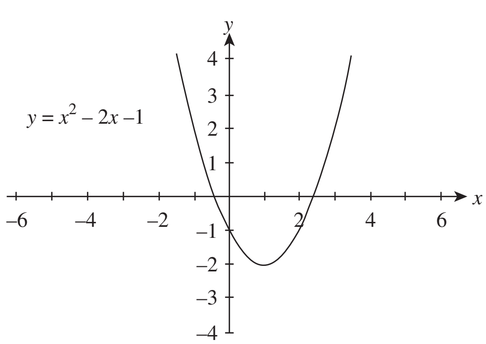
Symmetric about the origin

6.



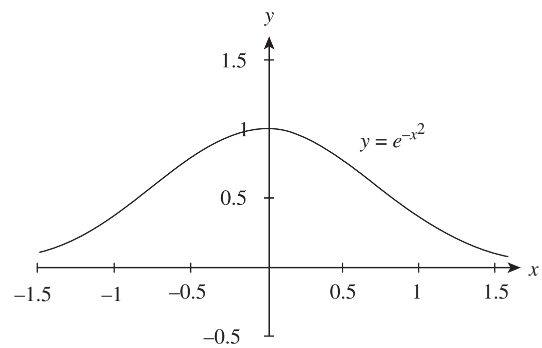
Symmetric about the y-axis

7.



Neither

8.



Symmetric about the $y-axis$

9. $y(-x) = (-x^2) + 1 = x^2 + 1 = y(x)$. Even.

10. $y(-x) = (-x)^5 - (-x)^3 - (-x) = -x^5 + x^3 + x = -y(x)$. Odd

11. $y = (-x) = 1 - \cos(-x) = 1 - \cos x = y(x)$. Even.

12. $y(-x) = \sec(-x)\tan(-x) = \dfrac{\sin(-x)}{\cos^2(-x)}$ Odd.

$$= \frac{-\sin x}{\cos^2 x} = -\sec x \tan x = -y(x).$$

13. $y(-x) = \dfrac{(-x)^4 + 1}{(-x)^3 - 2(-x)} = \dfrac{x^4 + 1}{-x^3 + 2x} = \dfrac{x^4 + 1}{x^3 - 2x} = -y(x)$ .Odd

14. $y(-x) = (-x) - \sin(-x) = (-x) + \sin x = -(x - \sin x) = -y(x)$. Odd

15. $y(-x) = -x + \cos(-x) = -x + \cos x$. Neither even nor odd.

16. $y(-x) = (-x)\cos(-x) = -x\cos x = -y(x)$. odd.

17. Since f and g are odd $\Rightarrow f(-x) = -f(x)$ and $g(-x) = -g(x)$

(a) $(f \cdot g)(-x) = f(-x)g(-x) = [-f(x)][-g(x)]$
$= f(x)g(x) = (f \cdot g)(x) \Rightarrow f \cdot g$ is even.

(b) $f^3(-x) = f(-x)f(-x)f(-x) = [-f(x)][-f(x)][-f(x)]$
$= -f(x) \cdot f(x) \cdot f(x) = -f^3(x) \Rightarrow f^3$ is odd.

(c) $f(\sin(-x)) = f(-\sin(x)) = -f(\sin(x)) \Rightarrow f(\sin(x))$ is odd

(d) $g(\sec(-x)) = g(\sec(x)) \Rightarrow g(\sec(x))$ is even.

(e) $|g-(-x)| = |-g(x)| = |g(x)| \Rightarrow |g|$ is even.

18. Let $f(a - x) = f(a + x)$ and define $g(x) = f(x + a)$. Then
$g(-x) = f((-x) + a) = f(a - x) = f(a + x)$
$= f(x + a) = g(x) \Rightarrow g(x) = f(x + a)$ is even

19. **(a)** The function is defined for all values of x, so the domain is $(-\infty, \infty)$.

(b) Since $|x|$ attains all nonnegative values, the range is $[-2, \infty)$.

20. **(a)** Since the square root requires $1-x\ge 0$, the domain is $(-\infty,1]$.

(b) Since $\sqrt{1-x}$ attains all nonnegative values, the range is $[-2,\infty)$.

21. **(a)** Since the square root requires $16-x^2\ge 0$, the domain is $[-4, 4]$

(b) For values of x in the domain, $0\le 16-x^2\le 16$, so $0\le\sqrt{16-x^2}\le 4$. The range is $[0, 4]$.

22. **(a)** The function is defined for all values of x, so the domain is $(-\infty,\infty)$

(b) Since 3^{2-x} attains all positive values, the range is $(1,\infty)$

23. **(a)** The function is defined for all values of x, so the domain is $(-\infty,\infty)$

(b) Since $2e^{-x}$ attains all positive values, the range is $(-3,\infty)$

24. **(a)** The function is equivalent to $y=\tan 2x$, so we require $2x\ne\frac{k\pi}{2}$ for odd integers k. The domain is given by $x\ne\frac{k\pi}{4}$ for odd integers k.

(b) Since the tangent function attains all values, the range is $(-\infty,\infty)$

25. **(a)** The function is defined for all values of x, so the domain $(-\infty,\infty)$

(b) The sine function attains values from –1 to 1 so $-2\le 2\sin(3x+\pi)\le 2$ and hence $-3\le 2\sin(3x+\pi)-1\le 1$. The range is $[-3, 1]$.

26. **(a)** The function is defined for all values of x, so the domain is $(-\infty,\infty)$

(b) The function is equivalent to $y=\sqrt[5]{x^2}$, which attains all nonnegative values. The range is $[0,\infty)$.

27. **(a)** The logarithm requires $x-3>0$, so the domain is $(3,\infty)$.

(b) The logarithm attains all real values, so the range is $(-\infty,\infty)$

28. **(a)** The function is defined for all values of x ,so the domain is $(-\infty,\infty)$

(b) The cube root attains all real values, so the range is $(-\infty,\infty)$.

29. **(a)** Increasing because volume increases as radius increases.

(b) Neither, since the greatest integer function is composed of horizontal (constant) line segments.

(c) Decreasing because as the height increases, the atmospheric pressure decreases.

(d) Increasing because the kinetic (motion) energy increases as the particles velocity increases.

30. **(a)** Increasing on $[2.\infty)$ (b) Increasing on $[-1.\infty)$

(c) Increasing on $[-\infty.\infty)$ (d) Increasing on $[\frac{1}{2},\infty)$

31. **(a)** The function is defined for $-4\le x\le 4$, so the domain is $[-4, 4]$.

(b) The function is equivalent to $y=\sqrt{|x|}, -4\le x\le 4$, which attains values from 0 to 2 for x in the domain. The range is $[0, 2]$.

32. **(a)** The function is defined for $-2\le x\le 2$, so domain is $[-2, 2]$.

(b) The range is $[-1, 1]$.

33. **First piece:** Line through (0, 1) and (1, 0).

$$m=\frac{0-1}{1-0}=\frac{-1}{1}\Rightarrow y=-x+1=1-x$$

Second piece: Line through (1, 1) and (2, 0).

$$m=\frac{0-1}{2-1}=\frac{-1}{1}=-1\Rightarrow y=-(x-1)+1=-x+2=2-x$$

$$f(x)=\begin{cases}1-x, & 0\le x<1\\ 2-x, & 1\le x\le 2\end{cases}$$

34. First piece: Line through (0, 0) and (2, 5).

$$m=\frac{5-0}{2-0}=\frac{5}{2}\Rightarrow y=\frac{5}{2}x$$

Second piece: Line through (2, 5) and (4, 0).

$$m=\frac{0-5}{4-2}=\frac{-5}{2}=-\frac{5}{2}\Rightarrow y=-\frac{5}{2}(x-2)+5=-\frac{5}{2}x+10=10-\frac{5x}{2}$$

$$f(x)=\begin{cases}\frac{5}{2}x, & 0\le x<2\\ 10-\frac{5x}{2}, & 2\le x\le 4\end{cases}$$

Note: $x=2$ can be included on either piece.)

35. **(a)** $(f\circ g)(-1)=f(g(-1))=f\left(\frac{1}{\sqrt{-1+2}}\right)=f(1)=\frac{1}{1}=1$

(b) $(g\circ f)(2)=g(f(2))=g\left(\frac{1}{2}\right)=\frac{1}{\sqrt{\frac{1}{2}+2}}=\frac{1}{\sqrt{2.5}}$ or $\sqrt{\frac{2}{5}}$

(c) $(f\circ f)(x)=f(f(x))=f\left(\frac{1}{x}\right)=\frac{1}{1/x}=x, x\ne 0$

(d) $(g\circ g)(x)=g(g(x))=g\left(\frac{1}{\sqrt{x+2}}\right)$

$$=\frac{1}{\sqrt{\frac{1}{\sqrt{x+2}}+2}}=\frac{\sqrt[4]{x+2}}{\sqrt{1+2\sqrt{x+2}}}$$

36. **(a)** $(f\circ g)(-1)=f(g(-1))=f(\sqrt[3]{-1+1}=f(0)=2-0=2$

(b) $(g\circ f)(2)=f(g(2))=g(2-2)=g(0)=\sqrt[3]{0+1}=1$

(c) $(f\circ f)(x)=f(f(x))=f(2-x)=2-(2-x)=x$

(d) $(g\circ g)(x)=g(g(x))=g\left(\sqrt[3]{x+1}\right)=\sqrt[3]{\sqrt[3]{x+1}+1}$

37. **(a)** $(f\circ g)(x)=f(g(x))=f(\sqrt{x+2})$

$$=2-\left(\sqrt{x+2}\right)^2=-x, x\ge -2.$$

$$(g\circ f)(x)=g(f(x))=g(2-x^2)=\sqrt{\left(2-x^2\right)+2}=\sqrt{4-x^2}$$

(b) Domain of $f\circ g:[-2,\infty)$,

Domain of $g\circ f:[-2,2]$.

(c) Range of $f\circ g:(-\infty,2]$.

Range of $g\circ f:[0,2]$.

38. **(a)** $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{1-x}\right) = \sqrt{\sqrt{1-x}} = \sqrt[4]{1-x}$.

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{1-\sqrt{x}}$

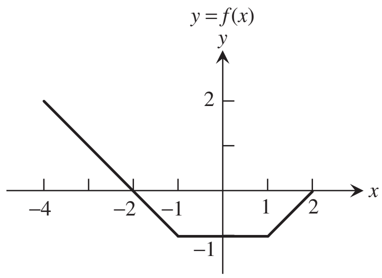
(b) Domain of $f \circ g : [-\infty, 1)$.

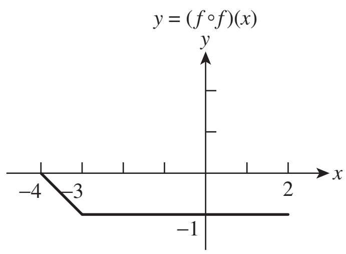
Domain of $g \circ f : [0,1]$.

(c) Range of $f \circ g : (0, \infty]$.

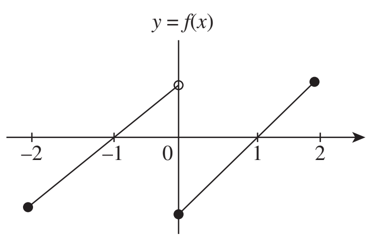
Range of $g \circ f : [0,1]$.

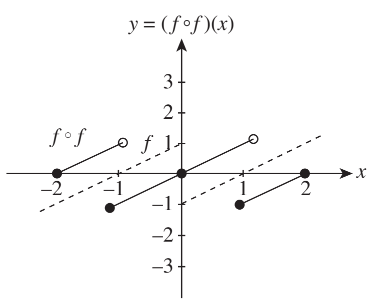
39.



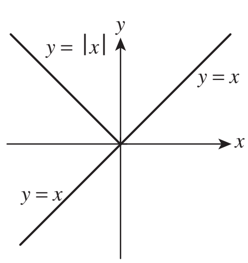


40.

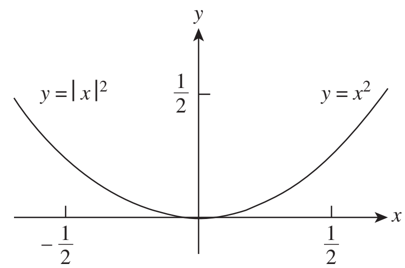




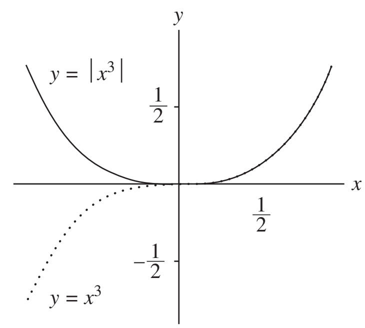
41. The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0$ across the y-axis.



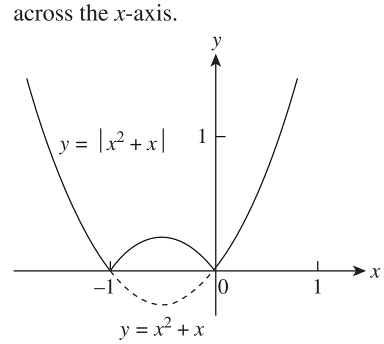
42. It does not change the graph.



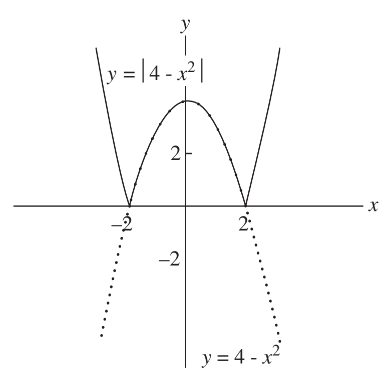
43. Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.



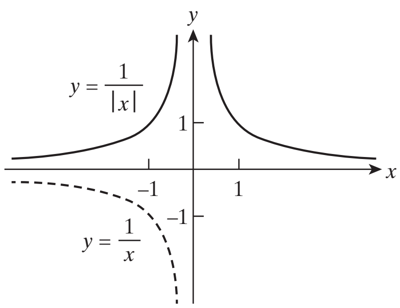
44. Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.



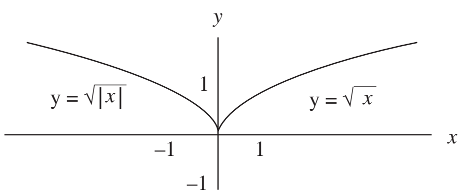
45. Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.



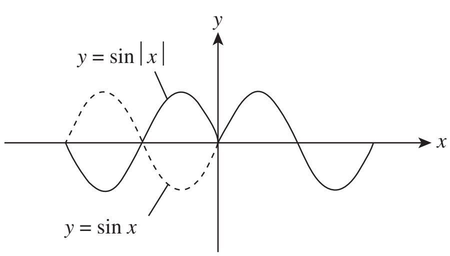
46. The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0,$ across the y-axis.



47. The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0$ across the y-axis.



48. The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0$ across the y-axis.

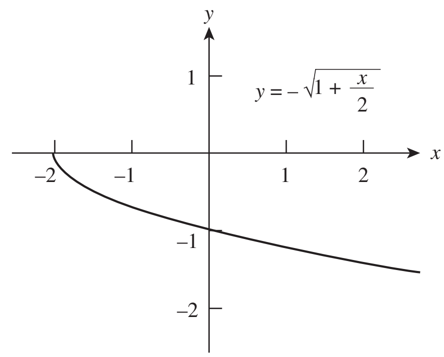


49. **(a)** $y = g(x-3) + \frac{1}{2}$ **(b)** $y = g\left(x + \frac{2}{3}\right) - 2$ **(c)** $y = g(-x)$

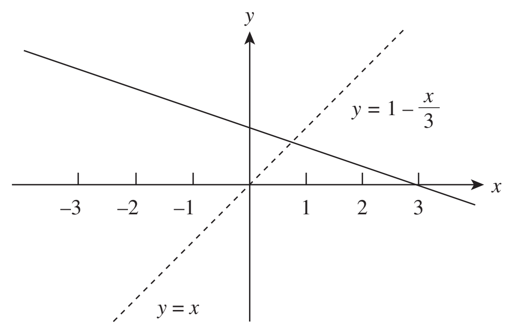
(d) $y = -g(x)$ **(e)** $y = 5 \cdot g(x)$ **(f)** $y = g(5x)$

50. **(a)** Shift the graph of f right 5 units
(b) Horizontally compress the graph of f by a factor of 4.
(c) Horizontally compress the graph of f by a factor of 3 and then reflect the graph about the y-axis.
(d) Horizontally compress the graph of f by a factor of 2 and then shift the graph left $\frac{1}{2}$ unit.
(e) Horizontally stretch the graph of f by a factor of 3 and then shift the graph down 4 units.
(f) Vertically stretch the graph of f by a factor of 3, then reflect the graph about x-axis, and finally shift the graph up $\frac{1}{4}$ unit.

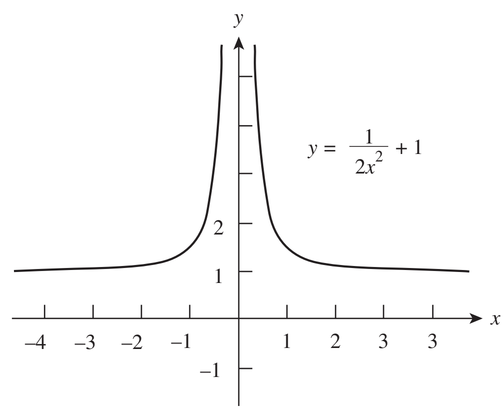
51. Reflection of the graph of $y = \sqrt{x}$ about the x-axis, followed by a horizontal compression by a factor of $\frac{1}{2}$ then shift left 2 units.



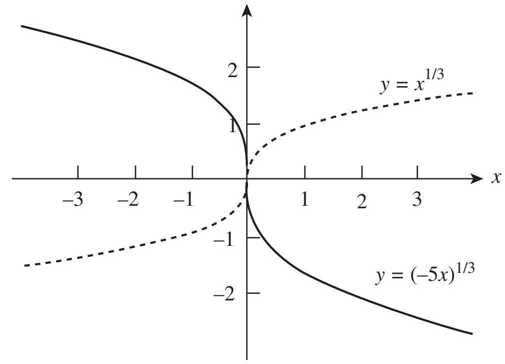
52. Reflect the graph of $y = x$ about the x-axis, followed by a vertical compression of the graph by a factor of 3, then shift the graph up 1 unit.



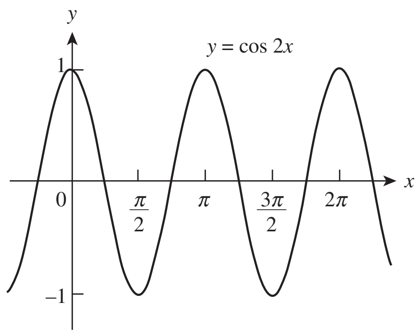
53. Vertical compression of the graph of $y = \frac{1}{x^2}$ by a factor of 2, then shift the graph up 1 unit.



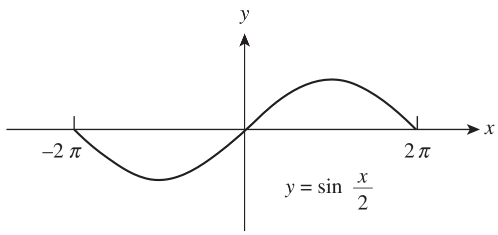
54. Reflect the graph of $y = x^{1/3}$ about the y-axis, then compress the graph horizontally by a factor of 5.



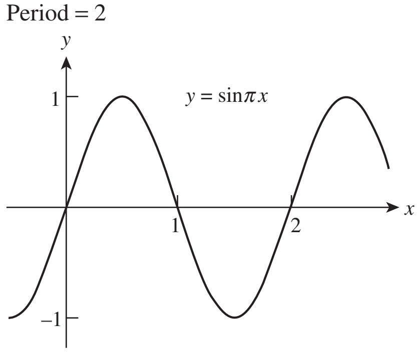
55. Period = π



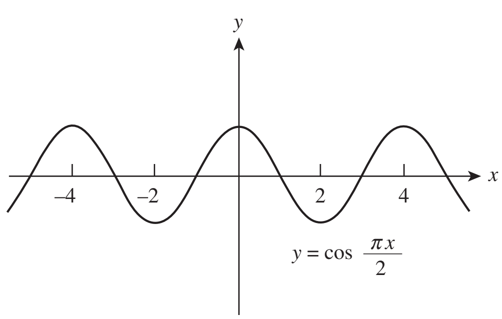
56. Period = 4π



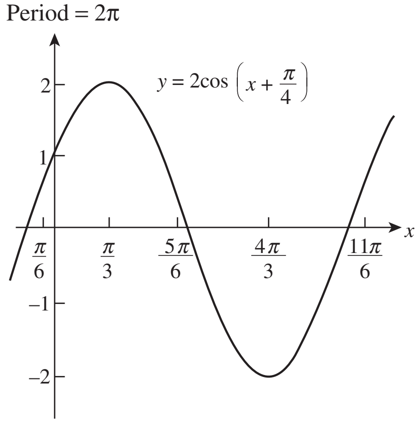
57. Period = 2



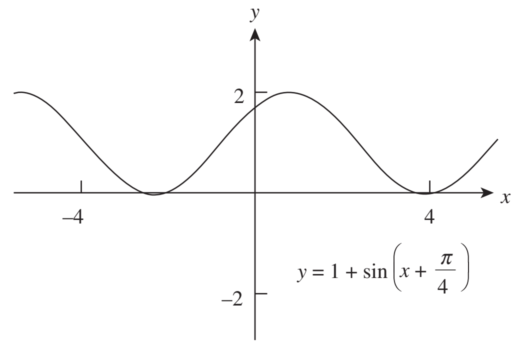
58. Period = 4



59. Period = 2π



60. Period = 2π



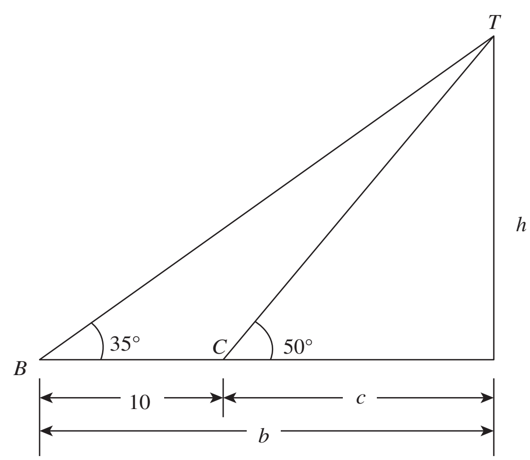
61. **(a)** $a = 1 \quad b = \sqrt{3}$ **(b)** $a = 2/\sqrt{3} \quad c = 4/\sqrt{3}$.

62. **(a)** $a = c \sin A$ **(b)** $a = b \tan A$

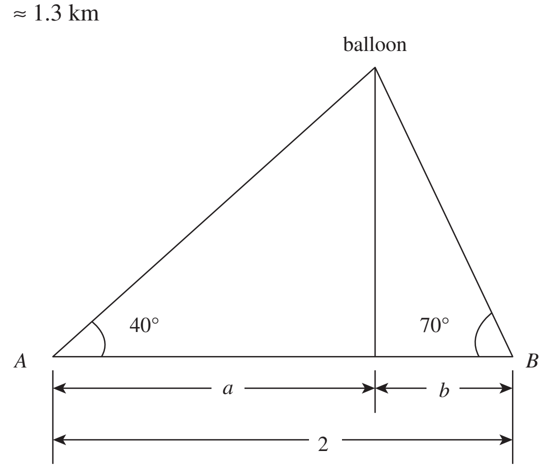
63. **(a)** $a = \dfrac{b}{\tan B}$ **(b)** $c = \dfrac{a}{\sin A}$

64. **(a)** $\sin A = \dfrac{a}{c}$ **(b)** $\dfrac{\sqrt{c^2 - b^2}}{c}$

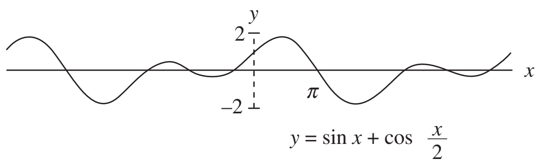
65. ≈ 16.98 m



66. ≈ 1.3 km



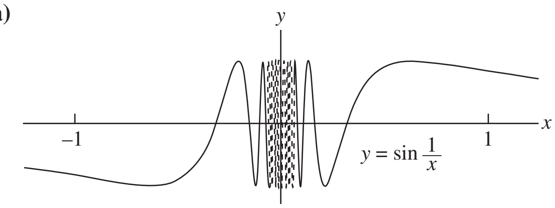
67. **(a)**



(b) The period appears to be 4π.

(c) $f(x+4\pi)=\sin(x+4\pi)+\cos\left(\frac{x+4\pi}{2}\right)=\sin(x+2\pi)$ $+\cos\left(\frac{x}{2}+2\pi\right)=\sin x+\cos\frac{x}{2}$ since the period of sine and cosine is 2π. Thus, $f(x)$ has period 4π.

68. **(a)**



(b) $D=(-\infty,0)\cup(0,\infty); R=[-1,1]$

(c) f is not periodic.

Answers to Single Choice Questions

1. (a)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (c)
11. (c)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (d)	**20.** (b)
21. (b)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (d)
31. (c)	**32.** (b)	**33.** (d)	**34.** (b)	**35.** (a)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (b)
41. (d)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (c)	**46.** (a)	**47.** (b)	**48.** (a)		

Answers to Multiple Choice Questions

1. (b) and (d)	**2.** (a) and (b)	**3.** (b) and (c)	**4.** (a), (b), (c) and (d)	**5.** (b) and (d)
6. (b) and (c)	**7.** (a), (b) and (d)	**8.** (a), (c) and (d)	**9.** (a), (b) and (c)	**10.** (a), (b) and (d)
11. (a) and (c)	**12.** (c) and (d)	**13.** (b) and (c)	**14.** (c) and (d)	**15.** (b) and (c)
16. (a), (b) and (d)	**17.** (a), (b) and (d)	**18.** (a) and (b)	**19.** (a) and (b)	**20.** (a), (c) and (d)
21. (a), (b), (c) and (d)	**22.** (b) and (c)	**23.** (a), (b), (c) and (d)	**24.** (c)	**25.** (a), (b) and (c)

Answers to Passage Type Questions

Passage 1
1. (a) **2.** (c) and (d)

Passage 2
1. (d) **2.** (d)

Passage 3
1. (a) **2.** (b)

Passage 4
1. (b) **2.** (a)

Passage 5
1. (b) **2.** (a) **3.** (b)

Passage 6
1. (d) **2.** (a)

Passage 7
1. (c) **2.** (c) **3.** (a)

Passage 8
1. (c) **2.** (d) **3.** (c)

Answers to Matrix Match Type Questions

1. (a) $\to$ (p)	(b) $\to$ (p) and (r)	(c) $\to$ (q) and (s)	(d) $\to$ (p)
2. (a) $\to$ (p)	(b) $\to$ (p), (q) and (r)	(c) $\to$ (q) and (r)	
3. (a) $\to$ (p) and (r)	(b) $\to$ (q) and (r)	(c) $\to$ (q) and (s)	(d) $\to$ (p) and (r)
4. (a) $\to$ (r)	(b) $\to$ (s)	(c) $\to$ (p)	(d) $\to$ (q)
5. (a) $\to$ (p)	(b) $\to$ (r)	(c) $\to$ (q)	(d) $\to$ (s)

Answers to Integer Type Questions

1. 8	**2.** 9	**4.** 12	**5.** 2525	**6.** 11
7. 5	**8.** 2	**9.** 2	**10.** 500	

Answers to Additional and Advanced Exercises

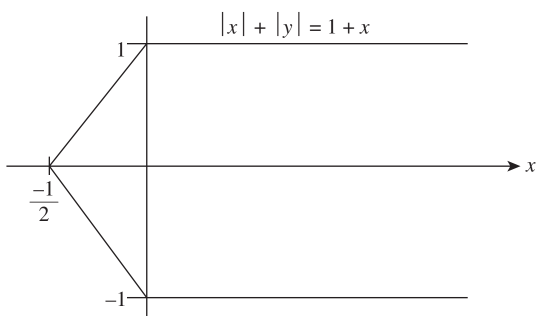
1. Yes, for instance: $f(x) = 1/x$ and $g(x) = 1/x$, or $f(x) = 2x$ and $g(x) = x/2$, or $f(x) = e^x$ and $g(x) = \ln x$.

2. Yes, there are many such function pairs. For example, $f(x) = 3x$ and $g(x) = 4x$ satisfy $f(g(x)) = f(4x) = 3(4x) = 12x = 4(3x) = g(3x) = g(f(x))$

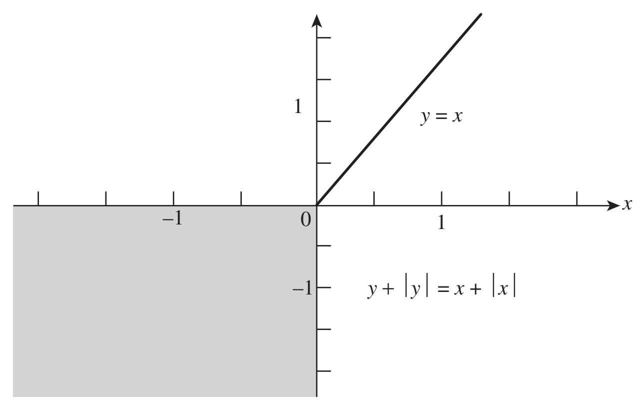
3. If $f(x)$ is odd, then $g(x) = f(x) - 2$ is not odd, Nor is $g(x)$ even, unless $f(x) = 0$ for all x. If f is even, then $g(x) = f(x) - 2$ is also even.

4. If g is odd and $g(0)$ is defined, then $g(0) = g(-0) = -g(0)$. Therefore, $2g(0) = 0 \Rightarrow g(0) = 0$

5.



6.



14. (a) Slope of OP (when P is mid point) = $\dfrac{b}{a}$

(b) Slope of OP (when P is perpendicular to AB) = $\dfrac{a}{b}$

16. $ad - bc + b - d = 0$

Answers to Exercises

Chapter 2

Exercises 2.1

1. **(a)** $\frac{-2}{\pi}$ **(b)** 0
2. **(a)** 4 **(b)** $y = 4x - 9$
3. **(a)** Slope is 0 **(b)** $y - 0 = 0(x - 2) \Rightarrow y = 0$
4.

PQ_1	PQ_2	PQ_3	PQ_4
12	13.7	14.7	16

(units are m/sec)

5. **(a)**

PQ_1	PQ_2	PQ_3	PQ_4
43	46	49	50

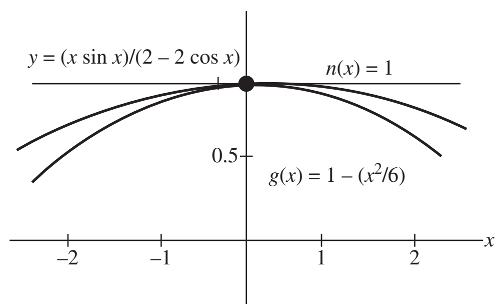
(units are m/sec)

(b) ≈ 50 m/sec or 180 km/hr

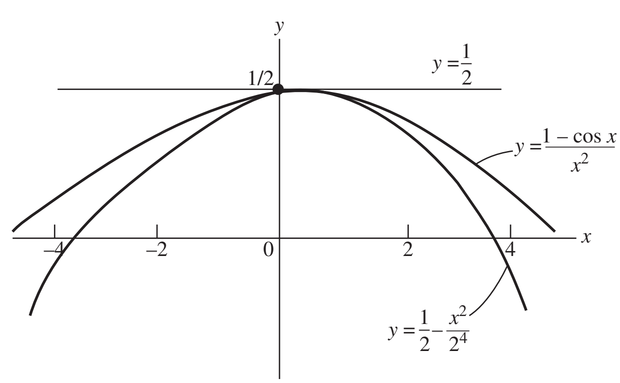
6. **(a)** –1.67, –2.2, –0.5
(b) –1.45 gal/day; –3 gal/day; –0.55 gal/day
(c) –1.45 gal/day; $t = 3.5$

Exercises 2.2

1. **(a)** 0 **(b)** –1
(c) Do not exist as Left hand limit = –1 and Right hand limit = 1
(d) 1
2. **(a)** True **(b)** True **(c)** False **(d)** False
(e) False **(f)** True **(g)** True
3. **(a)** False **(b)** False **(c)** True
(d) True **(e)** True
4. As x approaches 0 from left, $\frac{x}{|x|}$ approaches –1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. While $x \to 0$, $\frac{|x|}{x}$ does not approaches a unique value. Hence, limit do not exist.
5. As x approaches 1 from the left hand side or right hand side the expression $\frac{1}{x-1}$ does not approach a unique finite quantity.
So $\lim\limits_{x \to 1} \frac{1}{x-1}$ do not exist.
6. Nothing can be said
7. Nothing can be said
8. No
9. No
10. 16
11. 4
12. $\frac{-1}{3}$
13. $\frac{1}{2}$
14. $\frac{-3}{2}$
15. $\frac{3}{2}$
16. $\frac{5}{4}$
17. –1
18. 0
19. 1
20. 0
21. $\frac{1}{3}$
22. –1
23. $\sqrt{4 - \pi}$
24. $2\sqrt{2}$
25. **(a)** quotient rule
(b) power and product rules
(c) difference and constant multiple sites
26. **(a)** quotient rule
(b) power and product rules
(c) difference and constant multiple rules
27. **(a)** –10 **(b)** –20 **(c)** –1 **(d)** $\frac{5}{7}$
28. **(a)** 0 **(b)** 0 **(c)** 9 **(d)** 3
29. **(a)** 4 **(b)** –21 **(c)** –12 **(d)** $\frac{-7}{3}$
30. **(a)** 1 **(b)** 0 **(c)** $\frac{16}{3}$
31. 2
32. –4
33. 3
34. $\frac{-1}{4}$
35. $\frac{1}{2\sqrt{7}}$
36. $\frac{3}{2}$
37. $\sqrt{5}$
38. 2
39. According to Sandwich theorem $\lim\limits_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$



40. $\operatorname{lt}\limits_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$



41. $C = 0, 1$ or -1
42. Nothing can be concluded. $\lim\limits_{x \to 2} f(x) = -5$ as $f(x)$ is sandwiched between $g(x)$ and $h(x)$
43. 7
44. **(a)** 4 **(b)** –2
45. **(a)** 5 **(b)** 5

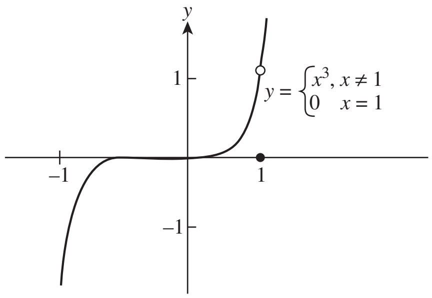
Exercises 2.4

1. **(a)** True **(b)** False **(c)** False **(d)** True
(e) True **(f)** True **(g)** True **(h)** True
(i) True **(j)** False **(k)** True

2. **(a)** No **(b)** Yes, 0 **(c)** No

3. **(a)** Yes, 0 **(b)** No **(c)** Yes, 0

4. **(a)**



(b) 1, 1 **(c)** Yes, 1

5. **(a)** 1 **(b)** $\frac{2}{3}$

6. **(a)** 0 **(b)** 1

7. 3 **8.** 1

9. Yes **10.** Yes

11. If f is odd function of x, then $f(-x) = -f(x)$. Given $\lim_{x\to 0^+} f(x) = 3$, then $\lim_{x\to 0^-} f(x) = -3$

12. Nothing can be said

Exercises 2.5

1. No, discontinuous at $x = 2$, not defined at $x = 2$.

2. No, discontinuous at $x = 3$, $1 = \lim_{x\to 3^-} g(x) \neq g(3) = 1.5$

3. No, discontinuous at $x = 1$

4. No, discontinuous at $x = 1$, $1.5 = \lim_{x\to 1^-} k(x) \neq \lim_{x\to 1^+} k(x) = 0$

5. **(a)** Yes **(b)** Yes, $\lim_{x\to -1^+} f(x) = 0$
(c) Yes **(d)** Yes

6. **(a)** Yes, $f(1) = 1$ **(b)** Yes, $\lim_{x\to 1} f(x) = 2$
(c) No **(d)** No

7. **(a)** No **(b)** No

8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

9. $f(2) = 0$, since $\lim_{x\to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x\to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. Non-removable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 0$). Removable discontinuity at $x = 0$ by assigning the number $\lim_{x\to 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.

12. Non-removable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist $\left(\lim_{x\to 1^-} f(x) = 2 \text{ and } \lim_{x\to 1^+} f(x) = 1\right)$. Removable discontinuity at $x = 2$ by assigning the number $\lim_{x\to 2} f(x) = 1$ to be the value of $f(2)$ rather than $f(2) = 2$.

13. $x = 2$

14. $x = -2$.

15. $x = 3$ or $x = 1$.

16. $x = 5$ or $x = -2$.

17. Continuous everywhere. ($|x - 1| + \sin x$ defined for all x; limits exist and are equal to functions values).

18. Continuous everywhere. ($|x| + 1 \neq 0$ for all x; limits exist and are equal to functions values)

19. Discontinuous only at $x = 0$.

20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

21. Discontinuous when $2x$ is an integer multiple of π, i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x.

22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow x = 2n-1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

24. Continuous everywhere since $x^4 + 1 \geq 1$ and $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 1 + \sin^2 x \geq 1$; limits exist and are equal to the function values.

25. Continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

26. Continuous on the interval $\left[\frac{1}{3}, \infty\right)$.

27. Continuous everywhere: $(2x-1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

28. Continuous everywhere: $(2x-1)^{1/5}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since
$$\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = 5 = g(3).$$

30. Discontinuous at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist while $f(-2) = 4$.

31. $\lim_{x\to\pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$.

32. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\cos(\tan(0))\right)$
$= \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.

33. $\lim_{y\to 1} \sec\left(y\sec^2 y - \tan^2 y - 1\right) = \lim_{y\to 1} \sec\left(y\sec^2 y - \sec^2 y\right)$
$= \lim_{y\to 1} \sec\left((y-1)\sec^2 y\right) = \sec\left((1-1)\sec^2 1\right) = \sec 0 = 1$, and function continuous at $y = 1$.

34. $\lim_{x\to 0} \tan\left[\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right]$
$= \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

35. $\lim_{t\to 0} \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 0}}\right]$
$= \cos\frac{\pi}{\sqrt{16}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

36. 3 and function continuous at $x = \dfrac{\pi}{6}$.

37. 0 and the function is not defined at $x = 0$ and thus is not continuous there.

38. 2 and the function is not defined at $x = 0$ and thus is not continuous there.

39. 6

40. 7

41. $\dfrac{3}{2}$

42. $\dfrac{8}{5}$

43. $a = \dfrac{4}{3}$.

44. $b = -\dfrac{1}{2}$.

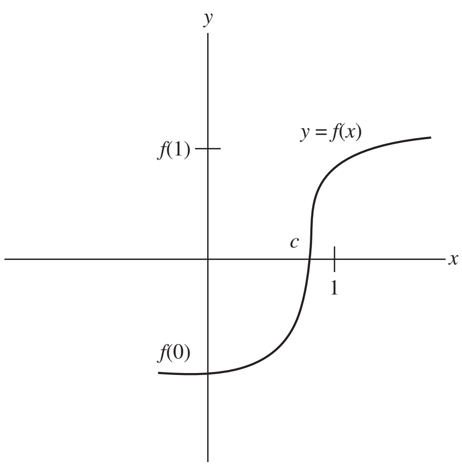
45. $a = 3$ or $a = -2$.

46. $b = 0$ or $b = -2$.

47. $a = \dfrac{5}{2}$ and $b = \dfrac{1}{2}$.

48. $a = -\dfrac{3}{2}$ and $b = -\dfrac{3}{2}$.

49. $f(x)$ is continuous on [0, 1] and $f(0) < 0, f(1) > 0 \Rightarrow$ by the Intermediate Value Theorem $f(x)$ takes on every value between $f(0)$ and $f(1) \Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



50. $\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\dfrac{\pi}{2}, \cos\left(-\dfrac{\pi}{2}\right) - \left(-\dfrac{\pi}{2}\right) > 0$. If $x = \dfrac{\pi}{2}, \cos\left(\dfrac{\pi}{2}\right) - \dfrac{\pi}{2} < 0$. Thus, $\cos x - x = 0$ for some x between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ according to the Intermediate Value Theorem, since the function $\cos x - x$ is continuous.

54. All five statements ask for the same information because of the intermediate value property of continuous functions.
(a) A root of $f(x) = x^3 - 3x - 1$ is a point c where $f(c) = 0$.
(b) The point where $y = x^3$ crosses $y = 3x + 1$ have the same y-co-ordinate, or $y = x^3 = 3x + 1 \Rightarrow f(x) = x^3 - 3x - 1 = 0$.
(c) $x^3 - 3x = 1 \Rightarrow x^3 - 3x - 1 = 0$. The solutions to the equation are the roots of $f(x) = x^3 - 3x - 1$.
(d) The points where $y = x^3 - 3x$ crosses $y = 1$ have common y-coordinates, or $y = x^3 - 3x = 1 \Rightarrow f(x) = x^3 - 3x - 1 = 0$.
(e) The solutions of $x^3 - 3x - 1 = 0$ are those points where $f(x) = x^3 - 3x - 1$ has value 0.

55. Answers may vary. For example, $f(x) = \dfrac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely) as $x \to 2$.

56. Answers may vary. For example, $g(x) = \dfrac{1}{x+1}$ has a discontinuity at $x = -1$ because $\lim\limits_{x \to -1} g(x)$ does not exist. $\left(\lim\limits_{x \to -1^-} g(x) = -\infty \text{ and } \lim\limits_{x \to -1^+} g(x) = +\infty\right)$.

57. **(b)** f is neither right-continuous nor left-continuous at any point.

58. Yes.

59. No.

60. Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = x + 1$. Both functions are continuous at $x = 0$. The composition $f \circ g = f(g(x)) = \dfrac{1}{(x+1)-1} = \dfrac{1}{x}$ is discontinuous at $x = 0$, since it is not defined there. Theorem 10 requires that $f(x)$ be continuous at $g(0)$, which is not in the case here since $g(0) = 1$ and f is undefined at 1.

61. Yes.

62. Let $f(x)$ be the new position of point x and let $d(x) = f(x) - x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive is x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x) = 0$ for some point in between. That is, $f(x) = x$ for some point x, which is then in its original position.

67. $x \approx 1.8794, -1.5321, -0.3473$

68. $x \approx 1.4516, -0.8547, 04030$

69. $x \approx 1.7549$

70. $x \approx 3.5156$

71. $x \approx 0.7391$

72. $x \approx -1.8955, 0, 1.8955$

Exercises 2.6

1. **(a)** $\lim\limits_{x \to 2} f(x) = 0$ **(b)** $\lim\limits_{x \to -3^+} f(x) = -2$
(c) $\lim\limits_{x \to -3^-} f(x) = 2$ **(d)** $\lim\limits_{x \to 3} f(x) =$ does not exist
(e) $\lim\limits_{x \to 0^+} f(x) = -1$ **(f)** $\lim\limits_{x \to 0^-} f(x) = +\infty$
(g) $\lim\limits_{x \to 0} f(x) =$ does not exist
(h) $\lim\limits_{x \to \infty} f(x) = 1$ **(i)** $\lim\limits_{x \to -\infty} f(x) = 0$

2. **(a)** $\lim_{x\to 4} f(x) = 2$ **(b)** $\lim_{x\to 2^+} f(x) = -3$
(c) $\lim_{x\to 2^-} f(x) = 1$ **(d)** $\lim_{x\to 2} f(x) =$ does not exist
(e) $\lim_{x\to -3^+} f(x) = +\infty$ **(f)** $\lim_{x\to -3^+} f(x) = +\infty$
(g) $\lim_{x\to -3} f(x) = +\infty$ **(h)** $\lim_{x\to 0^+} f(x) = +\infty$
(i) $\lim_{x\to 0^-} f(x) = -\infty$ **(j)** $\lim_{x\to 0} f(x) =$ does not exist
(k) $\lim_{x\to \infty} f(x) = 0$ **(l)** $\lim_{x\to -\infty} f(x) = -1$

Note: In these exercises we use the result $\lim_{x\to\pm\infty}\frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$ .This result follows immediately from Theorem 8 and the power rule in Theorem 1:

$$\lim_{x\to\pm\infty}\left(\frac{1}{x^{m/n}}\right) = \lim_{x\to\pm\infty}\left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^{m/n} = 0^{m/n} = 0.$$

3. **(a)** -3 **(b)** -3
4. **(a)** π **(b)** π
5. **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{2}$
6. **(a)** $\frac{1}{8}$ **(b)** $\frac{1}{8}$
7. **(a)** $-\frac{5}{3}$ **(b)** $-\frac{5}{3}$
8. **(a)** $\frac{3}{4}$ **(b)** $\frac{3}{4}$
9. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x\to\infty}\frac{\sin 2x}{x} = 0$ by the Sandwich Theorem
10. $-\frac{1}{3\theta} \le \frac{\cos\theta}{3\theta} \le \frac{1}{3\theta} \Rightarrow \lim_{x\to-\infty}\frac{\cos\theta}{3\theta} = 0$ by the Sandwich Theorem
11. -1 **12.** $\frac{1}{2}$
13. **(a)** $\frac{2}{5}$ **(b)** $\frac{2}{5}$ (same process as part (a))
14. **(a)** 2 **(b)** 2
15. **(a)** 0 **(b)** 0
16. **(a)** 0 **(b)** 0
17. **(a)** 7 **(b)** 7
18. **(a)** $\frac{9}{2}$ **(b)** $\frac{9}{2}$ (same process as part (a))
19. **(a)** 0 **(b)** 0
20. **(a)** ∞ **(b)** $-\infty$
21. **(a)** ∞ **(b)** ∞
22. **(a)** $-\infty$ **(b)** ∞
23. 2 **24.** $\frac{1}{2}$
25. ∞ **26.** 0
27. 0 **28.** -1
29. 1 **30.** ∞
31. ∞ **32.** $-\frac{5}{2}$
33. 1 **34.** -1
35. $\frac{1}{2}$ **36.** 3
37. ∞ $\left(\frac{\text{positive}}{\text{positive}}\right)$
38. $-\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$
39. $-\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$
40. ∞ $\left(\frac{\text{positive}}{\text{positive}}\right)$
41. $-\infty$ $\left(\frac{\text{negative}}{\text{positive}}\right)$
42. ∞ $\left(\frac{\text{negative}}{\text{negative}}\right)$
43. ∞ $\left(\frac{\text{positive}}{\text{positive}}\right)$
44. $-\infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{positive}}\right)$
45. **(a)** ∞ **(b)** $-\infty$
46. **(a)** ∞ **(b)** $-\infty$
47. ∞ **48.** ∞
49. ∞ **50.** ∞
51. $-\infty$
52. $\lim_{\theta\to 0^+}(2-\cot\theta) = -\infty$ and $\lim_{\theta\to 0^-}(2-\cot\theta) = \infty$, so the limit does not exist.
53. **(a)** $\infty\left(\frac{1}{\text{positive.positive}}\right)$
(b) $-\infty\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$
(c) $-\infty\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$
(d) $\infty\left(\frac{1}{\text{negative}\cdot\text{negative}}\right)$
54. **(a)** $\infty\left(\frac{\text{positive}}{\text{positive}\cdot\text{positive}}\right)$
(b) $-\infty\left(\frac{\text{positive}}{\text{positive}\cdot\text{negative}}\right)$
(c) $\infty\left(\frac{\text{negative}}{\text{positive}\cdot\text{negative}}\right)$
(d) $-\infty\left(\frac{\text{negative}}{\text{negative}\cdot\text{negative}}\right)$
55. **(a)** $-\infty\left(\frac{1}{\text{negative}}\right)$
(b) $\infty\left(\frac{1}{\text{positive}}\right)$
(c) 0
(d) $\left(\frac{3}{2}\right)$

56. **(a)** ∞ $\left(\frac{\text{positive}}{\text{positive}}\right)$

(b) $-\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$

(c) 0

(d) $\frac{-1}{4}$

57. **(a)** $-\infty$ $\left(\frac{\text{negative} \cdot \text{negative}}{\text{positive} \cdot \text{negative}}\right)$

(b) $\frac{1}{4}$

(c) $\frac{1}{4}$

(d) $\frac{1}{4}$

(e) $-\infty$ $\left(\frac{\text{negative} \cdot \text{negative}}{\text{positive} \cdot \text{negative}}\right)$

58. **(a)** $\frac{1}{8}$

(b) ∞ $\left(\frac{\text{negative}}{\text{negative} \cdot \text{positive}}\right)$

(c) ∞ $\left(\frac{\text{negative}}{\text{negative} \cdot \text{positive}}\right)$

(d) 0

(e) $\lim\limits_{x\to 0^+} \frac{x-1}{x(x+2)} = -\infty$ $\left(\frac{\text{negative}}{\text{positive} \cdot \text{positive}}\right)$

and $\lim\limits_{x\to 0^-} \frac{x-1}{x(x+2)} = \infty$ $\left(\frac{\text{negative}}{\text{negative.positive}}\right)$ so the function has no limit as $x \to 0$.

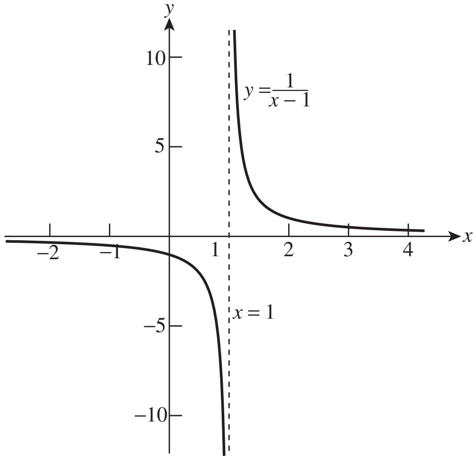
59. **(a)** $-\infty$ **(b)** ∞

60. **(a)** ∞ **(b)** $-\infty$

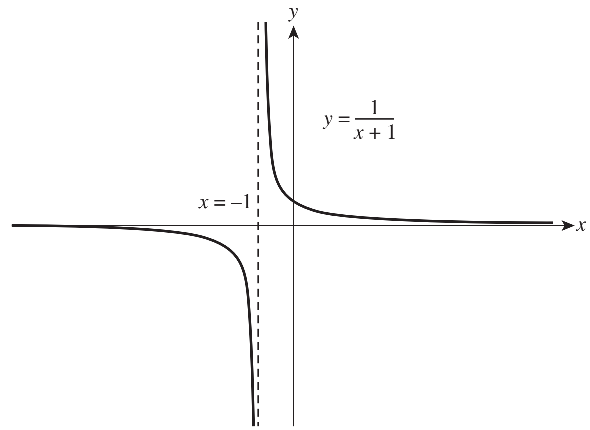
61. **(a)** ∞ **(b)** ∞
(c) ∞ **(d)** ∞

62. **(a)** ∞ **(b)** $-\infty$
(c) $-\infty$ **(d)** $-\infty$

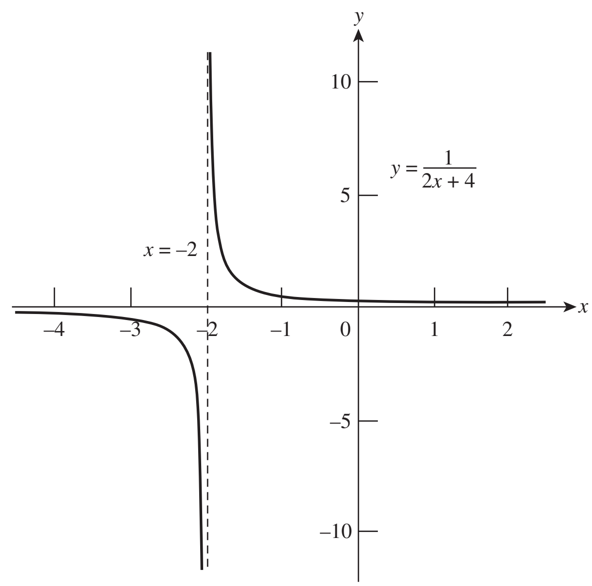
63. $y = \frac{1}{x-1}$



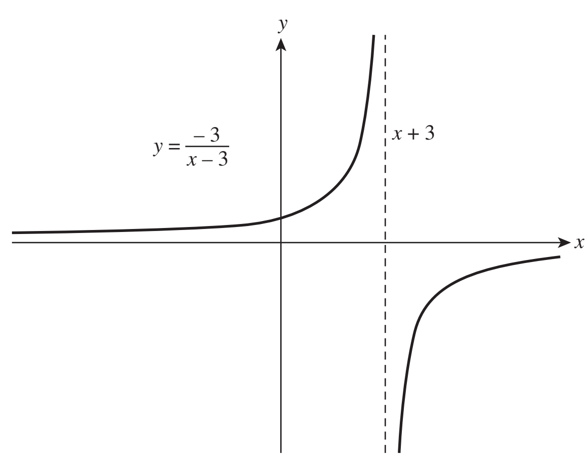
64. $y = \frac{1}{x+1}$



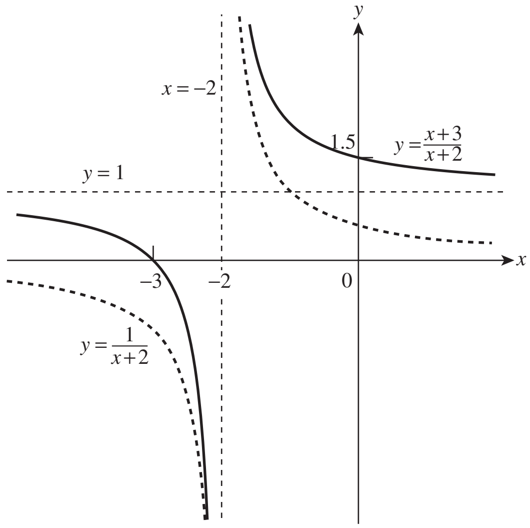
65. $y = \frac{1}{2x+4}$



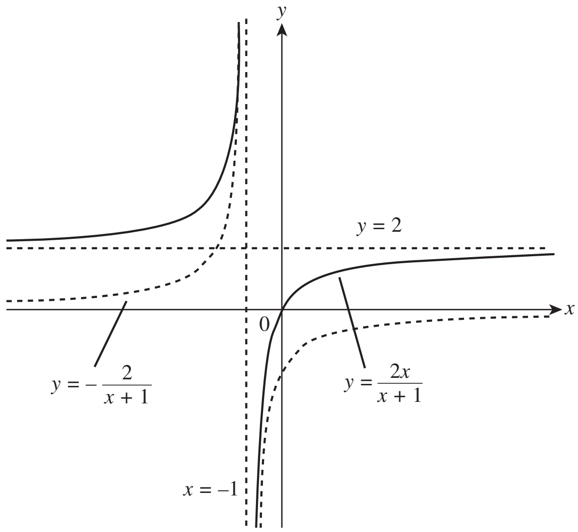
66. $y = \frac{-3}{x-3}$



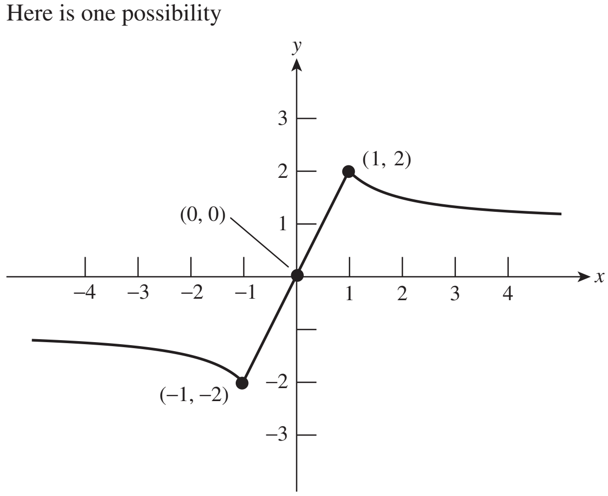
67. $y = \dfrac{x+3}{x+2} = 1 + \dfrac{1}{x+2}$



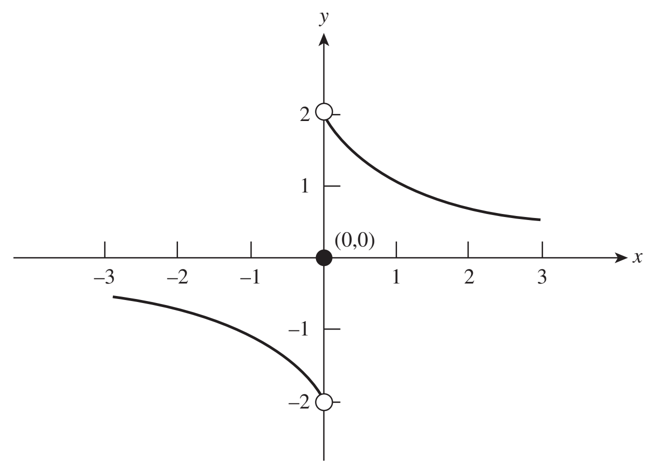
68. $y = \dfrac{2x}{x+1} = 2 - \dfrac{2}{x+1}$



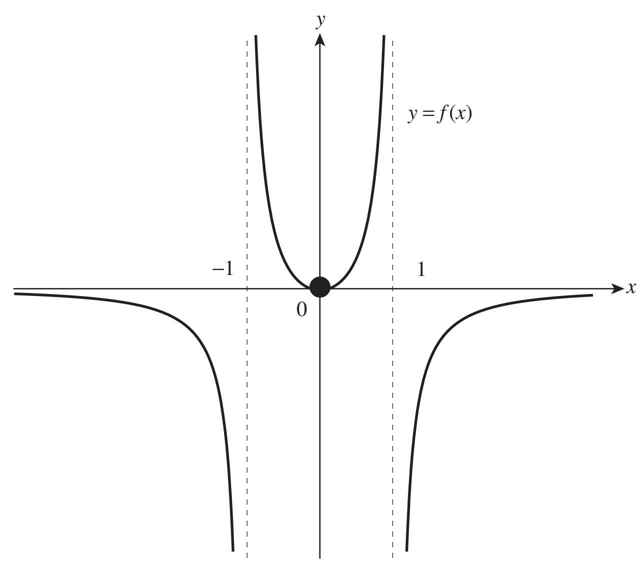
69. Here is one possibility



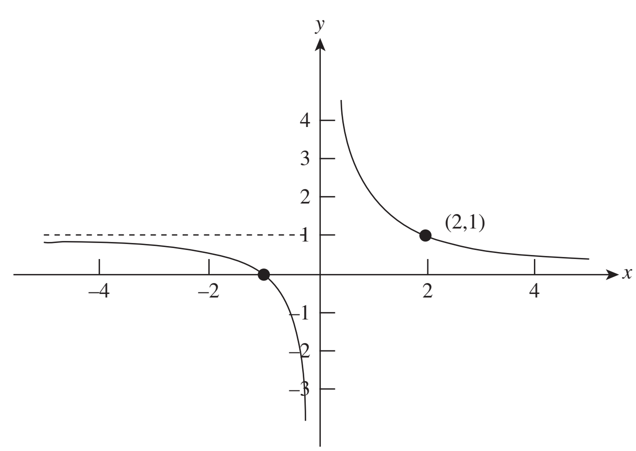
70. Here is one possibility



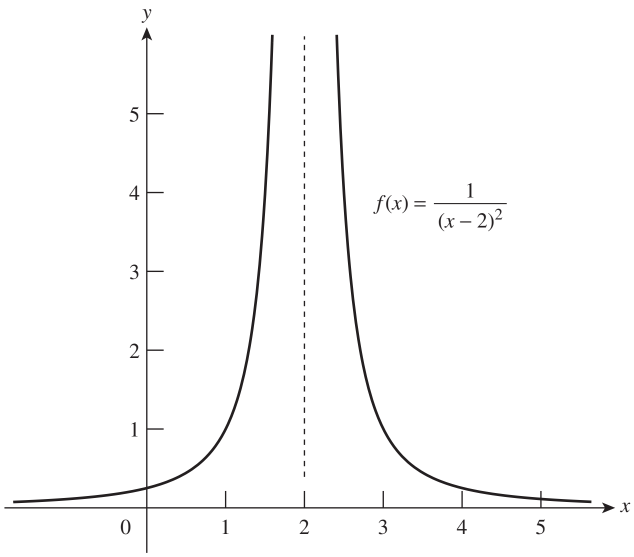
71. Here is one possibility.



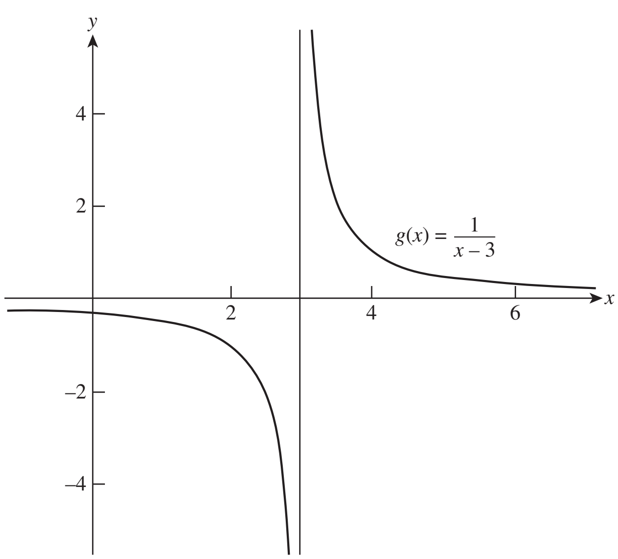
72. Here is one possibility



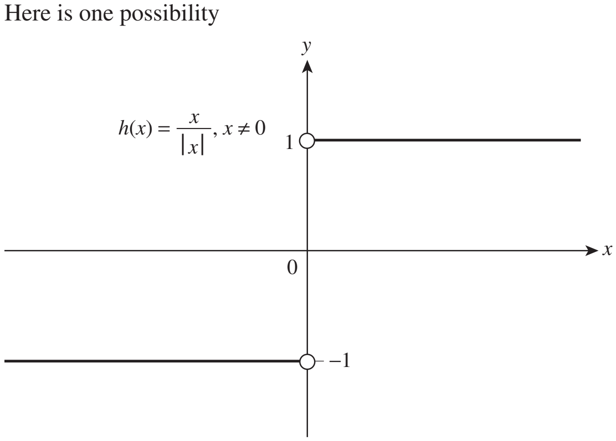
73. Here is one possibility



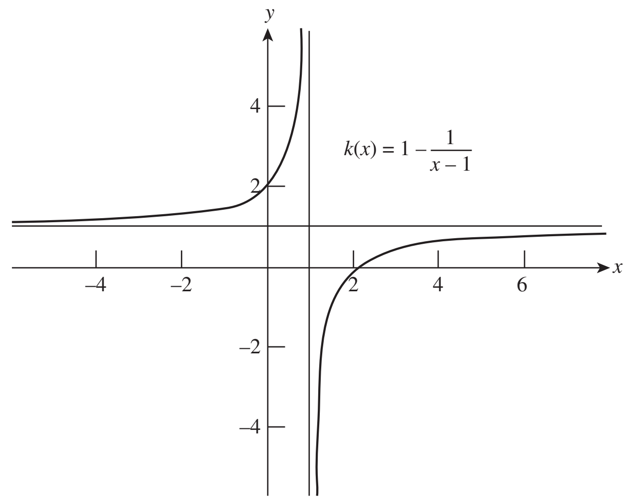
74. Here is one possibility



75. Here is one possibility



76. Here is one possibility



77. Yes

78. Yes

79. At most 1 horizontal asymptote

80. 0

81. 0

82. 0

83. $-\frac{3}{4}$

84. $-\frac{1}{6}$

85. $\frac{5}{2}$

86. 1

87. For any $\varepsilon > 0$, Take $N = 1$. Then for all $x > N$ we have that $|f(x) - k| = |k - k| = 0 < \varepsilon$.

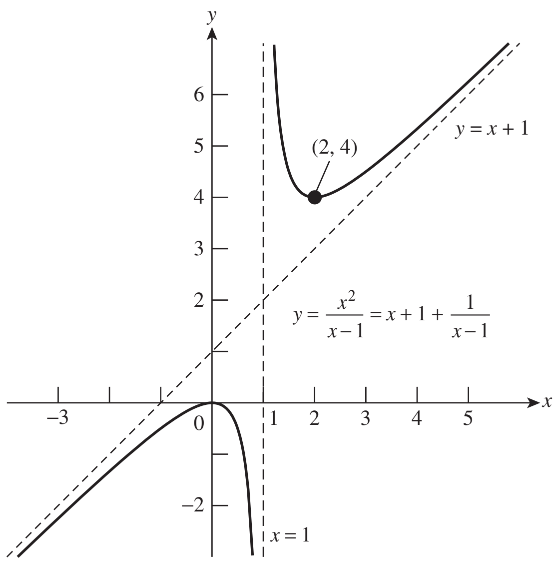
88. For any $\varepsilon > 0$, Take $N = 1$. Then for all $y < -N$ we have that $|f(x) - k| = |k - k| = 0 < \varepsilon$.

93. **(a)** We say that $f(x)$ approaches infinity as x approaches x_0 from the left, and write $\lim_{x \to x_0^-} f(x) = \infty$, if for every positive number B, there exists a corresponding number $\delta > 0$ such that for all $x, x_0 - \delta < x < x_0 \Rightarrow f(x) > B$.

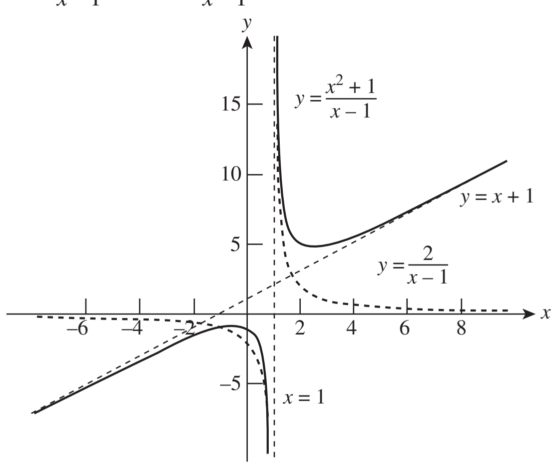
(b) We say $f(x)$ approaches minus infinity as x approaches x_0 from the right, and write $\lim_{x \to x_0^+} f(x) = -\infty$ if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all $x, x_0 < x < x_0 + \delta \Rightarrow f(x) < -B$.

(c) We say that $f(x)$ approaches infinity as x approaches x_0 from the left, and write $\lim_{x \to x_0^-} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all $x, x_0 - \delta < x < x_0 \Rightarrow f(x) < -B$.

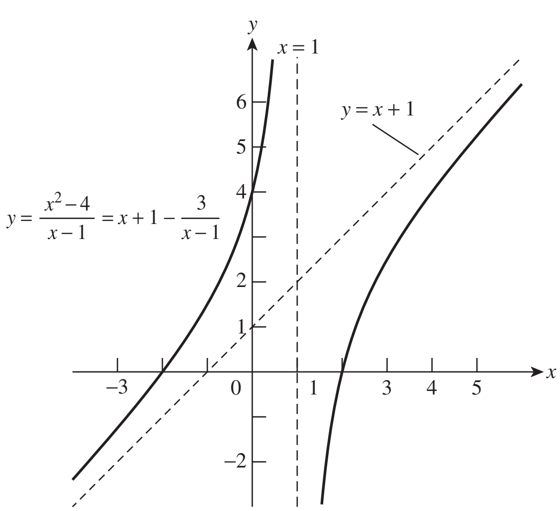
99. $y = \dfrac{x^2}{x^2-1} = x+1+\dfrac{1}{x-1}$



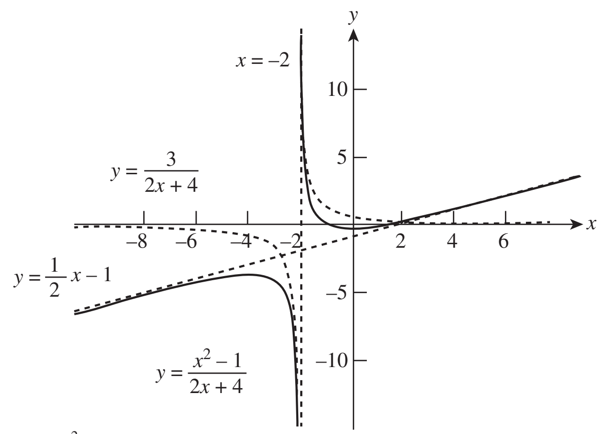
100. $y = \dfrac{x^2+1}{x-1} = x+1+\dfrac{2}{x-1}$



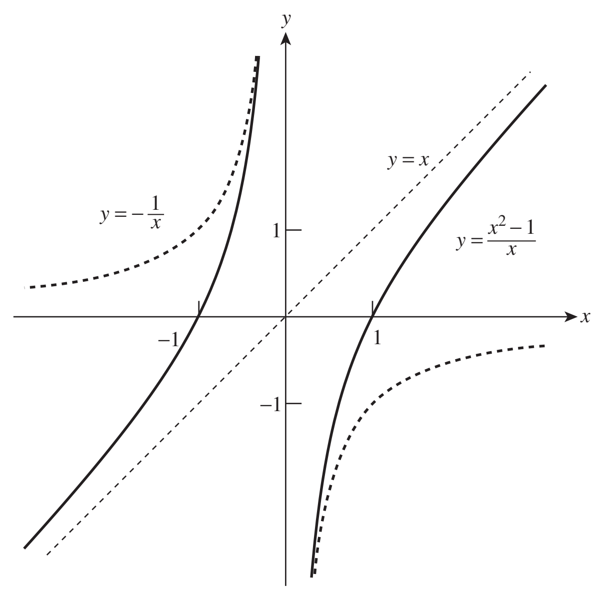
101. $y = \dfrac{x^2-4}{x-1} = x+1-\dfrac{3}{x-1}$



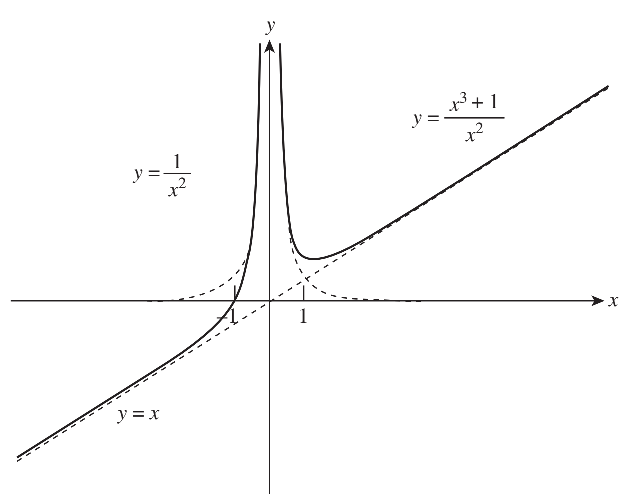
102. $y = \dfrac{x^2-1}{2x+4} = \dfrac{1}{2}x-1+\dfrac{3}{2x+4}$



103. $y = \dfrac{x^2-1}{x} = x-\dfrac{1}{x}$

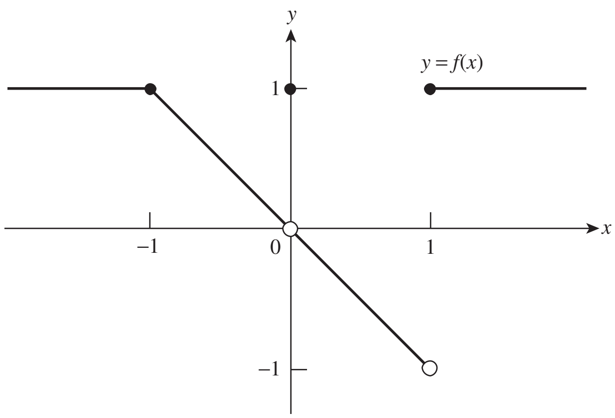


104. $y = \dfrac{x^3+1}{x^2} = x+\dfrac{1}{x^2}$

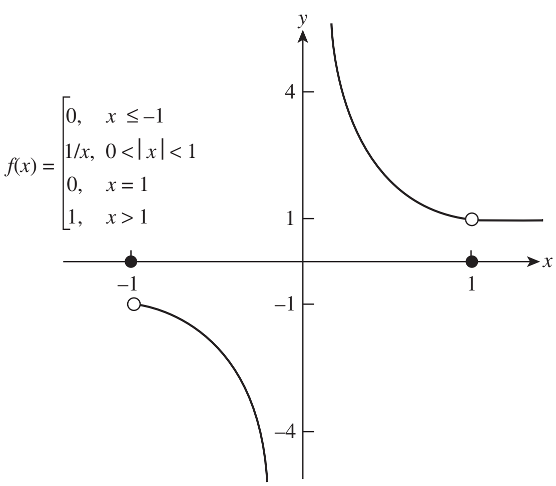


Answers to Practice Exercises

1. f is continuous at $x = -1$.
f is discontinuous at $x = 0$.
If we define $f(0) = 0$, then the discontinuity at $x = 0$ is removable.
f is discontinuous at $x = 1$.



2. f is discontinuous at $x = -1$.
f is discontinuous at $x = 0$.
f is discontinuous at $x = 1$.
If we define $f(1) = 1$, then the discontinuity at $x = 1$ is removable.



3. **(a)** -21 **(b)** 49
(c) 0 **(d)** 1
(e) 1 **(f)** 7
(g) -7 **(h)** $-\frac{1}{7}$.

4. **(a)** $-\sqrt{2}$ **(b)** $\frac{\sqrt{2}}{2}$
(c) $\frac{1}{2}+\sqrt{2}$. **(d)** 2
(e) $\frac{1}{2}$ **(f)** $-\frac{1}{2}$

5. 4 **6.** $-\frac{1}{2}$

7. **(a)** f is continuous on $(-\infty,\infty)$.
(b) g is continuous on $[0, \infty)$.
(c) h is continuous on $(-\infty, 0)$ and $(-\infty, \infty)$.
(d) k is continuous on $(0, \infty)$.

8. **(a)** $\bigcup_{n\in I}\left(\left(n-\frac{1}{2}\right)\pi, \left(n+\frac{1}{2}\right)\pi\right)$, where I = the set of all integers.
(b) $\bigcup_{n\in I}(n\pi, (n+1)\pi)$, where I = the set of all integers.
(c) $(-\infty, \pi) \cup (\pi, \infty)$
(d) $(-\infty, 0) \cup (0, \infty)$.

9. **(a)** The limit does not exist, because $\lim_{x\to 0^-}\frac{x-2}{x(x+7)} = \infty$ and $\lim_{x\to 0^+}\frac{x-2}{x(x+7)} = -\infty$.
(b) 0

10. $\lim_{x\to 0^+}\left[4g(x)\right]^{1/3} = 2 \Rightarrow \left[\lim_{x\to 0^+} 4g(x)\right]^{1/3} = 2 \Rightarrow \lim_{x\to 0^+} 4g(x) = 8$, since $2^3 = 8$. Then $\lim_{x\to 0^+} g(x) = 2$.

11. $\lim_{x\to\sqrt{5}} g(x) = \frac{1}{2} - \sqrt{5}$.

12. $\lim_{x\to 1} g(x) = 0$

13. $\lim_{x\to -2} g(x) = \infty$

14. $\frac{2}{5}$.

15. $\frac{2}{5}$.

16. 0
17. 0
18. $-\infty$
19. ∞
20. $\lim_{x\to\infty}\frac{\sin x}{\lfloor x \rfloor} = 0$.
21. 0
22. 1
23. 1
24. **(a)** $x = 3$ is a vertical asymptote.
(b) $x = 1$ is a vertical asymptote.
(c) $x = -4$ is a vertical asymptote.

25. **(a)** $y = -1$ is a horizontal asymptote.
(b) $y = 1$ is a horizontal asymptote.
(c) $y = 1$ and $y = -1$ are horizontal asymptotes.
(d) $y = \frac{1}{3}$ is a horizontal asymptote.

Answers to Single Choice Questions

1. (c)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (d)
11. (c)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (b)
31. (a)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (a)	**40.** (b)
41. (b)	**42.** (c)	**43.** (a)	**44.** (a)	**45.** (c)	**46.** (d)	**47.** (a)	**48.** (a)	**49.** (c)	**50.** (d)

Answers to Multiple Choice Questions

1. (b), (c) and (d)	**2.** (a), (b), (c) and (d)	**3.** (a) and (b)	**4.** (b) and (c)	**5.** (c)
6. (a), (b) and (c)	**7.** (a) and (b)	**8.** (a), (c) and (d)	**9.** (a), (b) and (d)	**10.** (a)
11. (a), (c) and (d)	**12.** (a), (b) and (c)	**13.** (b) and (c)	**14.** (a) and (c)	**15.** (a), (b) and (c)
16. (a), (c) and (d)	**17.** (a), (b) and (d)	**18.** (a) and (c)		

Answers to Passage Type Questions

Passage 1
1. (d) **2.** (a) **3.** (b)

Passage 2
1. (a) **2.** (c)

Passage 3
1. (d) **2.** (c)

Passage 4
1. (a) **2.** (b) **3.** (c)

Passage 5
1. (b) **2.** (a) **3.** (c)

Passage 6
1. (c) **2.** (b) **3.** (d)

Passage 7
1. (b) **2.** (d) **3.** (a)

Answers to Matrix Match Type Questions

1. (a) → (p), (q) and (r)	(b) → (p), (q) and (t)	(c) → (p), (q) and (t)	(d) → (p), (q) and (s)
2. (a) → (p)	(b) → (q)	(c) → (s)	(d) → (r)
3. (a) → (q) and (r)	(b) → (p), (r) and (s)	(c) → (p), (r) and (s)	
4. (a) → (r)	(b) → (p), (q), (r) and (s)	(c) → (p)	(d) → (q)
5. (a) → (r)	(b) → (p)	(c) → (q)	(d) → (t)

Answers to Integer Type Questions

1. (2)	**2.** (1)	**3.** (5)	**4.** (2)	**5.** (7)	**6.** (3)	**7.** (5)	**8.** (7)	**9.** (9)	**10.** (2)
11. (3)	**12.** (4)	**13.** (2)	**14.** (5)	**15.** (1)	**16.** (1)	**17.** (12)	**18.** (335)	**19.** (0)	**20.** (1)
21. (3)	**22.** (72)	**23.** (2)	**24.** (3)	**25.** (3)					

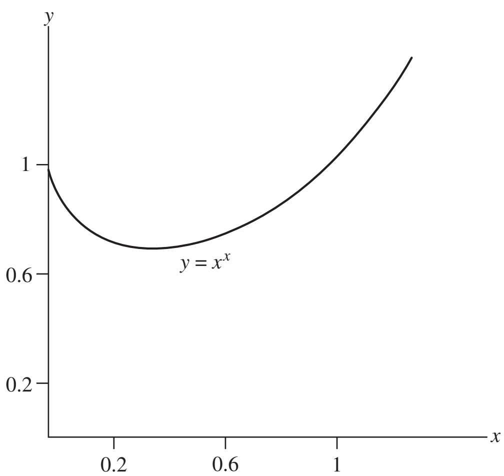
Answers to Additional and Advanced Exercises

1. (a)

x	0.1	0.01	0.001	0.0001	0.00001
x^x	0.7943	0.9550	0.9931	0.9991	0.9999

Apparently, $\lim_{x\to 0^+} x^x = 1$.

(b)

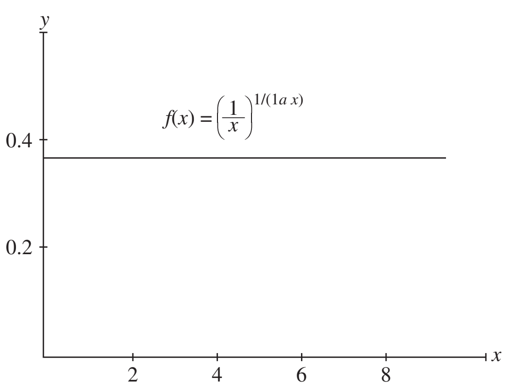


2. (a)

x	10	100	1000
$\left(\frac{1}{x}\right)^{1/(\ln x)}$	0.3679	0.3679	0.3679

Apparently, $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{1/(\ln x)} = 0.3678 = \frac{1}{e}$

(b)



3. **(a)** Since $x \to 0^+, 0 < x^3 < x < 1 \Rightarrow \left(x^3 - x\right) \to 0^-$

$\Rightarrow \lim_{x\to 0^+} f\left(x^3 - x\right) = \lim_{y\to 0^-} f\left(y\right) = B$ where $y = x^3 - x$.

(b) Since $x \to 0^-, -1 < x < x^3 < 0 \Rightarrow \left(x^3 - x\right) \to 0^+$

$\Rightarrow \lim_{x\to 0^-} f\left(x^3 - x\right) = \lim_{y\to 0^+} f\left(y\right) = A$ where $y = x^3 - x$.

(c) Since $x \to 0^+, 0 < x^4 < x^2 < 1 \Rightarrow \left(x^2 - x^4\right) \to 0^+$

$\Rightarrow \lim_{x\to 0^+} f\left(x^2 - x^4\right) = \lim_{y\to 0^+} f\left(y\right) = A$ where $y = x^2 - x^4$.

(d) Since $x \to 0^-, -1 < x < 0 \Rightarrow x^4 < x^2 < 1 \Rightarrow \left(x^2 - x^4\right) \to 0^+$

$\Rightarrow \lim_{x\to 0^+} f\left(x^2 - x^4\right) = A$ as in part (c).

4. **(a)** True, because if $\lim_{x\to a}(f(x) + g(x))$ exists then $\lim_{x\to a}(f(x) + g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a}[(f(x) + g(x)) - f(x)]$ $= \lim_{x\to a} g(x)$ exists, contrary to assumption.

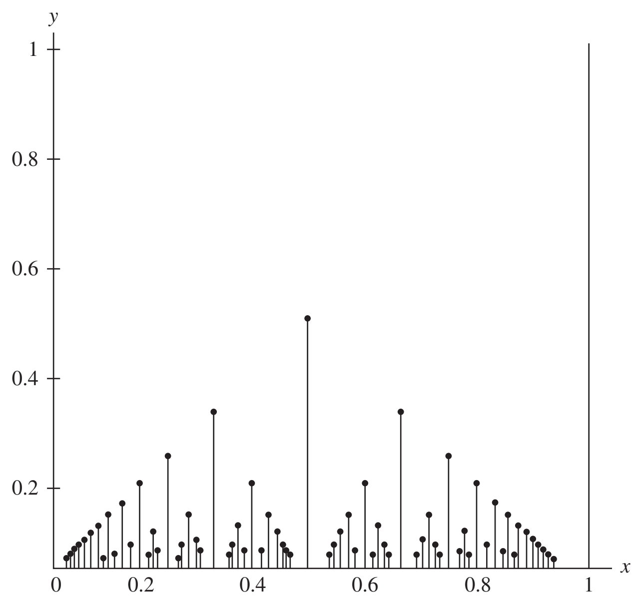
(b) False: for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$.
Then neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists, but

$\lim_{x\to 0}(f(x) + g(x)) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0$ exists.

(c) True, because $g(x)|x|$ is continuous $\Rightarrow g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions)

(d) False: for example let $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

8. **(c)** The graph looks like the marking on a typical ruler when the points $(x, f(x))$ on the graph $f(x)$ are connected to the x-axis with vertical lines.



$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

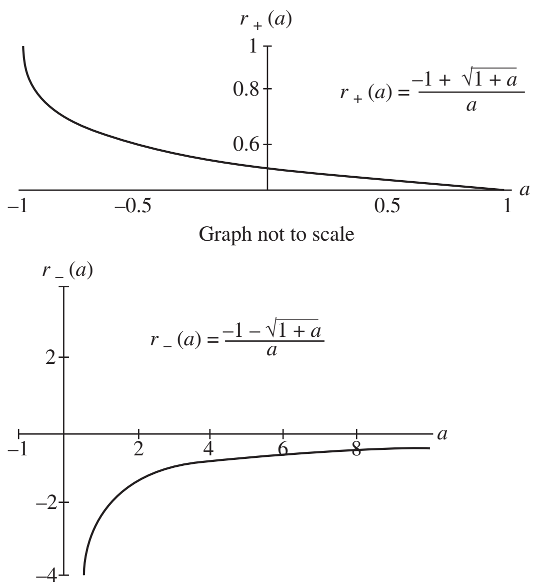
9. Yes

10. 2

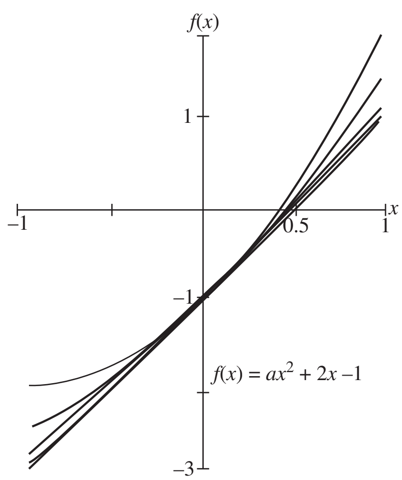
11. **(a)** $\frac{1}{2}$ and 1

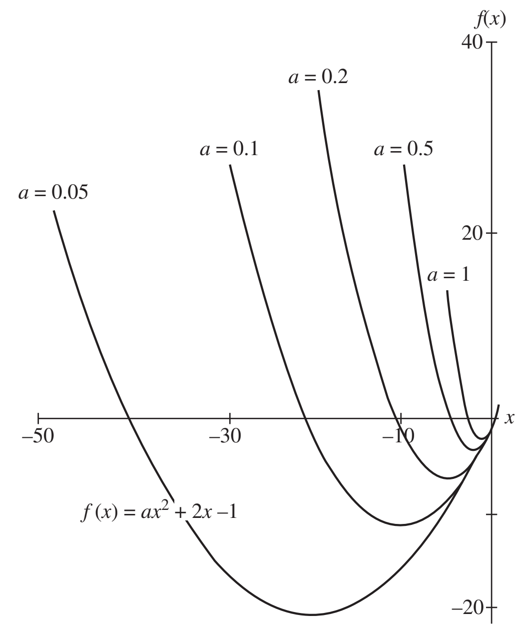
(b) $\lim_{a\to 0} r_-(a)$ does not exist and 1

(c)



(d)





14. **(b)** Let $\min\{a,b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$.

15. 0
16. 0
17. 1
18. 1
19. 4
20. $\frac{1}{6}$

Answers to Exercises

Chapter 3

Exercises 3.1

1. $P_1 : m_1 = 1$
$P_2 : m_2 = 5$

2. $P_1 : m_1 = -2$
$P_2 : m_2 = 0$

3. $P_1 : m_1 = 5/2$
$P_2 : m_2 = -1/2$

4. $P_1 : m_1 = 3$
$P_2 : m_2 = -3$

5. **(a)** It is the rate of change of the number of cells when $t = 5$. The units are the number of cells per hour.
(b) P′(3) is larger than P′(2) as slope is increasing.
(c) P′(5) = 51.72.

6. **(a)** From $t = 0$ to $t = 3$, the derivative is positive.
(b) At $t = 3$, the derivative appears to be 0. From $t = 2$ to $t = 3$, the derivative is positive but decreasing.

7. $y = -(x+1)$, $y = -(x-3)$

8. $y = \dfrac{x}{4} + 1$

9. 19.6 m/sec
10. 60 ft / sec
11. 6π
12. 16π
13. Yes
14. No
15. Yes
16. No
17. Nowhere
18. Nowhere
19. At $x = 0$
20. At $x = 0$
21. Nowhere
22. At $x = 0$
23. At $x = 1$
24. at $x = 0$ and $x = 1$
25. At $x = 0$
26. at $x = 4$

Exercises 3.2

1. $\dfrac{3}{2}q^{1/2}$

2. $\dfrac{-w}{(w^2-1)^{3/2}}$

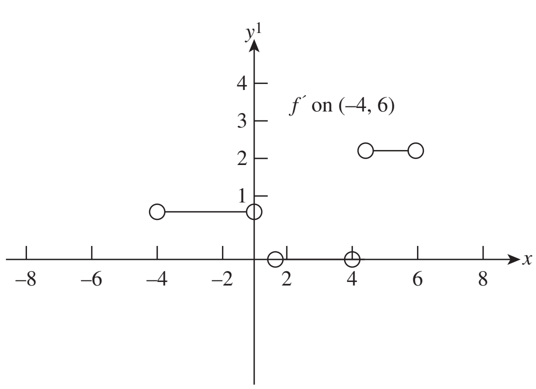
3. $\dfrac{-1}{(x-1)^2}$

4. $g'(x) = \dfrac{1}{2\sqrt{x}}$

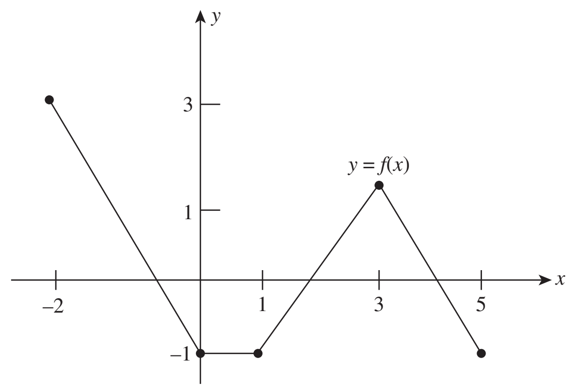
5. (b)
6. (a)
7. (d)
8. (c)

9. **(a)** $x = 0, 1, 4$

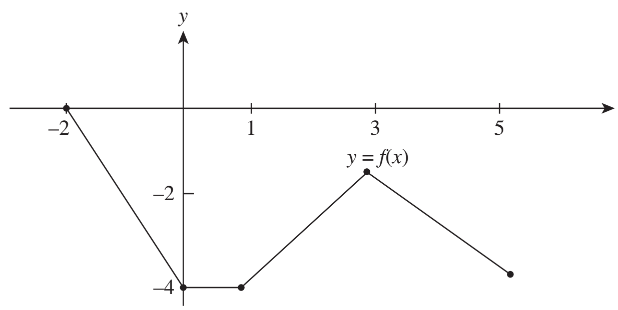
(b)



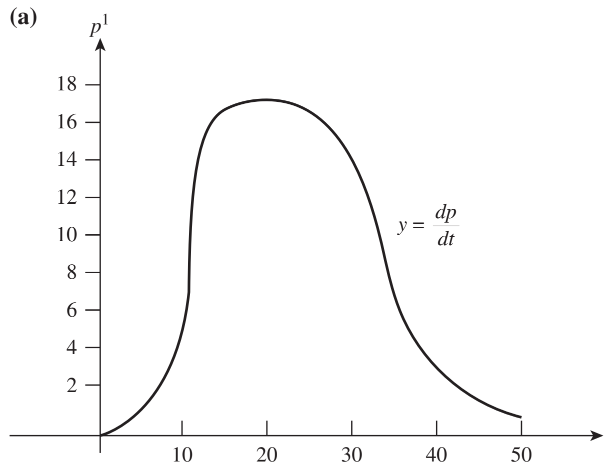
10. **(a)**



(b) Shift the graph in (a) down 3 units



11. **(a)**



(b) **Fastest:** 20th and 30th days
Slowest: 40th and 50th days

12. Since $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = 1$

While $\lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = 0$,

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist and $f(x)$ is not differentiate at $x = 0$

13. left hand derivative : 0
Rights hand derivative : $f'(1)$ does not exist.

14. Since $\lim\limits_{h\to 0^+}\dfrac{f(1+h)-f(1)}{h}=2$ while $\lim\limits_{h\to 0^-}\dfrac{f(1+h)-f(1)}{h}=\dfrac{1}{2}$,
$f'(1)=\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}$ does not exist and $f(x)$ is not differentiable at $x=1$

15. Left H.D : 1
R.H.D : $f'(1)$ does not exist

16. Since $f(x)$ is not continuous at $x=0$, $f(x)$ is not differentiable at $x=0$

17. L.H.D : $+\infty$
R.H.D : $g'(0)$ does not exist.

18. **(a)** $-3\le x\le 2$
(b) None
(c) None

19. **(a)** function is differentiable on its domain $-2\le x\le 3$ (it is smooth)
(b) None
(c) None

20. (a) $-3\le x<0,\ 0<x\le 3$
(b) None
(c) $x=0$

21. **(a)** f is differentiable on $-2<x<-1$, $-1<x<0$, $0<x<2$ and $2<x\le 3$
(b) $f'(-1)$ does not exist.
(c) At $x=0$ $\lim\limits_{x\to 0}f(x)$ does not exist at $x=2$ $\lim\limits_{x\to 2}f(x)\ne f(2)$

22. **(a)** $-1\le x<0,\ 0<x\le 2$
(b) $x=0$
(c) None

23. **(a)** f is differentiable on $-3\le x<-2$, $-2<x<2$ and $2<x\le 3$
(b) f is continuous but not differentiable at $x=-2$ and $x=2$; there are corners at those points.
(c) None.

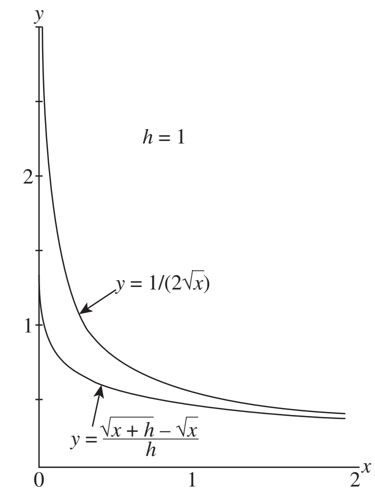
24. Yes

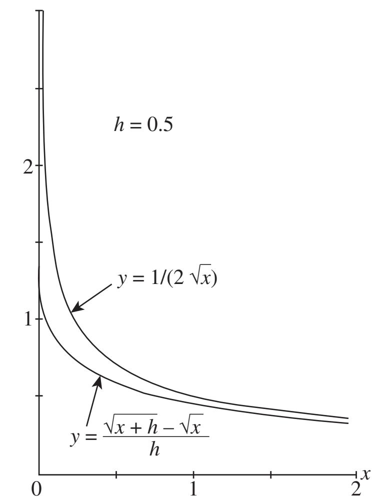
25. Yes

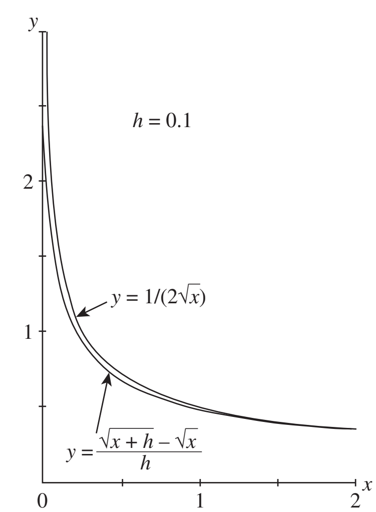
26. Yes, No

27. **(a)** $f'(0)=0$
(b) $f'(0)=0$

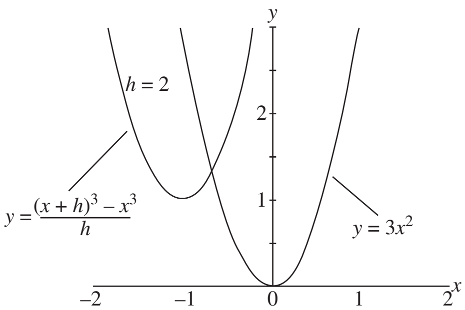
28. The graphs are shown below for $h=1, 0.5, 0.1$ The function $y=\dfrac{1}{2\sqrt{x}}$ is the derivative of the function $y=\sqrt{x}$, so that $\dfrac{1}{2\sqrt{x}}=\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$. The graphs reveal that $y=\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ gets closer to $y=\dfrac{1}{2\sqrt{x}}$ as h gets smaller and smaller.

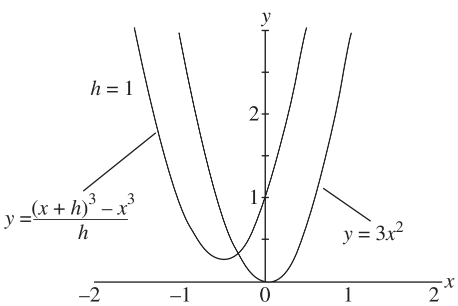


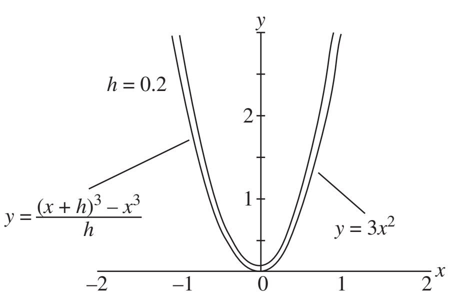




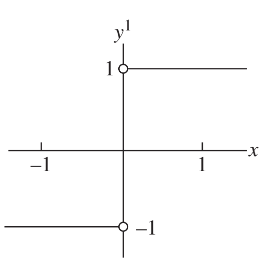
29. The graphs are shown below for $h = 2, 1, 0.5$. The function $y = 3x^2$ is the derivative of the function $y = x^3$ so that $3x^2 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$. The graphs reveal that $y = \frac{(x+h)^3 - x^3}{h}$ get closer to $y = 3x^2$ as h gets smaller and smaller.



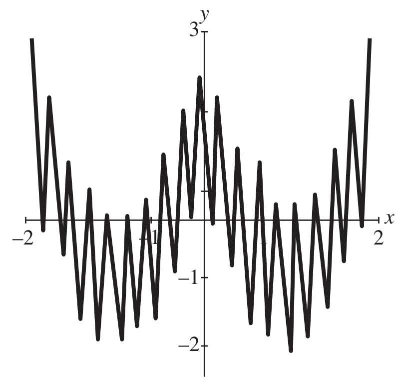




30. The graphs are the same. So we know that for $f(x) = |x|$, we have $f'(x) = \frac{|x|}{x}$.



31. Weierstrass's nowhere differentiable continuous function.



$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \ldots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$

Exercises 3.3

1. $y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$

2. $y' = \frac{3}{4x^{1/4}} + \frac{3}{x^4} + \frac{11}{4}x^{7/4} + \frac{1}{x^2}$

3. $y' = \frac{-4x^3 - 3x^2 + 1}{(x^2-1)^2(x^2+x+1)^2}$

4. $\frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

5. **(a)** 13 **(b)** −7
(c) 7/25 **(d)** 20

6. **(a)** −2 **(b)** $\frac{2}{25}$
(c) $\frac{-1}{2}$ **(d)** −7

7. **(a)** $y = -\frac{x}{8} + \frac{5}{4}$ **(b)** $m = -4$ at $(0,1)$

8. Horizontal $y' = 3x^2 - 3$ perpendicular $y = -4$. The line perpendicular to this line at $(1, -4)$ is $x = 1$

9. $y = 4x,\ y = 2$

10. $y = \frac{-x}{2} + 2$

11. $a = 1\ b = 1\ c = 0$

12. $a = -3\ b = 2\ c = 1$

13. (2, 4)

14. $\left(4, \frac{-5}{3}\right)$ or $\left(-1, \frac{-5}{6}\right)$

15. (0, 0), (4, 2)
16. (4, 16) or (2, 4)
17. **(a)** $y = 2x + 2$ **(c)** (2,6)
18. **(a)** $y = 5x$
19. 50
20. $\frac{-2}{9}$
21. $a = -3$
22. $b = -\frac{3}{2}$
23. $P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1$
24. $\frac{dR}{dM} = CM - M^2$
25. The Product Rule is then the constant Multiple Rule, so the latter is a special case of product Rule.
27. **(a)** $\frac{d}{dx}(uvw) = uvw' + uv'w + u'vw$

(b) $\frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$

(c) $\frac{d}{dx}(u_1 \ldots u_n) = u_1u_2 \ldots u_{n-1}u_n' + u_1u_2 \ldots u_{n-2}u_{n-1}'u_n + \ldots u_1'u_2 \ldots u_n$

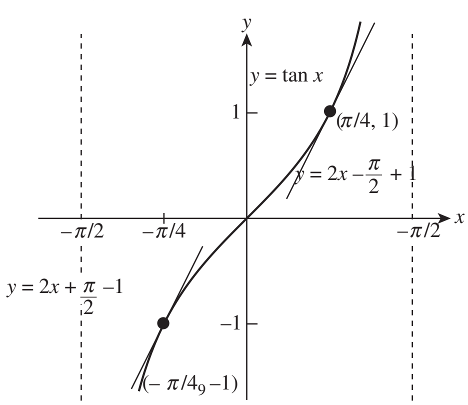
28. $\frac{d}{dx}\left(x^{-m}\right) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{-m \cdot x^{m-1}}{x^2 m} = -m \cdot x^{m-1-2m} = -m \cdot x^{-m-1}$

29. $\frac{dP}{dV} = -\frac{-nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

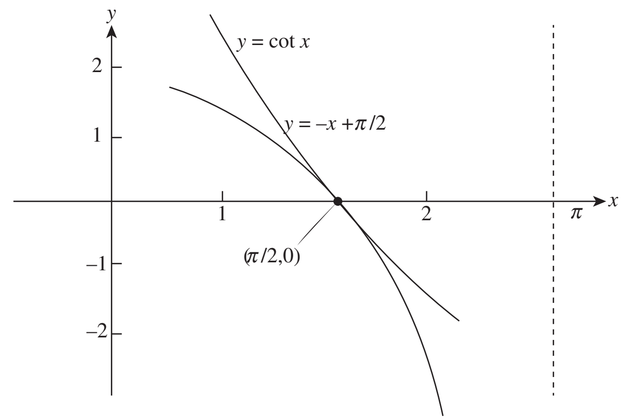
30. $\frac{dA}{dq} = -\frac{KM}{q^2} + \frac{h}{2}$ $\frac{d^2A}{dq^2} = \frac{2KM}{q^3}$

Exercises 3.4

1. $y = -x^2 \sin x$
2. $-x^3 \sin x + x^3 \cos^2 x + 3x^2 \sin x \cos x$
3. $g(x) = 2(2-x)\tan x\left(\sec^2 x - \tan x\right)$
4. **(a)** $2\csc^3 x - \csc x$ **(b)** $2\sec^3 x - \sec x$
5. **(a)** $y^{(4)} = -2\sin x$ **(b)** $y^{(4)} = 9\cos x$
6. $\left(-\frac{\pi}{4}, -1\right)$; $\left(\frac{\pi}{4}, 1\right)$



7. $\left(\frac{\pi}{2}, 0\right)$ is $y = -x + \frac{\pi}{2}$



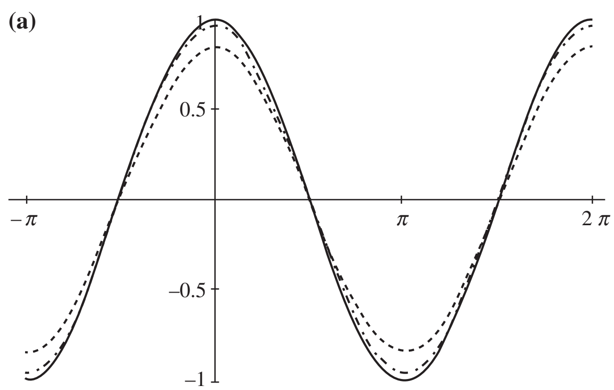
8. **(a)** $y = -x + \frac{\pi}{2} + 2$

(b) When $x = \frac{\pi}{3}$, then $y = 4 - \sqrt{3}$ is horizontal tangent.

9. **(a)** $y = -4x + \pi + 4$

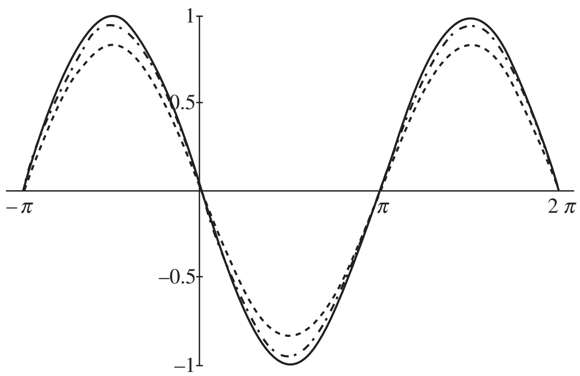
(b) when $x = \frac{3\pi}{4}$, then $y = 2$ is horizontal tangent

10. 0 **11.** $\sqrt{2}$
12. $\frac{\sqrt{3}}{2}$ **13.** 2
14. -1 **15.** -1
16. 0 **17.** -1
18. $c = 9$
19. The left and right-hand derivatives can never agree at $x = 0$, so g is not differentiable at $x = 0$ for any value of b (including $b = 1$)
20. $\sin x$
21. **(a)** $\sec x \tan x$ **(b)** $-\csc x \cot x$ **(c)** $-\csc^2 x$
22. **(a)** $-5\sqrt{2}\,cm$ **(b)** $-5\sqrt{2}\,\frac{\text{cm}}{\text{sec}}$
23. **(a)**



The dashed curves of $y = \frac{\sin(x+h) - \sin(x-h)}{2h}$ are closer to the black curve $y = \cos x$ than the corresponding dashed curves.

(b)



The dashed curves of $y = \dfrac{\cos(x+h)-\cos(x-h)}{2h}$ are closer to the black curve $y = -\sin x$ than the corresponding dashed curves.

24. Derivative of $f(x) = |x|$ does not exist at $x = 0$

Exercises 3.5

1. $\dfrac{-5}{x}\sin^{-6}x\cos x - \dfrac{1}{x^2}\sin^{-5}x + x\cos^2 x\sin x - \dfrac{1}{3}\cos^3 x$

2. $(3x-2)^5 - \dfrac{1}{x^3\left(4-\dfrac{1}{2x^2}\right)^2}$

3. $\dfrac{6}{(5-2x)^4} - \dfrac{\left(\dfrac{2}{x}+1\right)^3}{x^2}$

4. $\dfrac{(4x+3)^3(4x+7)}{(x+1)^4}$

5. $6(x^2-5x)^5 - \dfrac{2(x^2-5x)^6}{(2x-5)^2}$

6. $\dfrac{(3(x+7)\sec^2 3x - 4\tan 3x)}{(x+7)^5}$

7. $\dfrac{2\sin\theta}{(1+\cos\theta)^2}$

8. $\dfrac{-2-2\sin 3t - 9\cos 3t + 6t\cos 3t}{(1+\sin 3t)}$

9. $-2\sin(\theta^2)\sin(2\theta) + 2\theta\cos(2\theta)\cos(\theta^2)$

10. $(\sec\sqrt{\theta})\left[\dfrac{\tan\sqrt{\theta}\tan\left(\dfrac{1}{\theta}\right)}{2\sqrt{\theta}} - \dfrac{\sec^2\left(\dfrac{1}{\theta}\right)}{\theta^2}\right]$

11. $\left[1+\tan^4\left(\dfrac{t}{12}\right)\right]^2\left[\tan^3\left(\dfrac{t}{12}\right)\sec^2\left(\dfrac{t}{12}\right)\right]$

12. $3(2t^2-5)^3(18t^2-5)$

13. $\dfrac{1}{2\sqrt{3t+\sqrt{2\sqrt{1-t}}}}\left(\dfrac{12\sqrt{1-t}\sqrt{2+\sqrt{1-t}}-1}{4\sqrt{1-t}\sqrt{2+\sqrt{1-t}}}\right)$

14. $\dfrac{6}{x^3}\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)$

15. $\dfrac{1}{2x}(1-\sqrt{x})^{-3}\left(\dfrac{3}{2}-\dfrac{1}{2\sqrt{x}}\right)$

16. $2\csc^2(3x-1)\cot(3x-1)$

17. $2\sec^2\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)$

18. $16(2x+1)^2(5x+1)$

19. $2(x^3-1)^3(136x^6-47x^3+1)$

20. -5 **21.** 2

22. **(a)** $\dfrac{2}{3}$ **(b)** $2\pi+5$ **(c)** $15-8\pi$

(d) $\dfrac{37}{6}$ **(e)** 1 **(f)** $\dfrac{\sqrt{2}}{24}$

(g) $\dfrac{5}{32}$ **(h)** $-\dfrac{5}{3\sqrt{17}}$

23. **(a)** 1 **(b)** 6 **(c)** 1

(d) $-\dfrac{1}{9}$ **(e)** $-\dfrac{40}{3}$ **(f)** $-\dfrac{1}{3}$

(g) $-\dfrac{2}{9}$

Exercises 3.6

1. $\dfrac{1}{x}$ **2.** $1/x$ **3.** 2/t

4. $3/2t$ **6.** $-1/x$ **5.** $-1/x$

7. $\dfrac{1}{\theta+1}$ **8.** $\dfrac{1}{\theta+1}$ **9.** $\dfrac{3}{x}$

10. $\dfrac{3(\ln x)^2}{x}$ **11.** $(\ln t)^2 + 2\ln t$

12. $(\ln t)^{1/2} + \dfrac{1}{2(\ln t)^{1/2}}$ **13**. $x^3\ln x$

14. $4x^7(\ln x)^3 + 8x^7(\ln x)^4$ **15.** $\dfrac{1-\ln t}{t^2}$

16. $\dfrac{-\ln t}{t^2}$ **17.** $\dfrac{1}{x(1+\ln x)^2}$

18. $1-\dfrac{\ln x}{(1+\ln x)^2}$ **19.** $\dfrac{1}{x\ln x}$

20. $\dfrac{1}{x(\ln x)\ln(\ln x)}$

21. $2\cos(\ln\theta)$

22. $\sec\theta$

23. $\dfrac{-(3x+2)}{2x(x+1)}$

24. $\dfrac{1}{1-x^2}$

25. $\dfrac{2}{t(1-\ln t)^2}$

26. $\dfrac{1}{4t\sqrt{\ln\sqrt{t}}}$

27. $\dfrac{\tan(\ln\theta)}{\theta}$

28. $\dfrac{1}{2}\left[\cot\theta-\tan\theta-\dfrac{4}{\theta(1+2\ln\theta)}\right]$

29. $\dfrac{10x}{x^2+1}+\dfrac{1}{2(1-x)}$

30. $\dfrac{-5}{2}\left[\dfrac{3x+2}{(x+1)(x+2)}\right]$

31. $\dfrac{2x+1}{2\sqrt{x(x+1)}}$

32. $\dfrac{(2x^2-x+1)|x-1|}{\sqrt{x^2+1}(x-1)}$

33. $\dfrac{1}{2\sqrt{t}(t+1)^{3/2}}$

34. $\dfrac{-(2t+1)}{2(t^2+t)^{3/2}}$

35. $\dfrac{dy}{d\theta}=\sqrt{\theta+3}(\sin\theta)\left[\dfrac{1}{2(\theta+3)}+\cot\theta\right]$

36. $(\sec^2\theta)\sqrt{2\theta+1}+\dfrac{\tan\theta}{\sqrt{2\theta+1}}$

37. $3t^2+6t+2$

38. $\dfrac{-3t^2+6t+2}{(t^3+3t^2+2t)^2}$

39. $\dfrac{dy}{d\theta}=\left(\dfrac{\theta+5}{\theta\cos\theta}\right)\left(\dfrac{1}{\theta+5}-\dfrac{1}{\theta}+\tan\theta\right)$

40. $\dfrac{dy}{d\theta}=\dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\dfrac{1}{\theta}+\cot\theta-\dfrac{1}{2}\tan\theta\right)$

41. $y'=\dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\dfrac{1}{x}+\dfrac{x}{x^2+1}-\dfrac{2}{3(x+1)}\right]$

42. $y'=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}\left[\dfrac{5}{x+1}-\dfrac{5}{2x+1}\right]$

43. $y'=\dfrac{1}{3}\sqrt[3]{\dfrac{x(x-2)}{x^2+1}}\left(\dfrac{1}{x}+\dfrac{1}{x-2}-\dfrac{2x}{x^2+1}\right)$

44. $y'=\dfrac{1}{3}\sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\left(\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{2x}{x^2+1}-\dfrac{2}{2x+3}\right)$

45. $-\sqrt{2}$

46. -8

47. $\dfrac{-\sqrt{2}}{4}$

48. 0

49. -2

50. 1/4

51. $-3\sqrt{3}$

52. $-1/3$

53. $1/2$

54. 1

55. -4

56. -1

57. 108

58. $-1/8$

59. $-3/16$

60. $\dfrac{10\sqrt{3}}{9}$

61. 0

62. -4

63. 1

64. 1/2

Exercises 3.7

1. $y'=\dfrac{2-4x^3-9x^2y}{3x^3+1}$

2. $-y/x$

3. $\dfrac{-\cos^2(xy)-y}{x}$

4. $\dfrac{dy}{dx}=\dfrac{3x^2y^2-4x^3}{\cos y-2x^3y}$

5. $\dfrac{-y^2}{y\sin\left(\dfrac{1}{y}\right)-\cos\left(\dfrac{1}{y}\right)+xy}$

6. $-1/y^3$

7. $\dfrac{y^{1/3}}{3x^{4/3}}+\dfrac{1}{3y^{1/3}.x^{2/3}}$

8. $y''=\dfrac{y^2-(x+1)^2}{y^3}$

9. $y''=\dfrac{-1}{(y+1)^3}$

10. $\dfrac{1}{2(1+\sqrt{y})^3}$

11. $\dfrac{2y(x+y)}{(x+2y)^3}$

12. -2

13. $-1/4$

14. 1

15. 1, 1

16. -2

17. $y=-2x-3$

18. $\sqrt{3}$

19. $y=\dfrac{-1}{2}x+\dfrac{3}{2}$

20. $m=\dfrac{-27}{8}$

21. **(a)** $4/5$
(b) Horizontal tangent at $x=\sqrt[3]{54}$ and co-ordinate $(\sqrt[3]{54},\sqrt[3]{4})$
(c) Vertical tangent $\sqrt[3]{4}$ and co-ordinate $(\sqrt[3]{4},\sqrt[3]{54})$

22. $(3,-1)$

23. $\dfrac{p}{q}.x^{(p/q)-1}$

24. $a=\dfrac{3}{4}$

25. Tangents to the curves are perpendicular at (1, 1) and $(1,-1)$, therefore curves are orthogonal at these two points of intersection.

27. $b=\dfrac{28}{3}$

Exercises 3.8

1. $\dfrac{-2x}{\sqrt{1-x^4}}$

2. $\dfrac{1}{|x|\sqrt{x^2-1}}$

3. $\dfrac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $\dfrac{-1}{\sqrt{2t-t^2}}$

5. $\dfrac{1}{|2s+1|\sqrt{s^2+s}}$

6. $\dfrac{1}{|s|\sqrt{25s^2-1}}$

7. $\dfrac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$ **8.** $\dfrac{-2}{|x|\sqrt{x^2-4}}$

9. $\dfrac{-1}{\sqrt{1-t^2}}$ **10.** $\dfrac{-6}{t\sqrt{t^4-9}}$

11. $\dfrac{-1}{2\sqrt{t}\,(1+t)}$ **12.** $\dfrac{-1}{2t\sqrt{t-1}}$

13. $\dfrac{1}{(\tan^{-1}x)(1+x^2)}$ **14.** $\dfrac{1}{x\left[1+(\ln x)^2\right]}$

15. $\dfrac{-1}{\sqrt{e^{2t}-1}}$ **16.** $\dfrac{e^{-t}}{\sqrt{1-e^{-2t}}}$

17. $\dfrac{-2s^2}{\sqrt{1-s^2}}$ **18.** $\dfrac{s|s|-1}{|s|\sqrt{s^2-1}}$

19. 0 **20.** 0

21. $\sin^{-1}x$ **22.** $-\tan^{-1}\left(\dfrac{x}{2}\right)$

23. **(a)** $\alpha=\cos^{-1}\left(\dfrac{x}{15}\right)-\cot^{-1}\left(\dfrac{x}{3}\right)$

(b) $x=3\sqrt{5}\approx 6.7082\,ft$

24. 42.22°

25. Smallest angle α has a tangent of $1\Rightarrow\alpha=\tan^{-1}1$
Middle angle β has a tangent of $2\Rightarrow\beta=\tan^{-1}2$
Largest angle γ has a tangent at $3\Rightarrow\gamma=\tan^{-1}3$

26. **(a)** $\sec^{-1}(-x)$ **(b)** $\pi-\sec^{-1}x$

Exercises 3.9

1. $D: f\left(f^{-1}(x)\right)=\left(x^{1/5}\right)^5=x$

$R: f^{-1}\left(f(x)\right)=\left(x^5\right)^{1/5}=x$

2. $D: f\left(f^{-1}(x)\right)=\left(x^{1/4}\right)^4=x$

$R: f^{-1}\left(f(x)\right)=\left(x^4\right)^{1/4}=x$

3. $D: f\left(f^{-1}(x)\right)=\left((x-1)^{1/3}\right)^3+1=(x-1)+1=x$

$R: f^{-1}\left(f(x)\right)=\left((x^3+1)-1\right)^{1/3}=\left(x^3\right)^{1/3}=x$

4. $D: f\left(f^{-1}(x)\right)=\dfrac{1}{2}(2x+7)-\dfrac{7}{2}==\left(x+\dfrac{7}{2}\right)-\dfrac{7}{2}=x$

$R: f^{-1}\left(f(x)\right)=2\left(\dfrac{1}{2}x-\dfrac{7}{2}\right)+7=(x-7)+7=x$

5. $D: f\left(f^{-1}(x)\right)=\dfrac{1}{\left(\dfrac{1}{\sqrt{x}}\right)^2}=\dfrac{1}{\left(\dfrac{1}{x}\right)}=x$

$R: f^{-1}\left(f(x)\right)=\dfrac{1}{\sqrt{\dfrac{1}{x^2}}}=\dfrac{1}{\left(\dfrac{1}{x}\right)}=x$

6. $D: f\left(f^{-1}(x)\right)=\dfrac{1}{\left(x^{-1/3}\right)^3}=x$

$R: f^{-1}\left(f(x)\right)=\left(\dfrac{1}{x^3}\right)^{-1/3}=x$

7. $D: f\left(f^{-1}(x)\right)=\dfrac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)}=x$

$R: f^{-1}\left(f(x)\right)=\dfrac{2(x+3)+3(x-2)}{(x+3)-(x-2)}=x$

8. Domain of $f^{-1}:(-\infty,\,0]\cup(1,\infty)$,

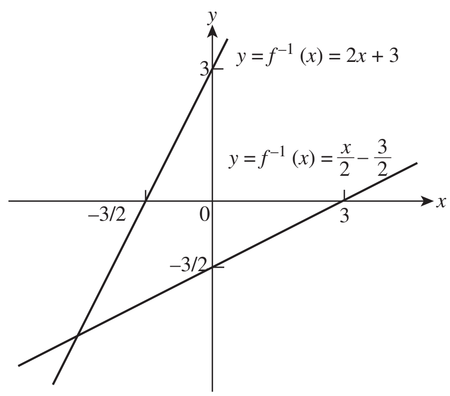
Range of $f^{-1}:[0,9)\cup(9,\infty)$; $f\left(f^{-1}(x)\right)=\dfrac{\sqrt{\left(\dfrac{3x}{x-1}\right)^2}}{\sqrt{\left(\dfrac{3x}{x-1}\right)^2-3}}$

9. Domain of $f^{-1}:[-1,\,\infty)$, Range of $f^{-1}:(-\infty,1]$

10. Domain of $f^{-1}:(-\infty,\infty)$, Range of $f^{-1}:(-\infty,\infty)$;

11. **(a)** $f^{-1}(x)=\dfrac{x}{2}-\dfrac{3}{2}$

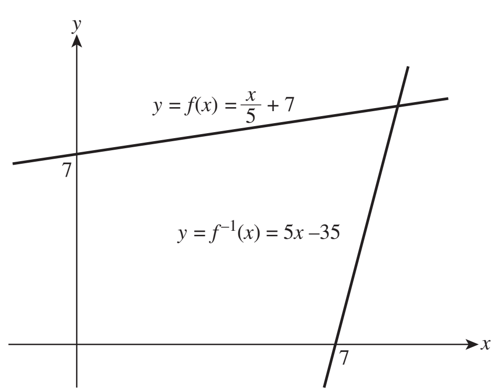
(b)



(c) $\left.\dfrac{df}{dx}\right|_{x=-1}=2,\left.\dfrac{df^{-1}}{dx}\right|_{x=1}=\dfrac{1}{2}$

12. **(a)** $f^{-1}(x)=5x-35$

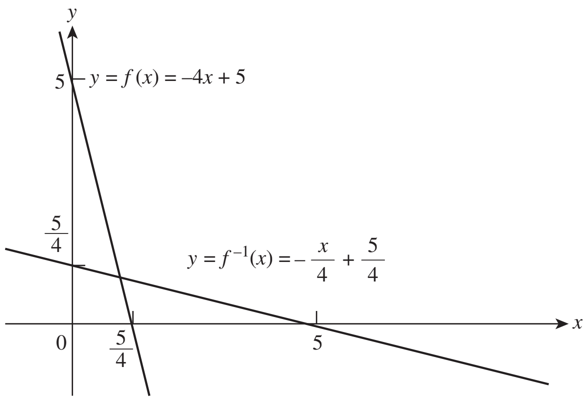
(b)



(c) $\left.\dfrac{df}{dx}\right|_{x=-1}=\dfrac{1}{5},\left.\dfrac{df^{-1}}{dx}\right|_{x=34/5}=5$

13. **(a)** $f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$

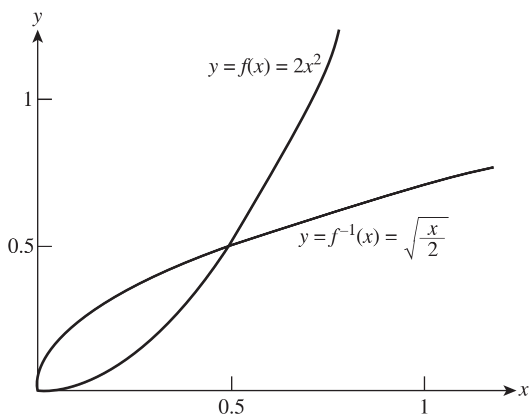
(b)



(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4, \left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

14. **(a)** $f^{-1}(x) = \sqrt{\frac{x}{2}}$

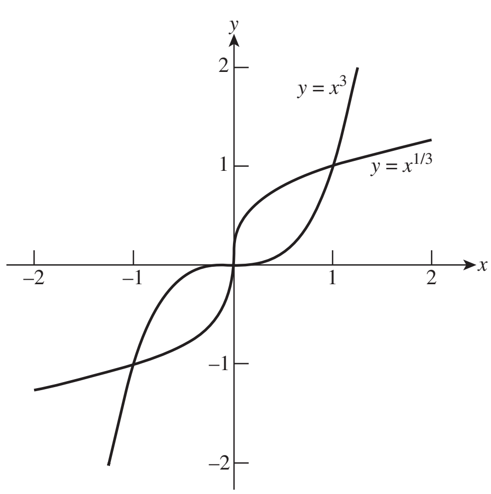
(b)



(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x\big|_{x=5} = 20$

$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \frac{1}{2\sqrt{2}}x^{-1/2}\Big|_{x=50} = \frac{1}{20}$

15. (b)

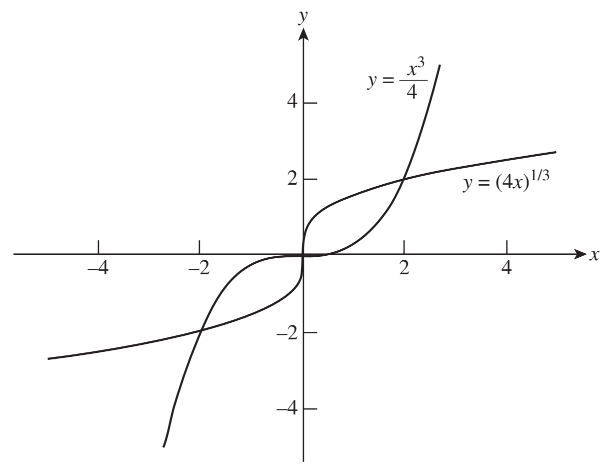


(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3, f'(-1)3;$

$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}, g'(-1) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $f(x) = x^3$ at (0, 0); the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at (0, 0)

16. **(b)**



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3, h'(-2) = 3$

$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}, k'(-2) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at (0, 0); the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at (0, 0)

17. $\frac{1}{9}$ **18.** $\frac{1}{6}$

19. 3 **20.** $\frac{1}{2}$

Exercises 3.10

1. $\frac{1}{4}$	**2.** 5	**3.** $5/7$
4. $3/11$	**5.** $1/2$	**6.** 0
7. $1/4$	**8.** -10	**9.** $-23/7$
10. $9/11$	**11.** $5/7$	**12.** $-2/3$
13. 0	**14.** $5/2$	**15.** -16
16. $-1/6$	**17.** -2	**18.** 3
19. $1/4$	**20.** $\frac{1}{1+\pi}$	**21.** 2
22. $1/2$	**23.** 3	**24.** 2
25. -1	**26.** 1	**27.** $\ln 3$
28. $-1\ln 2$	**29.** $\frac{1}{\ln 2}$	**30.** $\frac{\ln 3}{\ln 2}$
31. $\ln 2$	**32.** $\frac{\ln 3}{\ln 2}$	**33.** 1

34. 1 **35.** 1/2

36. $\lim_{y\to 0}\frac{a}{2\sqrt{ay+a^2}}=\frac{1}{2}, a>0$ **37.** $\ln 2$

38. 0 **39.** $-\infty$ **40.** 3

41. $-1/2$ **42.** 1 **43.** -1

44. $1/2$ **45.** 1 **46.** 0

47. 0 **48.** 1 **49.** 2

50. $1/2$ **51.** $\frac{1}{e}$ **52.** e

53. 1 **54.** $(e)^{\frac{1}{e}}$ **55.** $\frac{1}{e}$

56. e **57.** $e^{1/2}$ **58.** e^2

59. $e^0=1$ **60.** 1 **61.** e^3

62. 1 **63.** 0 **64.** 0

65. 1 **66.** 0 **67.** 3

68. 1 **69.** 1 **70.** 1

71. 0 **72.** -1 **73.** ∞

74. ∞

75. Part (b) correct

76. Part (b) is correct

77. Part (d) is correct

78. **(a)** $=-1$ **(b)** $\frac{b+a}{2}$ **(c)** $\frac{-1+\sqrt{37}}{3}$

79. $27/10$ **80.** $-8/3$ **81.** e^r

82. **(a)** $x=e$ **(b)** $x=\sqrt{e}$

(c) $x=\sqrt[n]{e}$ **(d)** 1

83. **(a)** $x\to\infty$ and $x\to-\infty$

(b) $y=3/2$ as $x\to-\infty$

$y=0$ as $x\to\infty$

84. 0

Answers to Practice Exercises

1. $y'=-\frac{y+2}{x+3}$ **2.** $\frac{dy}{dx}=\frac{5-2x-y}{x+2y}$

3. $\frac{dy}{dx}=\frac{2-3x^2-4y}{4x-4y^{1/3}}$ **4.** $-\frac{1}{3(xy)^{1/5}}$

5. $\frac{dy}{dx}=-\frac{y}{x}$ **6.** $\frac{dy}{dx}=-\frac{y}{x}$

7. $\frac{dy}{dx}=\frac{1}{2y(x+1)^2}$ **8.** $\frac{dy}{dx}=\frac{1}{2y^3(1-x)^2}$

9. $\frac{dp}{dq}=\frac{6q-4p}{3p^2+4q}$ **10.** $\frac{dp}{dq}=-\frac{\left(5p^2+2p\right)^{5/2}}{3\left(5p+1\right)}$

11. **(a)** $\frac{-2xy^3-2x^4}{y^5}$ **(b)** $\frac{-2xy^2-1}{y^3x^4}$

12. **(a)** $\frac{x}{y}$ **(b)** $\frac{-1}{y^3}$

13. **(a)** 7 **(b)** -2

(c) $\frac{5}{12}$ **(d)** $\frac{1}{4}$

(e) 12 **(f)** $\frac{9}{2}$

(g) $\frac{3}{4}$

14. **(a)** $-\frac{13}{10}$ **(b)** $-\frac{1}{3}$

(c) $\frac{1}{10}$ **(d)** -1

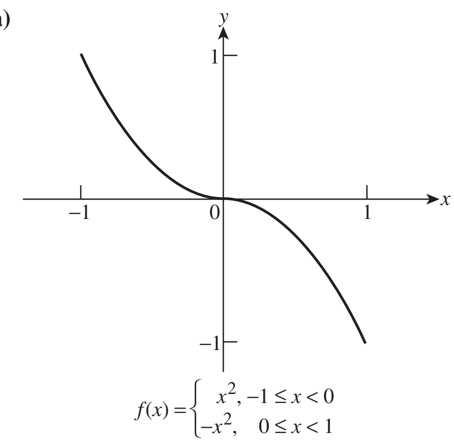
(e) $-\frac{2}{3}$ **(f)** -12

15. 0 **16.** $\frac{9}{2}$

17. $\sqrt{3}$ **18.** $-\frac{1}{6}$

19. $-\frac{1}{2}$ **20.** $\frac{1}{6}$

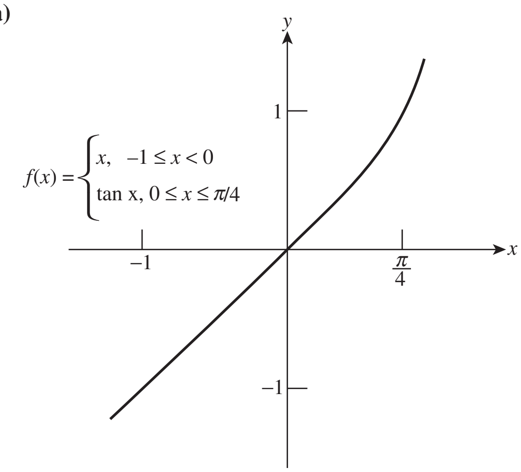
21. **(a)**



(b) f is continuous at $x=0$

(c) f is differentiable at $x=0$

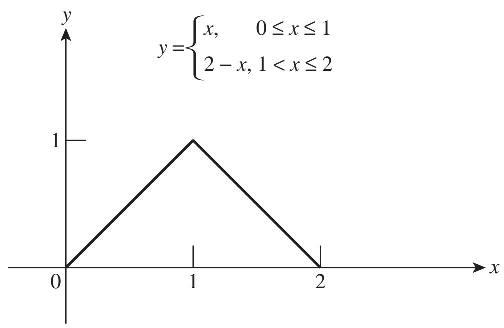
22. **(a)**



(b) f is continuous at $x=0$

(c) f is differentiable at $x=0$

23. **(a)**



(b) f is continuous at $x = 1$
(c) f is not differentiable at $x = 1$

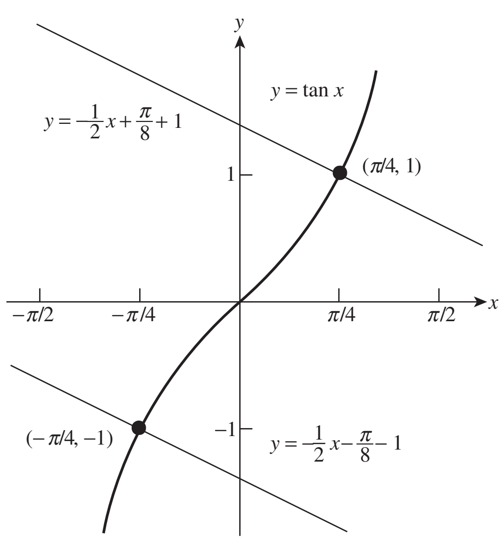
24. **(a)** f is continuous at $x = 0$ for all values of m
(b) f is differentiable at $x = 0$ provided that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \Rightarrow m = 2$.

25. **(a)** $\left(\frac{5}{2}, \frac{5}{9}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ are points on the curve where the slope is $-\frac{3}{2}$

26. (2, 0) and (–1, 27) are points on the curve where the tangent is parallel to the x-axis

27. **(a)** (–2, 16) and (3, 11) are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$
(b) (0, 20) and (1, 7) are points where the tangent is parallel to $y = \sqrt{2} - 12x$

28. $\left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$ are points where the normal is parallel to $y = -\frac{x}{2}$

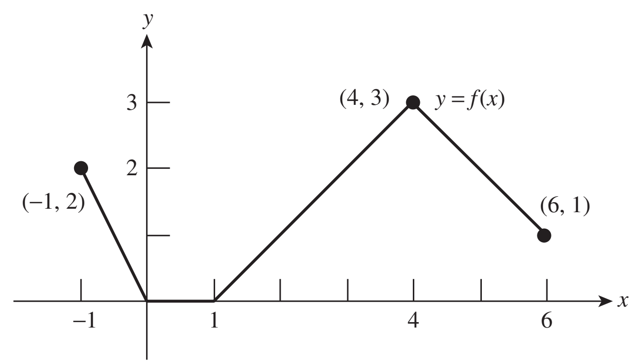


29. The curve has horizontal tangents at the x-axis for the x-values -2π, 0, and 2π (which are even integer multiples of π) $\Rightarrow$ the curve has an infinite number of horizontal tangents.

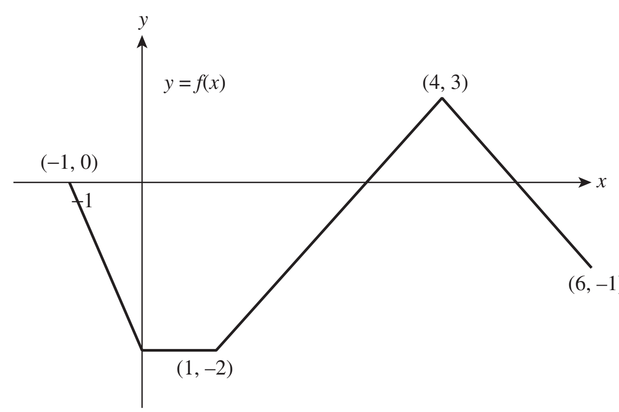
30. B = graph of f, A = graph of f'. Curve B cannot be the derivative of A because A has only negative slopes while some of B's values are positive

31. A = graph of f, B = graph of f'. Curve A cannot be the derivative of B because B has only negative slopes while A has positive values for $x > 0$

32.



33.



34. **(a)** 0, 0
(b) largest 1700, smallest about 1400

35. rabbits/day and foxes/day

36. –1

37. –2

38. $\frac{1}{2}$

39. 1

40. 4

41. $-\frac{2}{5}$

42. 1

43. $\frac{1}{2}$

Answers to Single Choice Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (c)
11. (d)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (a)	**19.** (c)	**20.** (d)
21. (c)	**22.** (b)	**23.** (a)	**24.** (d)	**25.** (a)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (d)	**30.** (c)
31. (c)	**32.** (c)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (d)	**37.** (d)	**38.** (c)	**39.** (a)	**40.** (a)
41. (c)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (c)	**47.** (d)	**48.** (a)	**49.** (a)	**50.** (b)

Answers to Multiple Choice Questions

1. (a), (b) and (d)	**2.** (b), (c) and (d)	**3.** (c) and (d)	**4.** (a), (b) and (c)	**5.** (a), (b) and (c)
6. (a), (c) and (d)	**7.** (b) and (c)	**8.** (a) and (d)	**9.** (a), (b) and (d)	**10.** (a) and (b)
11. (a), (c) and (d)	**12.** (a), (c) and (d)	**13.** (b), (c) and (d)	**14.** (a), (b), (c) and (d)	**15.** (a) and (b)
16. (a) and (d)	**17.** (a) and (b)	**18.** (a) and (b)		

Answers to Passage Type Questions

Passage 1

1. (d) **2.** (b) **3.** (b)

Passage 2

1. (a) **2.** (d) **3.** (b)

Passage 3

1. (c) **2.** (d) **3.** (a)

Passage 4

1. (d) **2.** (d) **3.** (b)

Passage 5

1. (c) **2.** (a)

Passage 6

1. (c) **2.** (d)

Passage 7

1. (b) **2.** (d)

Passage 8

1. (a) **2.** (d)

Passage 9

1. (d) **2.** (c)

Answers to Matrix Match Type Questions

1. (a) → (p), (r) and (t)	(b) → (p) and (t)	(c) → (q) and (s)	(d) → (p), (r) and (t)
2. (a) → (q)	(b) → (q)	(c) → (p)	(d) → (q)
3. (a) → (3)	(b) → (4)	(c) → (4)	(d) → (2)
4. (a) → (r)	(b) → (s)	(c) → (p)	(d) → (q)
5. (a) → (p) and (q)	(b) → (r) and (q)	(c) → (p) and (q)	(d) → (p) and (s)

Answers to Integer Type Questions

1. (1)	**2.** (2)	**3.** (4)	**4.** (11)	**5.** (6)	**6.** (2)	**7.** (60)	**8.** (5)	**9.** (1)	**10.** (3)
11. (6)	**12.** (2)	**13.** (9)	**14.** (2)	**15.** (1)	**16.** (0)	**17.** (15)	**18.** (6)	**19.** (0)	**20.** (25)
21. (8)	**22.** (5)	**23.** (28)	**24.** (64)	**25.** (4)	**26.** (6)	**27.** (14)	**28.** (2)	**29.** (0)	

Answers to Additional and Advanced Exercises

1. **(a)** $\cos 2\theta = \cos^2\theta - \sin^2\theta$
(b) $\sin 2\theta = 2\sin\theta\cos\theta$

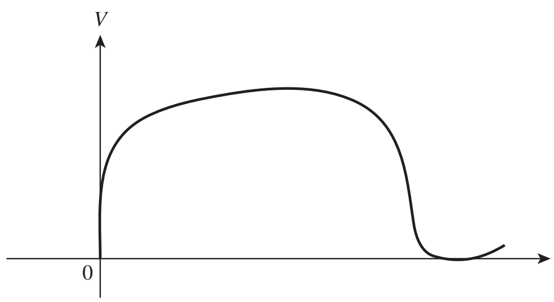
2. Not applicable

3. **(a)** $g(x) = 1 - \frac{1}{2}x^2$
(b) $g(x) = \sin x \cos a + \cos x \sin a$

5. $a = \frac{5\sqrt{5}}{2}$

6. $2x = 3ft$ wide

7.



8. **(a)** $m = \frac{-b}{\pi}$
(b) $m = -1$ and $b = \pi$

9. $1/2$

10. **(a)** $a = \frac{3}{4}$ and $b = \frac{9}{4}$

11. **(a)** $a = -1/2$

12. f' is even

13. f' is odd

16. No, Yes

Answers to Exercises

Chapter 4

Exercises 4.1

1. Mars: $\approx$ 7.5 sec, Jupiter: $\approx$ 1.2 Sec

2. **(a)** -1.6 m/sec^2
(b) $t = 15$ sec
(c) 180 m
(d) 4.39 sec going up and 25.6 sec going down
(e) Twice the time it took to reach its highest point or 30 sec.

3. $g_s = 0.75$ m/sec^2

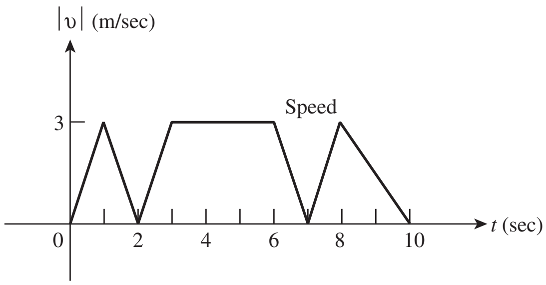
4. Time (on Moon) = 320 sec
Time (on Earth) = 52 sec
Height (above Moon's surface) = 66,560 ft.
Height (above Earth's surface) = 10,816 ft.

5. **(a)** $v = -32t$, $|v| = 32t$ ft/sec, $a = -32$ ft/sec^2
(b) $t \approx 3.3$ sec
(c) $v \approx -107.0$ ft/sec

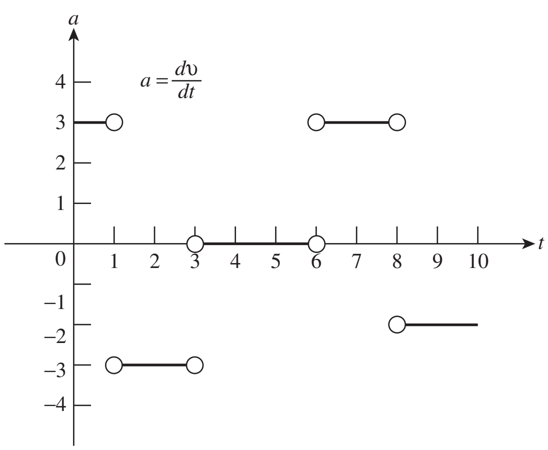
6. **(a)** 9.8t m/sec in free fall
(b) $a = 9.8$ m/sec^2

7. **(a)** $t = 2, t = 7$ **(b)** $3 \le t \le 6$

(c)

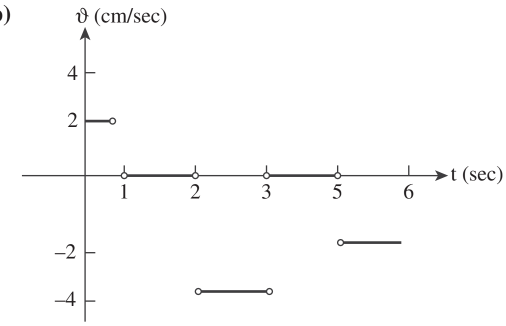


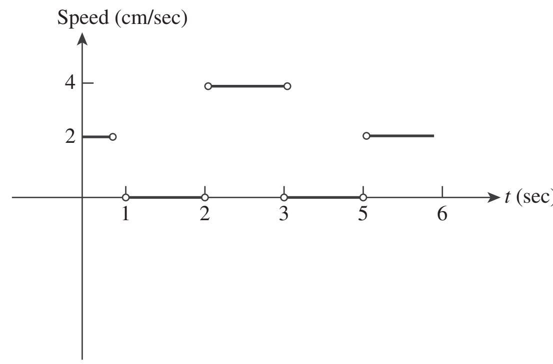
(d)



8. **(a)** P is moving to left when $2 < t < 3$ or $5 < t < 6$; P is moving to right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$

(b)



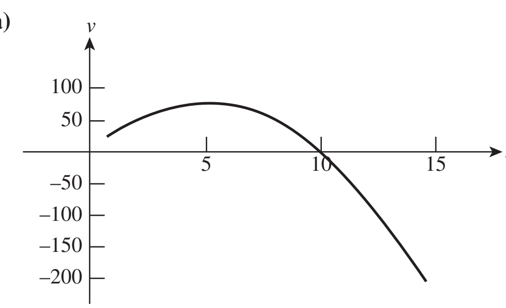


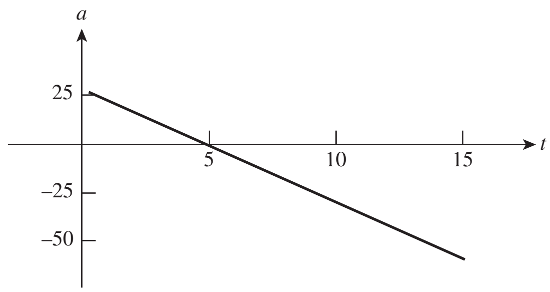
9. **(a)** 190 ft/sec **(b)** 2 sec **(c)** 8 sec, 0 ft/sec
(d) 10.8 sec, 90 ft/sec **(e)** 2.8 sec
(f) Greatest acceleration happens 2 sec after launch
(g) Constant acceleration between 2 and 10.8 sec, -32 ft/sec^2

10. **(a)** Forward: $0 \le t < 1$ and $5 < t < 7$;
Backward: $1 < t < 5$;
Speed up: $1 < t < 2$ and $5 < t < 6$
Slow down: $0 \le t < 1$, $3 < t < 5$ and $6 < t < 7$
(b) Positive: $3 < t < 6$
Negative: $0 \le t < 2$ and $6 < t < 7$; Zero: $2 < t < 3$ and $7 < t < 9$
(c) $t = 0$ and $2 \le t \le 3$
(d) $7 \le t \le 9$

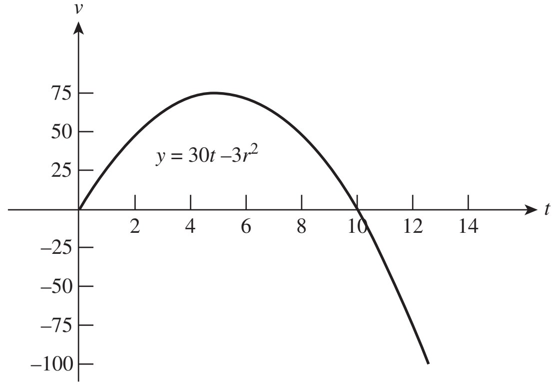
11. **(a)** $\frac{4}{7}$ sec, 280 cm/sec **(b)** 560 cm/sec, 980 cm/sec^2
(c) 29.75 flashes / sec

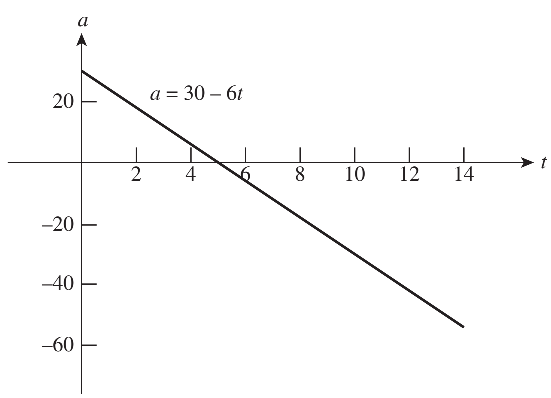
12. **(a)**





(b)





13. C = position, A = velocity, B = acceleration

14. A = acceleration
B = Velocity
C = Position

15. **(a)** \$110/machine **(b)** \$80 **(c)** \$79.90

16. **(a)** \$2 **(b)** \$1.96 **(c)** 0

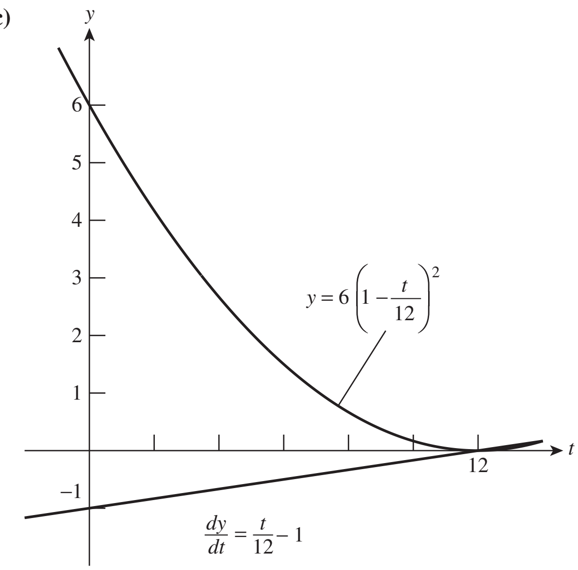
17. **(a)** $b'(0) = 10^4$ bacteria/h
(b) $b'(5) = 0$ bacteria/h
(c) $b'(10) = -10^4$ bacteria/h

18. S increases more rapidly at lower weights where the derivative is greater.

19. **(a)** $\dfrac{dy}{dt} = \dfrac{t}{12} - 1$

(b) The largest value of $\dfrac{dy}{dt}$ is 0 m/h when $t = 12$ and the smallest value of $\dfrac{dy}{dt}$ is –1 m/h when $t = 0$.

(c)



20. –8000 gallon/min is rate the water is running at the end of 10 min.
–10000 gallons/min is the average rate that the water flows during first 10 minutes. The negative sign indicates water is leaving the tank.

21. 4.88 ft, 8.66 ft, additional ft to stop car for 1 mph speed increase

22. **(a)** 16π ft^3/ft **(b)** 11.09 ft^3

23. $t = 25$ sec, $D = \dfrac{6250}{9}$ m

24. 238 mph

Exercises 4.2

1. $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ **2.** $8\pi r\dfrac{dr}{dt}$

3. $\dfrac{1}{6}$

4. **(a)** $\pi r^2\dfrac{dh}{dt}$ **(b)** $2\pi rh\dfrac{dr}{dt}$

(c) $\dfrac{dV}{dt}\pi r^2\dfrac{dh}{dt} + 2\pi rh\dfrac{dr}{dt}$

5. **(a)** $\dfrac{1}{3}\pi r^2\dfrac{dh}{dt}$ **(b)** $\dfrac{2}{3}\pi rh\dfrac{dr}{dt}$

(c) $\dfrac{1}{3}\pi r^2\dfrac{dh}{dt} + \dfrac{2}{3}\pi rh\dfrac{dr}{dt}$

6. **(a)** 1 Volt/sec **(b)** $\dfrac{-1}{3}$ amp /sec

(c) $\dfrac{1}{I}\left(\dfrac{dV}{dt} - \dfrac{V}{I}\dfrac{dI}{dt}\right)$

7. **(a)** $I^2\dfrac{dR}{dt} + 2RI\dfrac{dI}{dt}$ **(b)** $\dfrac{-2P}{I^3}\dfrac{dI}{dt}$

8. **(a)** $\dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt}$ **(b)** $\dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt} + \dfrac{y}{\sqrt{x^2+y^2}}\dfrac{dy}{dt}$

(c) $\dfrac{-y}{x}\dfrac{dy}{dt}$

9. (a) $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}+2z\dfrac{dz}{dt}$

(b) $\dfrac{ds}{dt}=\dfrac{y}{\sqrt{x^2+y^2+z^2}}\dfrac{dy}{dt}+\dfrac{z}{\sqrt{x^2+y^2+z^2}}\dfrac{dz}{dt}$

(c) $\dfrac{dx}{dt}+\dfrac{y}{x}\dfrac{dy}{dt}+\dfrac{z}{x}\dfrac{dz}{dt}=0$

10. **(a)** $\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}$

(b) $\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}+\dfrac{1}{2}b\sin\theta\dfrac{da}{dt}$

(c) $\dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}+\dfrac{1}{2}b\sin\theta\dfrac{da}{dt}+\dfrac{1}{2}a\sin\theta\dfrac{db}{dt}$

11. π cm^2/min

12. **(a)** 14 cm^2/sec, increasing

(b) 0 cm/sec

(c) $\dfrac{-14}{13}$ cm/sec, decreasing

13. **(a)** 2m^3 / sec

(b) $\dfrac{ds}{dt}=(2y+2z)\dfrac{dx}{dt}+(2x+2z)\dfrac{dy}{dt}+(2x+2y)\dfrac{dz}{dt}$

(c) 0 m/sec

14. **(a)** –12 ft/sec, ladder is sliding down the wall

(b) –59.5 ft^2/sec

(c) –1 rad/sec

15. –614 knots

16. 20 ft/sec.

17. 0.0239 in^3/min

18. **(a)** 11.19 cm/sec **(b)** 14.92 cm/sec

19. **(a)** –1.13 cm/min **(b)** –8.49 cm/sec

20. **(a)** $\dfrac{-1}{24\pi}$ m/min **(b)** $r=\sqrt{26y-y^2m}$

(c) $\dfrac{-5}{288\pi}m$ / min.

22. 1 ft/min balloon's radius increasing. 40 πft^2/min surface area increasing

23. **(a)** –2.5 ft/sec **(b)** $\dfrac{-3}{20}$ rad/sec

24. 11 ft/sec

25. **(a)** $\dfrac{10}{9\pi}$ in/min **(b)** $\dfrac{-8}{5\pi}$ in/min

26. 0.2772 L/min **27.** 1 rad/sec

28. –5 m/sec **29.** 1 rad/sec

30. –1500 ft/sec **31.** ≈7.1 in / min.

32. $\dfrac{10}{3}in^2$ / min **33.** 80 mph

34. **(a)** ~8.875 ft/sec. **(b)** $\dfrac{-8}{65}$ *rad* / sec

(c) $\dfrac{-1}{6}$ *rad* / sec

35. 29.5 knots apart **36.** –5.5°/min

37. 132, 183 ft^3/hr

Exercises 4.3

1. $L(x) = x + 1$ at $x = 8$ **2.** $L(x) = \dfrac{1}{12}x+\dfrac{4}{3}$ at $x=8$

3. $L(x) = \dfrac{1}{4}x + \dfrac{1}{4}$ at $x = 1$ **4.** $1 + kx$

5. **(a)** $1 – 6x$ **(b)** $2 + 2x$

(c) $1-\dfrac{x}{2}$ **(d)** $\sqrt{2}\left(1+\dfrac{x^2}{4}\right)$

(e) $4^{1/3}\left(1+\dfrac{x}{4}\right)$ **(f)** $1-\dfrac{2x}{6+3x}$

6. **(a)** 1.01 **(b)** 1.003

7. $L_h(x) = x$, linearization of sum is equal to sum of lineriazations.

8. $dy=\left(3x^2-\dfrac{3}{2\sqrt{x}}\right)dx$ **9.** $\dfrac{\left(1-2x^2\right)}{\sqrt{1-x^2}}dx$

10. $\dfrac{2-2x^2}{\left(1+x^2\right)^2}dx$

11. **(a)** 0.41 **(b)** 0.4 **(c)** 0.01

12. **(a)** 0.02 **(b)** 0 **(c)** 0.02

13. $4\pi r_0^{\,2}\,dr$ **14.** $dV = 3x_0^2dx$

15. $12x_0\,dx$

16. **(a)** 0.08πm^2 **(b)** 2%

17. 10 in^2 **18.** ≈565.5 in^3

19. **(a)** 1% **(b)** 1.5%

20. $\dfrac{1}{3}$%

21. **(a)** 0.5% **(b)** Within 5%

22. $3\%\dfrac{\frac{10^4\pi}{2}}{\frac{10^6\pi}{6}}$ **23.** 1%

24. 38 times **25.** ≈0.584 mg/mL

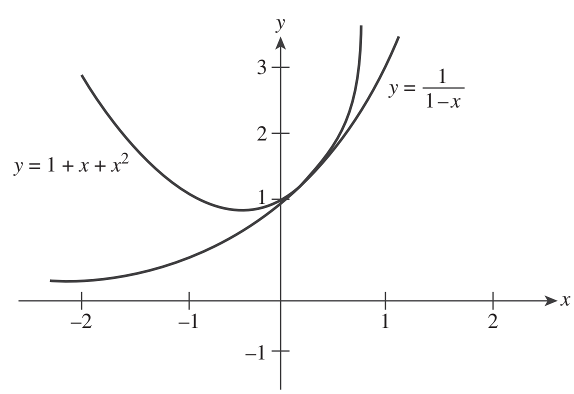
26. 40%

27. **(a)** $-\pi\sqrt{Lg}^{\frac{-3}{2}}\,dg$ (b) Both increase

(c) ≈979 cm/sec^2

28. **(a)** $b_0 = f(a)$ $b_1 = f'(a)$ $b_2 = \dfrac{f''(a)}{2}$

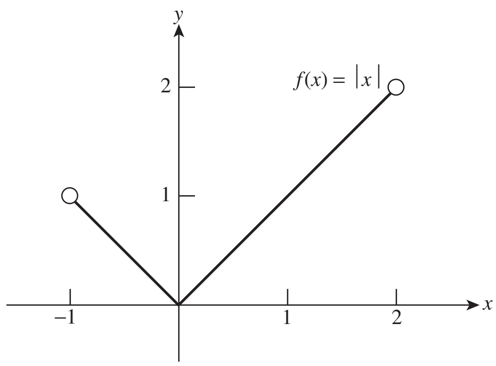
$b_0 = 1$ $b_1 = 1$, $b_2 = 1$ **(b)** $Q(x) = 1 + x + x^2$



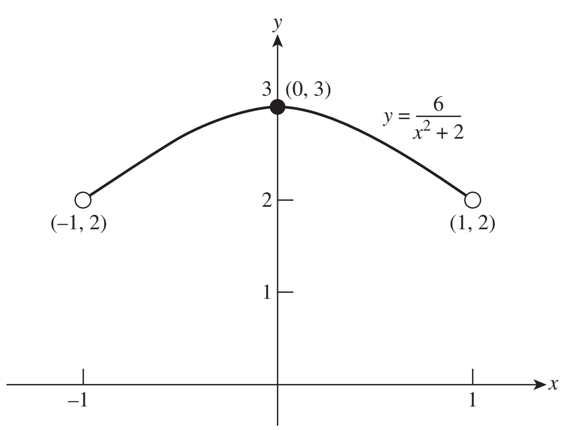
[–2.35,2.35] by [–1.25,3.25]

Exercises 4.4

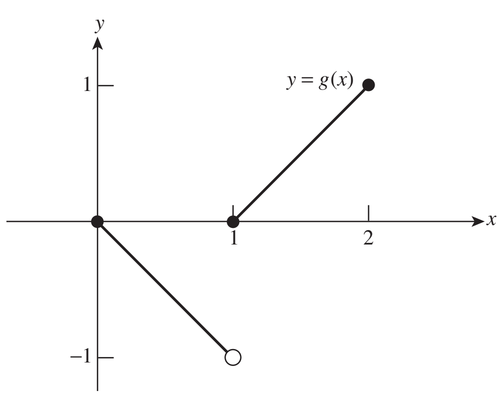
1. Absolute minimum at $x = c_2$; absolute maximum at $x = b$
2. Absolute min $x = b$; absolute max. $x = c$
3. Absolute maximum at $x = c$; no absolute minimum
4. No absolute extreme
5. Absolute minimum at $x = a$; absolute maximum at $x = c$
6. Absolute min at $x = c$; Absolute max at $x = a$.
7. No absolute minimum; no absolute maximum
8. Minimum at (–2, 0) and (2, 0) Maximum at (0, 2)
9. Absolute maximum at (0, 5)
10. Local maximum at (–3, 0) Local minimum at (2, 0) Maximum (1, 2); Minimum at (0, –1)
11. (c)
12. graph (b)
13. (d)
14. graph (a)
15. Absolute minimum at $x = 0$; no absolute maximum



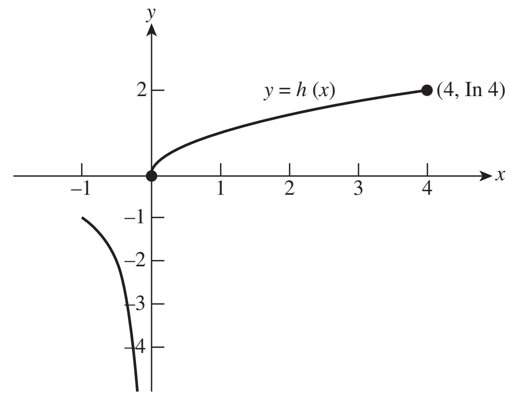
16. Absolute maximum at $x = 0$ but no absolute minimum



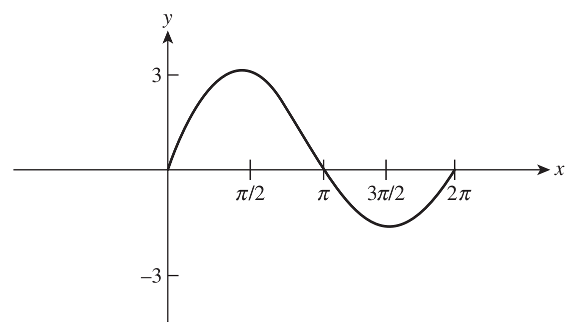
17. Absolute maximum at $x = 2$; no absolute minimum



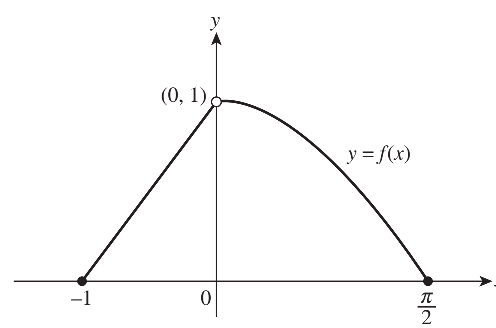
18. Absolute maximum at $x = 4$ but no absolute minimum



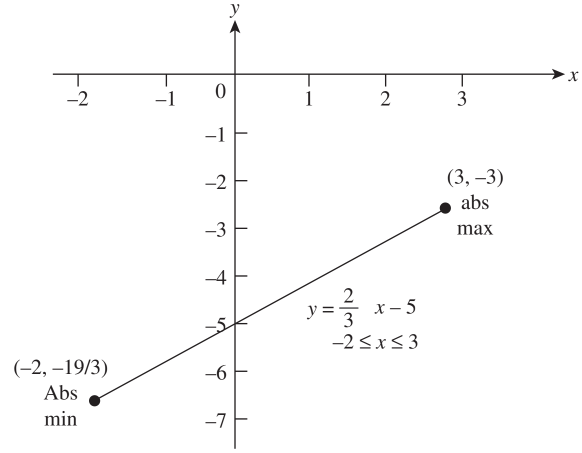
19. Absolute maximum at $x = \frac{\pi}{2}$; Absolute minimum at $x = \frac{3\pi}{2}$



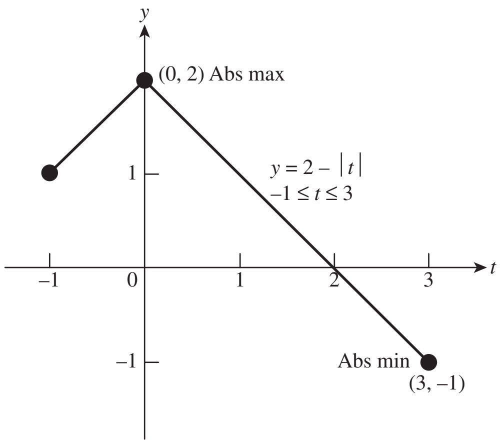
20. Absolute maximum at $x = 0$ and absolute minimum at $x = \frac{\pi}{2}$ and $x = -1$



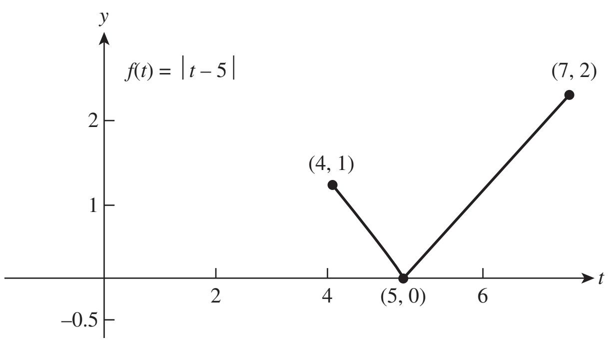
21. Absolute maximum: –3; Absolute minimum: $-\frac{19}{3}$



22. Abs. max is 2 at $t = 0$; abs. min. is –1 at $t = 3$.



23. absolute maximum is 2 at $t = 7$
Absolute minimum is 0 at $t = 5$



24. Increasing on (0, 8), decreasing on (–1, 0); absolute maximum;16 at $x = 8$; absolute minimum: 0 at $x = 0$

25. Absolute maximum is 32 at $x = 8$
Absolute minimum is –1 at $x = -1$

26. Increasing on (–32, 1); absolute maximum: 1 at $\theta = 1$; absolute minimum: –8 at $\theta = -32$

27. Absolute maximum is 27 at $\theta = -27$
Absolute minimum is 0 at $\theta = 0$

28. $x = 3$

29. $x = 0$ and $x = 4$

30. $x = 1, x = 4$

31. $x = 1, x = 2$ and $x = 3$

32. $x = 1$

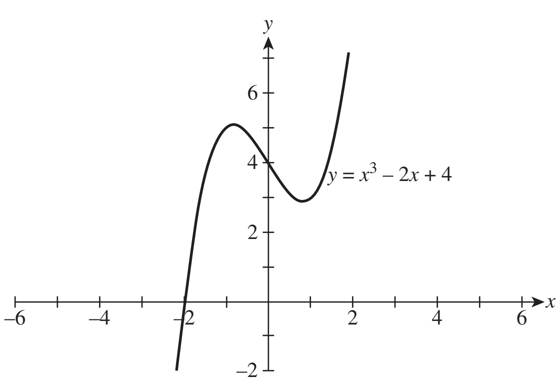
33. $x = 0$ and $x = 4$

34. $x = 0$ and $x = 4$

35. $x = 0, x = 1$ and $x = 2$

36. Minimum value is 1 at $x = 2$

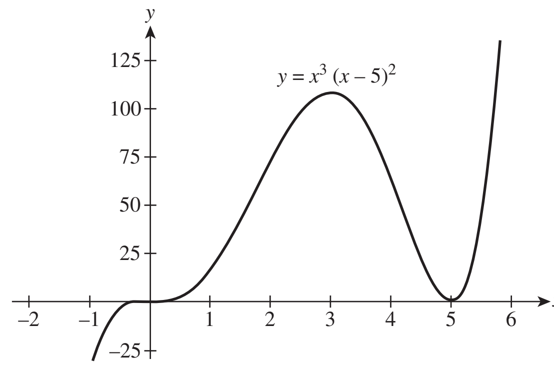
37. Local maximum ≈ (–0.816, 5.089)
Local minimum ≈ (0.816, 2.911)



[–6, 6] by [–2, 7]

38. Local maximum at (–2, 17); local minimum at $\left(\frac{4}{3}, \frac{41}{27}\right)$

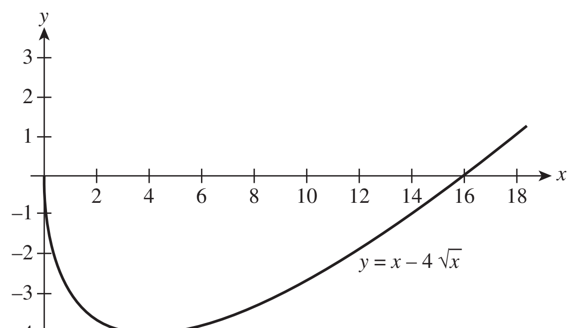
39. Local maximum at (3, 108)
Local minimum at (5, 0); (0, 0) is neither maximum nor minimum.



[–2, 6] by [–25, 125]

40. Minimum value is 0 at $x = -1$ and $x = 1$

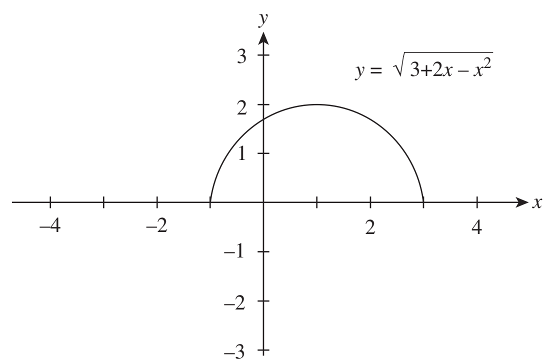
41. Local maximum at (0, 0)
Local minimum at (4, –4)



[–4, 3] by [0, 18]

42. There is a local minimum at (0, 1).

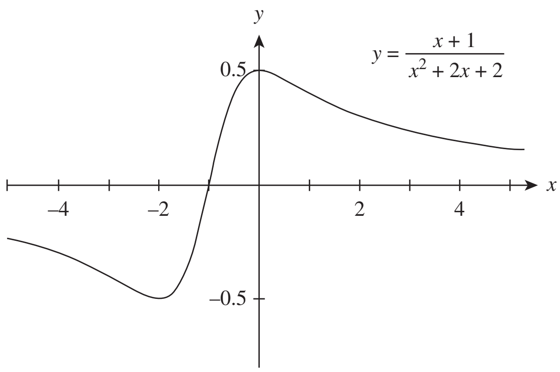
43. Maximum value is 2 at $x = 1$
Minimum value is 0 at $x = -1$ and $x = 3$



[–4.7,4.7] by [–3.1,3.1]

44. Maximum value is ½ at $x = 1$; minimum value is –½ at $x = -1$.

45. Maximum value is ½ at $x = 0$
Minimum value is –½ at $x = -2$



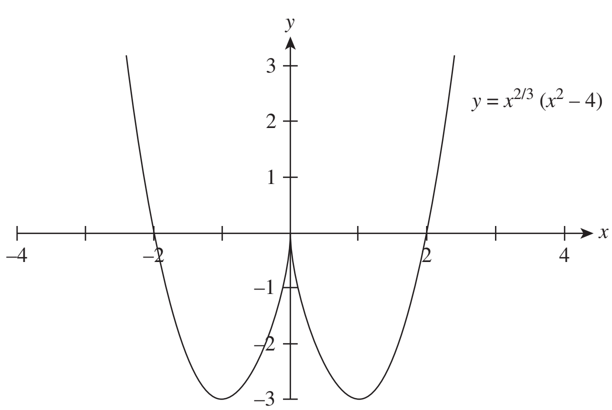
[–5,5] by [–0.8, 0.6]

46.

Crit. point or end point	Derivative	Extremum	Value
$x = -\frac{4}{5}$	0	Local max.	$\frac{12}{25}10^{\frac{1}{3}} \approx 1.034$
$x = 0$	Undefined	Local min.	0

47. $y' = x^{\frac{2}{3}}(2x) + \frac{2}{3}x^{-\frac{1}{3}}(x^2-4) = \frac{8x^2-8}{3\sqrt[3]{x}}$

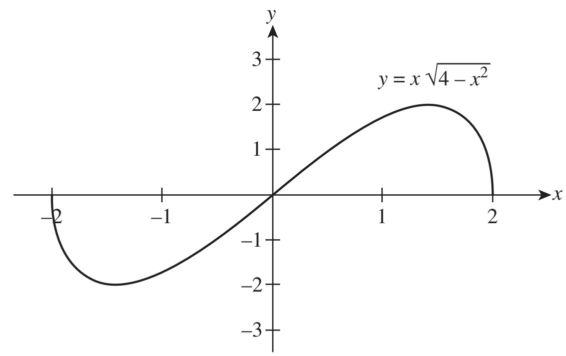
Crit. pt.	Derivative	Extremum	Value
$x = -1$	0	Minimum	–3
$x = 0$	Undefined	Local max	0
$x = 1$	0	Minimum	3



[–4, 4] by [–3, 3]

48. $y' = x\frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4-x^2} = \frac{-x^2+\left(4-x^2\right)}{\sqrt{4-x^2}} = \frac{\left(4-2x^2\right)}{\sqrt{4-x^2}}$

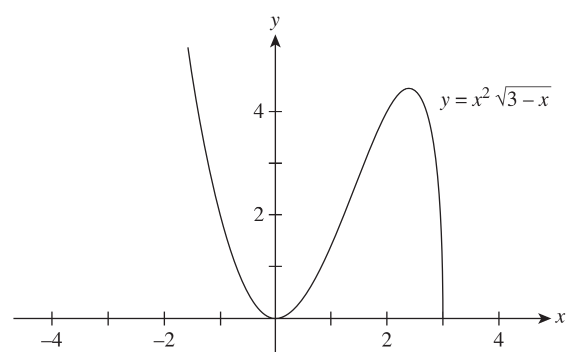
Crit. pt.	Derivative	Extremum	Value
$x = -2$	Undefined	Local max	0
$x = -\sqrt{2}$	0	Minimum	–2
$x = \sqrt{2}$	0	Maximum	2
$x = 2$	Undefined	Local min	0



[–2.35, 2.35] by [–3.5, 3.5]

49. $y' = x^2\frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3-x} = \frac{-x^2+(4x)(3-x)}{2\sqrt{3-x}} = \frac{-5x^2+12x}{2\sqrt{3-x}}$

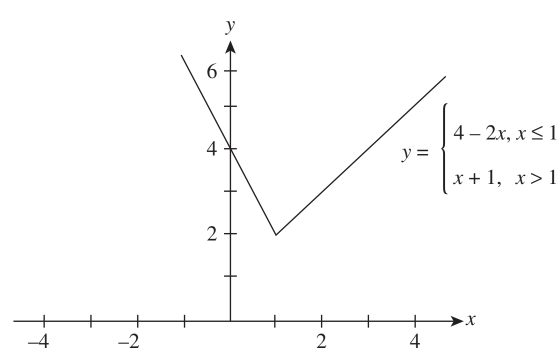
Crit. pt.	Derivative	Extremum	Value
$x = 0$	0	Minimum	0
$x = \frac{12}{5}$	0	Local max	$\frac{144}{125}15^{\frac{1}{2}} \approx 4.462$
$x = 3$	Undefined	Minimum	0



[–4.7, 4.7] by [–1, 5]

50. $y' = \begin{pmatrix} -2, & x < 1 \\ 1, & x > 1 \end{pmatrix}$

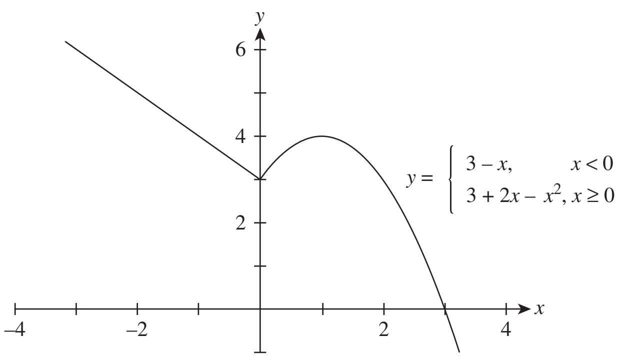
Crit. pt.	Derivative	Extremum	Value
$x = 1$	Undefined	Minimum	2



[–4.7, 4.7] by [0, 6.2]

51. $y' = \begin{pmatrix} -1, & x<0 \\ 2-2x, & x>0 \end{pmatrix}$

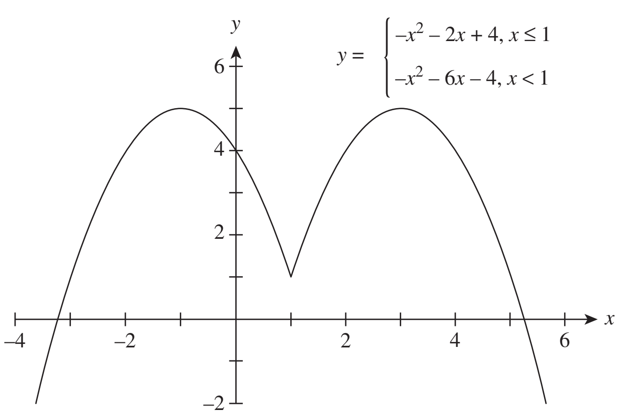
Crit. pt.	Derivative	Extremum	Value
$x = 1$	Undefined	Local min	3
x = 1	0	Local max	4



[–4, 4] by [–1, 6]

52. $y' = \begin{pmatrix} -2x-2, & x<1 \\ -2x+6, & x>1 \end{pmatrix}$

Crit. pt.	Derivative	Extremum	Value
$x = -1$	0	Maximum	5
$x = 1$	Undefined	Local min	1
$x = 3$	0	Maximum	5



[–4, 6] by [–2, 6]

53. We begin by determining $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{pmatrix} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{pmatrix}$$

Clearly, $f'(x) = \frac{-1}{2}x - \frac{1}{2}$ if $x < 1$, and $\lim\limits_{h\to 0^-} f'(1+h) = -1$. Also, $f'(x) = 3x^2 - 12x + 8$; if $x > 1$, and $\lim\limits_{h\to 0^+} f'(1+h) = -1$. Since f is continuous at $x = 1$, we have that $f'(1) = -1$.

Thus, $f'(x) = \begin{pmatrix} \frac{-1}{2}x - \frac{1}{2} & x \le 1 \\ 3x^2 - 12x + 8 & x > 1 \end{pmatrix}$

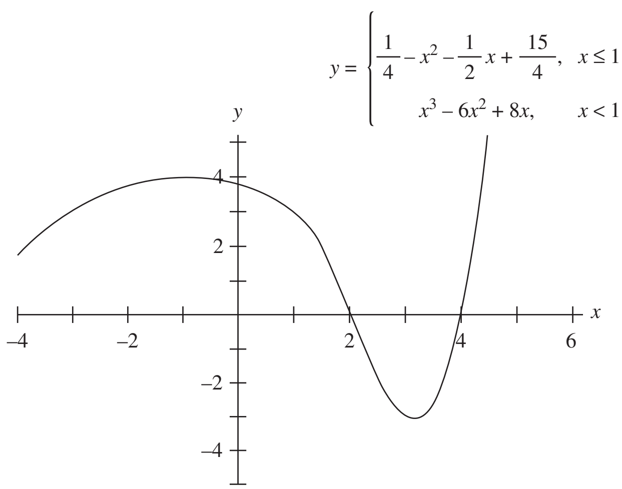
Note that $\frac{-1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and $3x^2 - 12x + 8 = 0$

when $x = \frac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)} = \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$.

But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the critical points occur at

$x = -1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155$.

Crit. pt.	Derivative	Extremum	Value
$x = -1$	0	Local max	4
$x \approx 3.155$	0	Local min	≈ -3.079



[–4, 6] by [–5, 5]

54. **(a)** No
(c) No
(d) The answers are the same

55. Note that $f(x) = \begin{pmatrix} -x^3 + 9x, & x \le -3 \text{ or } 0 \le x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \ge 3 \end{pmatrix}$. Therefore,

$f'(x) = \begin{pmatrix} -3x^3 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^3 - 9, & -3 < x < 0 \text{ or } x > 3 \end{pmatrix}$

(a) No
(b) No
(c) No

56. Yes.

57. f assumes local maximum at point $-c$.

58. g assumes a local maximum at $-c$

59. No extreme values.

60. **(a)** Maximum value is 144 at $x = 2$.
(b) The largest volume of the box is 144 cubic units, and it occurs when $x = 2$.

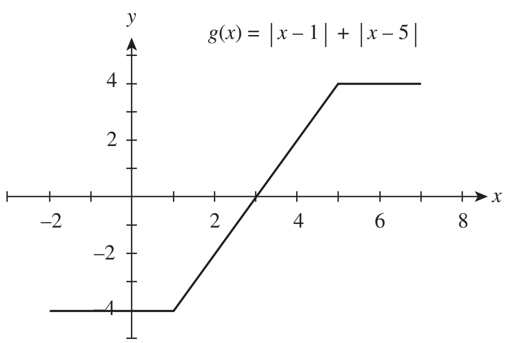
61. **(b)** The function can have either two local extreme values or no extreme values (If there is only one critical point, the cubic function has no extreme values).

62. $\dfrac{v_0^{\,2}}{2g}+s_0$

63. $2\sqrt{2}$ amps

64. Maximum value is 11 at $x = 5$; minimum value is 5 on the interval $(-3, 2)$; local maximum at $(-5, 9)$.

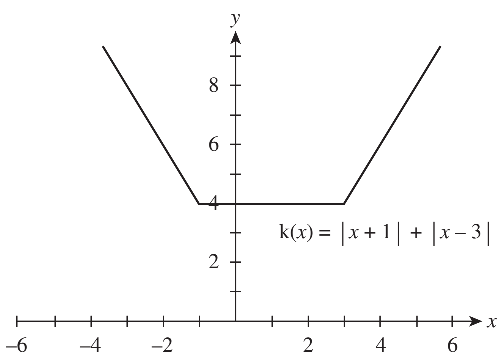
65. Max. value is 4 on $[5, 7]$; Min value is -4 on $[-2, 1]$.



[–3,8] by [–5,5]

66. Maximum value is 5 on the interval $(3, \infty)$; minimum value is -5 on the interval $(-\infty, -2)$.

67. Min value is 4 on $[-1, 3]$.



[–6,6] by [0,9]

Exercises 4.5

1. $c = \dfrac{1}{2}$

2. $c = 8/27$

3. $c = 1$

4. $c = 3/2$

5. ≈ -0.549

6. $c = 3/2$

7. Does not

8. Does

9. Does

10. Does not

11. Does not

12. Does

13. Theorem does not apply

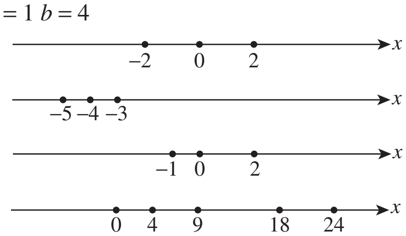
14. $a = 3$ $m = 1$ $b = 4$

15. (a) (i)

(ii)

(iii)

(iv)



27. $C = 3 \Rightarrow f(x) = 3$ for all x.

28. $f(x) = g(x) = 2x + 5$ for all x

29. $g(x) = x^2 \Rightarrow g'(x) = 2x = f'(x)$ for all x. By Corollary 2, $f(x) = g(x) + C$.
(a) 4 **(b)** 3 **(c)** 3

30. $g(x) = mx \Rightarrow g'(x) = m$, a constant. If $f'(x) = m$, then by Corollary 2, $f(x) = g(x) + b = mx + b$ where b is a constant. Therefore all functions whose derivatives are constant can be graphed as straight lines $y = mx + b$.

31. $(a).\ y = \dfrac{x^2}{2}+C$ $\quad (b).\ y = \dfrac{x^3}{x}+C$ $\quad (c).\ y = \dfrac{x^4}{4}+C$

32. **(a)** $y = x^2 + C$ **(b)** $y = x^2 - x + C$ **(c)** $y = x^3 + x^2 - x + C$

33. **(a)** $y' = -x^{-2} \Rightarrow y = \dfrac{1}{x}+C$ **(b)** $y = x + \dfrac{1}{x}+C$

(c) $y = 5x - \dfrac{1}{x}+C$

34. **(a)** $y' = \dfrac{1}{2}x^{-\frac{1}{2}} \Rightarrow y = x^{\frac{1}{2}}+C$ **(b)** $y = 2\sqrt{x}+C$

(c) $y = 2x^2 - 2\sqrt{x}+C$

35. **(a)** $y = -\dfrac{1}{2}\cos 2t + C$ **(b)** $y = 2\sin\dfrac{t}{2}+C$

(c) $y = -\dfrac{1}{2}\cos 2t + 2\sin\dfrac{t}{2}+C$

36. **(a)** $y = \tan\theta + C$

(b) $y' = \theta^{1/2} \Rightarrow y = \dfrac{2}{3}\theta^{3/2}+C$ **(c)** $y = \dfrac{2}{3}\theta^{3/2} - \tan\theta + C$

37. $f(x) = x^2 - x$

38. $g(x) = -\dfrac{1}{x} + x^2 - 1$

39. $r\theta = 8\theta + \cot\theta - 2\pi - 1$

40. $r(t) = \sec t - t - 1$

41. $s = 4.9t^2 + 5t + 10$

42. $s = 16t^2 - 2t + 1$

43. $s = \dfrac{1-\cos(\pi t)}{\pi}$

44. $s = \sin\left(\dfrac{2t}{\pi}\right) + 1$

45. $s = 16t^2 + 20t + 5$

46. $s = 4.9t^2 - 3t$

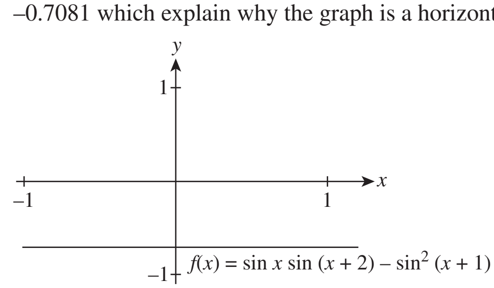
47. $s = \sin(2t) - 3$

48. $s = -\cos\left(\dfrac{3t}{\pi}\right)$

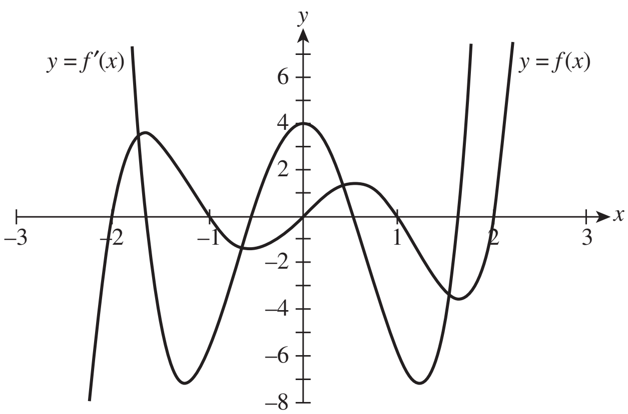
49. $c = \sqrt{ab}$

50. $c = \dfrac{a+b}{2}$

51. $f'(x) = [\cos x \sin(x + 2) + \sin x \cos(x + 2)] - 2\sin(x + 1)\cos(x + 1)$ $= \sin(x + x + 2) - \sin 2(x + 1) = \sin(2x + 2) - \sin(2x + 2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explain why the graph is a horizontal line.



52. **(a)** $f(x) = (x + 2)(x + 1)x(x - 1)(x - 2) = x^5 - 5x^3 + 4x$ is one possibility.
(b) Graphing $f(x) = x^5 - 5x^3 + 4x$ and $f'(x) = 5x^4 - 15x^2 + 4$ on $[-3, 3]$ by $[-7, 7]$ we see that each x-intercept of $f'(x)$ lies between a pair of x-intercepts of $f(x)$, as expected by Rolle's Theorem.



(c) Yes, since sin is continuous and differentiable on $(-\infty, \infty)$

59. Yes

62. The condition is that f' should be continuous over $[a, b]$.

63. $1.09999 \le f(0.1) \le 1.1$.

64. $2.1 \le f(0, 1) \le 2.10001$.

65. **(b)** Yes

Exercises 4.6

1. **(a)** 0, 1
(b) Increasing on $(-\infty, 0)$ and $(1, \infty)$; decreasing on $(0, 1)$
(c) Local maximum at $x = 0$; local minimum at $x = 1$

2. **(a)** –2, 1
(b) increasing on $(-\infty, -2)$ and $(1, \infty)$ decreasing on $(-2, 1)$
(c) local maximum at $x = -2$ local minimum at $x = 1$

3. **(a)** −2, 1
(b) Increasing on $(-2, 1)$ and $(1, \infty)$; decreasing on $(-\infty, -2)$
(c) No local maximum; local minimum at $x = -2$

4. (a) –2, 1
(b) increasing $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$ never decreasing
(c) No local extrema

5. **(a)** −2, 1, 3
(b) increasing on $(-2, 1)$ and $(3, \infty)$, decreasing on $(-\infty, -2)$ and $(1, 3)$
(c) local maximum at $x = 1$, local minimum at $x = -2, 3$

6. **(a)** –5,–1 and 7
(b) increasing on $(-5, -1)$ and $(7, \infty)$ decreasing on $(-\infty, -5)$ and $(-1, 7)$
(c) local maximum at $x = 1$ local minimum at $x = -2$ and $x = 3$

7. **(a)** 0, 1
(b) Increasing on $(-\infty, -2)$ and $(1, \infty)$; decreasing on $(-2, 0)$ and $(0, 1)$
(c) Local minimum at $x = 1$

8. **(a)** $x = 2, x = -4, x = -1$ and $x = 3$
(b) increasing on $(-\infty, -4), (-1, 2)$ and $(3, \infty)$ decreasing on $(-4, -1)$ and $(2, 3)$
(c) local maximum at $x = -4$ and $x = 2$

9. **(a)** −2, 2
(b) Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 0)$ and $(0, 2)$
(c) Local maximum at $x = -2$; local minimum at $x = 2$

10. **(a)** $x = 4$ and $x = 0$
(b) increasing on $(4, \infty)$, decreasing on $(0, 4)$
(c) Local minimum at $x = 4$

11. **(a)** −2, 0
(b) Increasing on $(-\infty, -2)$ and $(0, \infty)$; decreasing on $(-2, 0)$
(c) Local maximum at $x = -2$; local minimum at $x = 0$

12. **(a)** $x = 0$ and $x = 3$
(b) increasing on $(3, \infty)$ decreasing on $(0, 3)$
(c) No local maximum and a local minimum at $x = 3$

13. **(a)** $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$
(b) Increasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$; decreasing on $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{4\pi}{3}, 2\pi\right)$
(c) Local maximum at $x = 0$ and $x = \frac{4\pi}{3}$; local minimum at $x = \frac{2\pi}{3}$ and $x = 2\pi$

14. **(a)** $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}$ and $x = \frac{7\pi}{4}$
(b) increasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, decreasing on $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4,}, \frac{5\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$
(c) Local maximum at $x = 0$, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, local minimum at $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$ *and* $x = 2\pi$

15. **(a)** Increasing on $(-2, 0)$ and $(2, 4)$; decreasing on $(-4, -2)$ and $(0, 2)$
(b) Absolute maximum at $(-4, 2)$; local maximum at $(0, 1)$ and $(4, -1)$; absolute minimum at $(2, -3)$; local minimum at $(-2, 0)$

16. **(a)** Increasing on $(-4, -3.25)$, $(-1.5, 1)$ and $(2, 4)$ decreasing on $(-3.25, -1.5)$ and $(1,2)$
(b) Absolute maximum at $(4,2)$, local maximum at $(-3.25, 1)$ and $(1, 1)$; Absolute minimum at $(-1.5, -1)$ local minimum at $(-4,0)$ and $(2,0)$

17. **(a)** Increasing on $(-4, -1)$, $(1/2, 2)$, and $(2, 4)$; decreasing on $(-1, 1/2)$
(b) Absolute maximum at $(4, 3)$; local maximum at $(-1, 2)$ and $(2, 1)$; no absolute minimum; local minimum at $(-4, -1)$ and $(1/2, -1)$

18. **(a)** Increasing on $(-4, -2.5)$, $(-1, 1)$ and $(3, 4)$ decreasing on $(-2.5, -1)$ and $(1,3)$
(b) No absolute maximum, local maximum at $(-2.5, 1)$, $(1, 2)$ and $(4, 2)$; No absolute minimum, local minimum at $(-1, 0)$ and $(3, 1)$

19. **(a)** Increasing on $(-\infty, -1.5)$; decreasing on $(-1.5, \infty)$
(b) Local maximum: 5.25 at $t = -1.5$; absolute maximum: 5.25 at $t = -1.5$

20. **(a)** increasing on $\left(-\infty, \frac{3}{2}\right)$, decreasing on $\left(\frac{3}{2}, \infty\right)$
(b) local maximum value of $g\left(\frac{3}{2}\right) = \frac{47}{4}$ at $t = \frac{3}{2}$ absolute maximum is $\frac{47}{4}$ at $t = \frac{3}{2}$

21. **(a)** Decreasing on $(-\infty, 0)$; increasing on $(0, 4/3)$; decreasing on $(4/3, \infty)$
(b) Local minimum at $x = 0$ $(0, 0)$; local maximum at $x = 4/3$ $(4/3, 32/27)$; no absolute extrema

22. **(a)** critical points at $x = \pm\sqrt{3}$ increasing on $\left(-\infty, -\sqrt{3}\right)$ and $(\sqrt{3}, \infty)$ decreasing on $(-\sqrt{3}, \sqrt{3})$
(b) a local maximum is $h(-\sqrt{3}) = 12\sqrt{3}$ at $x = -\sqrt{3}$; local minimum is $h(\sqrt{3}) = -12\sqrt{3}$ at $x = \sqrt{3}$, No absolute extrema

23. **(a)** Decreasing on $(-\infty, 0)$; increasing on $(0, 1/2)$; decreasing on $(1/2, \infty)$
(b) Local minimum at $\theta = 0$ $(0, 0)$; local maximum at $\theta = 1/2$ $(1/2, 1/4)$; no absolute extrema

24. **(a)** $f(\theta) = 6\theta - \theta^3 \Rightarrow f'(\theta) = 6 - 3\theta^2 = 3(\sqrt{2} - \theta)(\sqrt{2} + \theta) \Rightarrow$ critical points at $\theta = \pm\sqrt{2} \Rightarrow f' = .- - - \underset{-\sqrt{2}}{|} + + + \underset{\sqrt{2}}{|} - - -$ increasing on $(-\sqrt{2}, \sqrt{2}$ decreasing on $(-\infty, \sqrt{2})$ and $(\sqrt{2}, \infty)$
(b) A local maximum is $f(\sqrt{2}) = 4\sqrt{2}$ at $\theta = \sqrt{2}$, a local maximum is $f(-\sqrt{2}) = 4\sqrt{2}$ at $\theta = -\sqrt{2}$ no absolute extrema.

25. **(a)** Increasing on $(-\infty, \infty)$; never decreasing
(b) No local extrema; no absolute extrema

26. **(a)** $h(r) = (r + 7)^3 \Rightarrow h'(r) = 3(r + 7)^2 \Rightarrow$ a critical point at $r = -7$ $\Rightarrow h' = + + + \underset{-7}{|} + + +$, increasing on $(-\infty, -7) \cup (-7, \infty)$, never decreasing
(b) No local extrema, no absolute extrema

27. **(a)** Increasing on $(-2, 0)$ and $(2, \infty)$; decreasing on $(-\infty, -2)$ and $(0, 2)$
(b) Local maximum: 16 at $x = 0$; local minimum: 0 at $x = \pm 2$; no absolute maximum; absolute minimum: 0 at $x = \pm 2$

28. **(a)** Critical points at $x = 0, 1, 2$ increasing on $(0, 1)$ and $(2, \infty)$ decreasing on $(-\infty, 0)$ and $(1, 2)$
(b) A local maximum is $g(1) = 1$ at $x = 1$, local minima are $g(0) = 0$ at $x = 0$ and $g(2) = 0$ at $x = 2$, no absolute maximum; absolute minimum is 0 at $x = 0, 2$

29. **(a)** Increasing on $(-\infty, -1)$; decreasing on $(-1, 0)$; increasing on $(0, 1)$; decreasing on $(1, \infty)$
(b) Local maximum: 0.5 at $x = \pm 1$; local minimum: 0 at $x = 0$; absolute maximum: $1/2$ at $x = \pm 1$; no absolute minimum

30. **(a)** $t = 0, \pm 3$ increasing on $(-3, 0) \cup (0, 3)$, decreasing on $(-\infty, -3)$ and $(3, \infty)$
(b) a local maximum is $K(3) = 162$ at $t = 3$, a local minimum is $k(-3) = -162$ at $t = -3$, no absolute extrema.

31. **(a)** Increasing on $(10, \infty)$; decreasing on $(1, 10)$
(b) Local maximum: 1 at $x = 1$; local minimum: -8 at $x = 10$; absolute minimum: -8 at $x = 10$

32. **(a)** $x = 1$ and $x = 0$; increasing on $(0, 1)$, decreasing on $(1, \infty)$
(b) a local minimum is $f(0) = 3$, a local maximum is $f(1) = 6$, absolute maximum of 6 at $x = 1$

33. **(a)** Decreasing on $\left(-2\sqrt{2}, -2\right)$; increasing on $(-2, 2)$; decreasing on $\left(2, 2\sqrt{2}\right)$
(b) Local minima: $g(-2) = -4$, $g\left(2\sqrt{2}\right) = 0$; local maxima: $g\left(-2\sqrt{2}\right) = 0$, $g(2) = 4$; absolute maximum: 4 at $x = 2$; absolute minimum: -4 at $x = -2$

34. **(a)** $x = 0, 4$ and 5; increasing on $(0, 4)$, decreasing on $(-\infty, 0)$ and $(4, 5)$
(b) a local maximum is $g(4) = 16$ at $x = 4$, a local minimum is 0 at $x = 0$ and $x = 5$, no absolute maximum; absolute minimum is 0 at $x = 0, 5$.

35. **(a)** Increasing on $(-\infty, 1)$; decreasing when $1 < x < 2$, decreasing when $2 < x < 3$; discontinuous at $x = 2$; increasing on $(3, \infty)$
(b) Local minimum at $x = 3$ $(3, 6)$; local maximum at $x = 1$ $(1, 2)$; no absolute extrema

36. **(a)** $x = 0$; increasing on $(-\infty, 0) \cup (0, \infty)$, and never decreasing
(b) No local extrema; no absolute extrema

37. **(a)** Increasing on $(-2, 0)$ and $(0, \infty)$; decreasing on $(-\infty, -2)$
(b) Local minimum: $-6\sqrt[3]{2}$ at $x = -2$; no absolute maximum; absolute minimum: $-6\sqrt[3]{2}$ at $x = -2$

38. **(a)** $x = -2$ and $x = 0$; increasing on $(-\infty, -2)$ and $(0, \infty)$, decreasing on $(-2, 0)$
(b) local maximum is $g(-2) = 3\sqrt[3]{4} \approx 4.762$ at $x = -2$, a local minimum is $g(0) = 0$ at $x = 0$, no absolute extrema

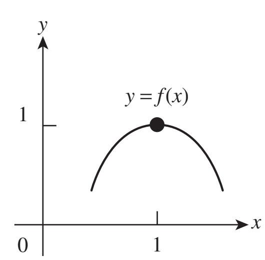
39. **(a)** Increasing on $\left(-\infty, -2/\sqrt{7}\right)$ and $\left(2/\sqrt{7}, \infty\right)$; decreasing on $\left(-2/\sqrt{7}, 0\right)$ and $\left(0, 2/\sqrt{7}\right)$
(b) Local maximum: $24\sqrt[3]{2}/7^{7/6} \approx 3.12$ at $x = -2/\sqrt{7}$; local minimum: $-24\sqrt[3]{2}/7^{7/6} \approx -3.12$ at $x = 2/\sqrt{7}$; no absolute extrema

40. **(a)** Increasing on $(-1, 0)$ and $(1, \infty)$; decreasing on $(-\infty, -1)$ and $(0, 1)$
(b) local maximum is $k(0) = 0$ at $x = 0$, local minima are $k(\pm 1) = -3$ at $x = \pm 1$, no absolute maximum; absolute minimum is -3 at $x = \pm 1$

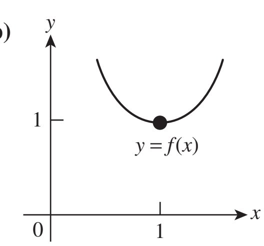
41. $h'(\theta) = 3\cos\dfrac{\theta}{2} \Rightarrow h'(\theta)\dfrac{-3}{2}\sin\left(\dfrac{\theta}{2}\right) \Rightarrow h' = \underset{0}{[}- - - \underset{2\pi}{]}$, $(0, 3)$ and $(2\pi, -3) \Rightarrow$ a local maximum is 3 at $\theta = 0$, a local minimum is -3 at $\theta = 2\pi$

42. $h(\theta) = 5\sin\left[\dfrac{\theta}{2}\right] \Rightarrow h'(\theta) = \dfrac{5}{2}\cos\left[\dfrac{\theta}{2}\right] \Rightarrow h' = \underset{0}{[}+ + + \underset{\pi}{]}$, $(0, 0)$ and $(\pi, 5) \Rightarrow$ a local maximum is 5 at $\theta = \pi$, local minimum is 0 at $\theta = 0$.

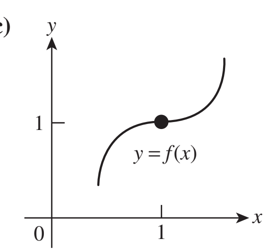
43. **(a)**



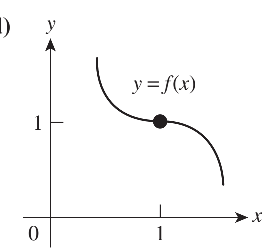
(b)



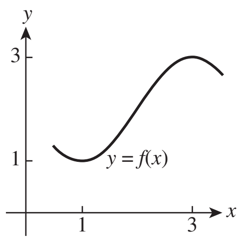
(c)



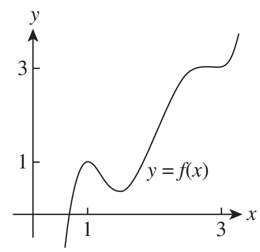
(d)



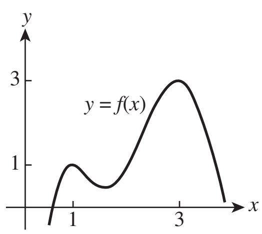
44. **(a)**



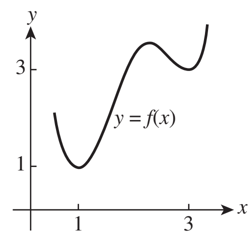
(b)



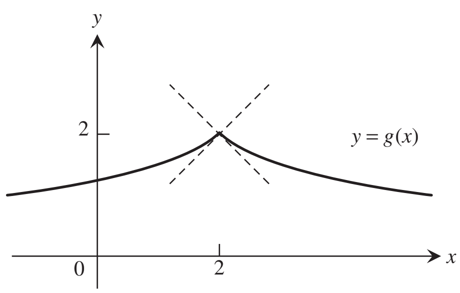
(c)



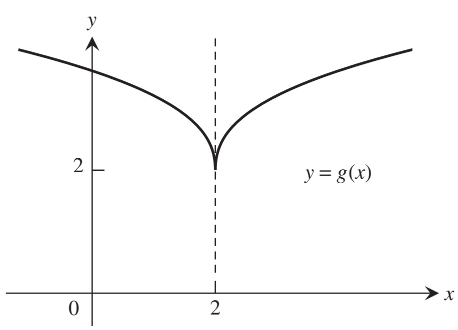
(d)



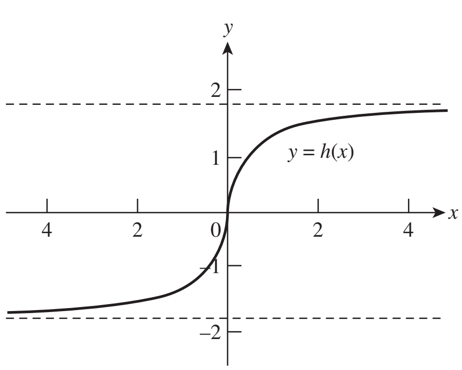
45. **(a)**



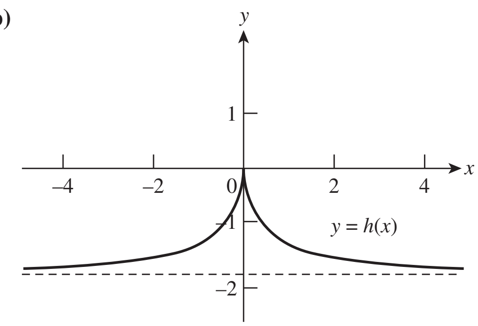
(b)



46. **(a)**



(b)



47. The function $f(x) = x\sin\left(\frac{1}{x}\right)$ has an infinite number of local maxima and minima on its domain, which is $(-\infty,0)\cup(0,\infty)$. The function sin x has the following properties: (a) it is continuous on $(-\infty,\infty)$; (b) It is periodic; and (c) its range is [–1, 1]. Also, for $a > 0$, the function $\frac{1}{x}$ has range of $(-\infty,-a) \cup [a,\infty)$ on $\left[\left(-\frac{1}{a},0\right)\cup\left(0,\frac{1}{a}\right)\right]$. In particular, if $a = 1$, then $\frac{1}{x}\le -1$ or $\frac{1}{x}\ge 1$ when x is in (–1, 0) ∪ (0, 1). This means $\sin\left(\frac{1}{x}\right)$ takes on the values of 1 and –1 infinitely many times on (–1, 0) ∪ (0, 1), namely at $\frac{1}{x}=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots.\Rightarrow x=\pm\frac{2}{\pi},\pm\frac{2}{3\pi}.\pm\frac{2}{5\pi},\ldots.$ Thus, $\sin\left(\frac{1}{x}\right)$ has infinitely many local maxima and minima in (–1, 0) ∪ (0, 1). On the interval (0, 1), $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and since, $x > 0$ we have $-x \le x\sin\left(\frac{1}{x}\right) \le x$. On the interval (–1, 0), $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and since $x > 0$ we have $-x \le \sin\left(\frac{1}{x}\right) \le x$. Thus, $f(x)$ is bounded by the lines $y = x$ and $y = -x$. Since $\sin\left(\frac{1}{x}\right)$ oscillates between 1 and –1 infinitely many times on on (–1, 0) ∪ (0, 1) then f will oscillate between $y = x$ and $y = -x$ infinitely many times. Thus f has infinitely many local maxima and minima. We can see from the graph that $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)=1$ and $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)=1$. The graph of f does not have any absolute maxima, but it does have two absolute minima.

48. $f(x)=ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$, a parabola whose vertex is at $x = -\frac{b}{2a}$. Thus when $a > 0$, f is increasing on $\left(\frac{-b}{2a},\infty\right)$ and decreasing

on $\left(-\infty, \frac{-b}{2a}\right)$; When a < 0, f is increasing on $\left(-\infty, \frac{-b}{2a}\right)$ and decreasing on $\left(\frac{-b}{2a}, \infty\right)$. Also note that

$f'(x) = 2ax + b = 2a\left(x + \frac{b}{2a}\right)$ *for* $a > 0$, $f' = - - - \underset{-\frac{b}{2a}}{|} + + +$;

for a < 0, $f' = + + + \underset{\frac{-b}{2a}}{|} - - -$.

49 $f(x) = ax^2 + bx \Rightarrow f'(x) = 2ax + b, f'(1) = 2 \Rightarrow a + b = 2, f'(1) = 0 \Rightarrow 2a + b = 0 \Rightarrow a = -2, b = 4 \Rightarrow f(x) = -2x^2 + 4x$

50. $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c, f(0) = 0 \Rightarrow d = 0, f(1) = -1 \Rightarrow a + b + c + d = -1, f'(0) = 0 \Rightarrow c = 0, f'(1) = 0 \Rightarrow 3a + 2b + c = 0 \Rightarrow a = 2, b = -3, c = 0, d = 0 \Rightarrow f(x) = 2x^3 - 3x^2$.

Exercises 4.7

1. Local maximum: $3/2$ at $x = -1$; local minimum: -3 at $x = 2$; point of inflection at $(1/2, -3/4)$; rising on $(-\infty, -1)$ and $(2, \infty)$; falling on $(-1, 2)$; concave up on $(1/2, \infty)$; concave down on $(-\infty, 1/2)$

2. The graph is rising on (–2,0) and (2,∞), falling on (–∞, –2) and (0,2) concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and concave down on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Consequently, a local maximum is 4 at $x = 0$, local minima are 0 at $x = \pm 2$, and $\left(-\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$ are points of inflection.

3. Local maximum: $3/4$ at $x = 0$; local minimum: 0 at $x = \pm 1$; points of inflection at $\left(-\sqrt{3}, \frac{3\sqrt[3]{4}}{4}\right)$ and $\left(\sqrt{3}, \frac{3\sqrt[3]{4}}{4}\right)$; rising on $(-1, 0)$ and $(1, \infty)$; falling on $(-\infty, -1)$ and $(0, 1)$; concave up on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$; concave down on $\left(-\sqrt{3}, \sqrt{3}\right)$

4. The graph is rising on (–∞, –1) and (1, ∞), falling on (–1, 1) $\Rightarrow$ a local maximum is $\frac{27}{7}$ at $x = -1$, a local minimum is $-\frac{27}{7}$ at $x = 1$; the graph is concave up on (0, ∞), concave down on (–∞, 0) $\Rightarrow$ a point of inflection at (0, 0).

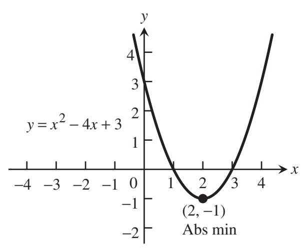
5. Local maxima: $\frac{-2\pi}{3} + \frac{\sqrt{3}}{2}$ at $x = -2\pi/3$, $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ at $x = \pi/3$; local minima: $-\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = -\pi/3$, $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = 2\pi/3$; points of inflection at $(-\pi/2, -\pi/2)$, $(0, 0)$, and $(\pi/2, \pi/2)$; rising on $(-\pi/3, \pi/3)$; falling on $(-2\pi/3, -\pi/3)$ and $(\pi/3, 2\pi/3)$; concave up on $(-\pi/2, 0)$ and $(\pi/2, 2\pi/3)$; concave down on $(-2\pi/3, -\pi/2)$ and $(0, \pi/2)$

6. The graph is rising on $\left(-\frac{\pi}{2}, \frac{-\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, falling on $\left(-\frac{\pi}{2}, \frac{\pi}{3}\right) \Rightarrow$ a local maximum is $-\sqrt{3} + \frac{4\pi}{3}$ at $x = \frac{-\pi}{3}$, a local minimum is $\sqrt{3} - \frac{4\pi}{3}$ at $x = \frac{\pi}{3}$, the graph is concave up on $\left(0, \frac{\pi}{2}\right)$, concave down on $\left(-\frac{\pi}{2}, 0\right) \Rightarrow$ a point of inflection at (0, 0).

7. Local maxima: 1 at $x = -\pi/2$ and $x = \pi/2$, 0 at $x = -2\pi$ and $x = 2\pi$; local minima: -1 at $x = -3\pi/2$ and $x = 3\pi/2$, 0 at $x = 0$; points of inflection at $(-\pi, 0)$ and $(\pi, 0)$; rising on $(-3\pi/2, -\pi/2)$, $(0, \pi/2)$, and $(3\pi/2, 2\pi)$; falling on $(-2\pi, -3\pi/2)$, $(-\pi/2, 0)$, and $(\pi/2, 3\pi/2)$; concave up on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$; concave down on $(-\pi, 0)$ and $(0, \pi)$

8. Rising on $\left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$, falling on $\left(-\pi, -\frac{3\pi}{4}\right)$ and $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right) \Rightarrow$ a local maxima are $-2 + \pi\sqrt{2}$ at $x = -\pi$, $\sqrt{2} + \frac{\pi\sqrt{2}}{4}$ at $x = -\frac{\pi}{4}$ and $-\frac{3\pi\sqrt{2}}{2}$ at $x = \frac{3\pi}{2}$, and local minima are $-\sqrt{2} + \frac{3\pi\sqrt{2}}{4}$ at $x = -\frac{3\pi}{4}$ and $-\sqrt{2} - \frac{5\pi\sqrt{2}}{4}$ at $x = \frac{5\pi}{4}$; concave up on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ concave down on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$ points of inflection at $\left(-\frac{\pi}{2}, \frac{\sqrt{2}\pi}{2}\right)$ and $\left(\frac{-\pi}{2}, \frac{\sqrt{2}\pi}{2}\right)$.

9.



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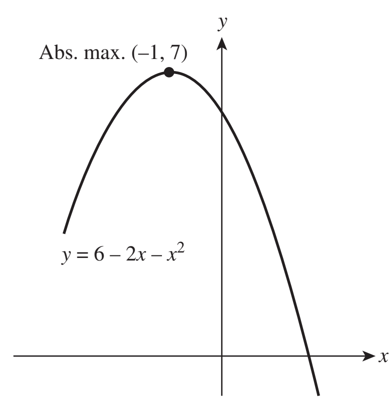

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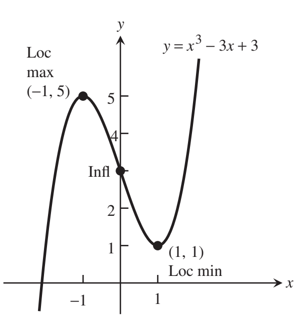

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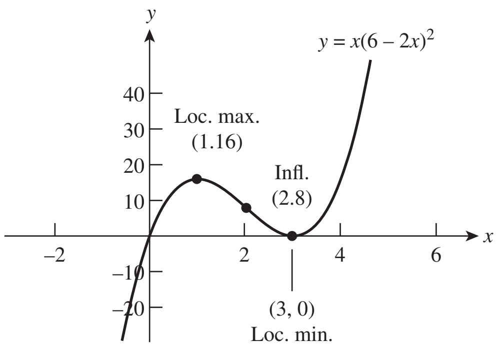

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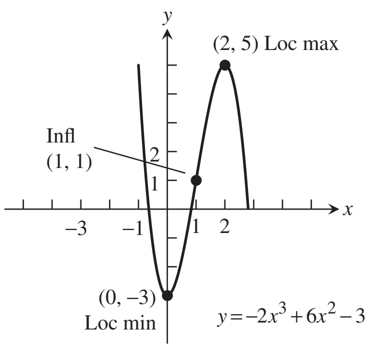

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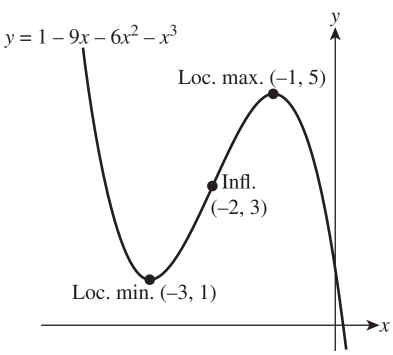

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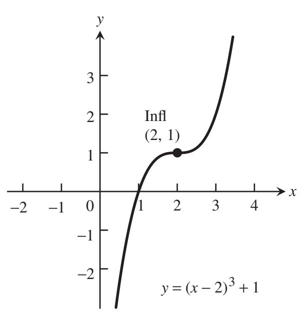

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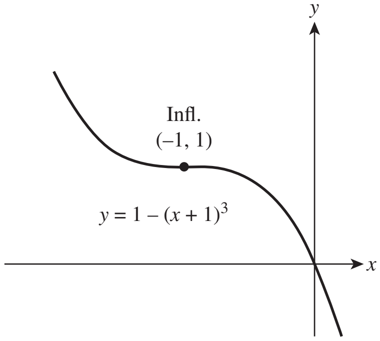

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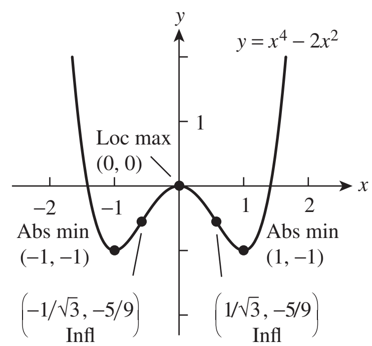

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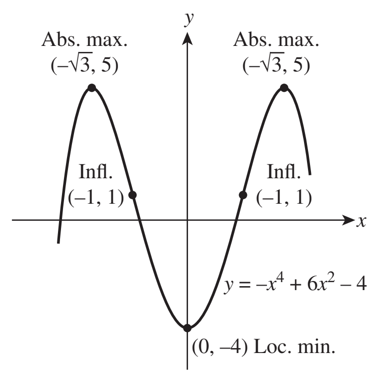

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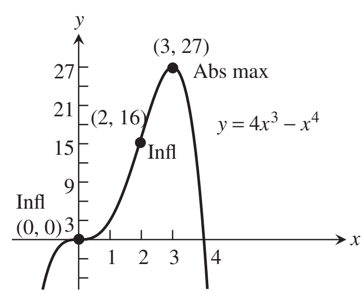

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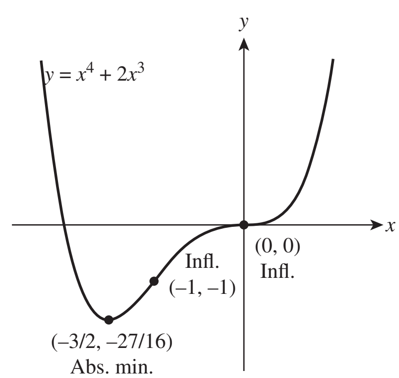

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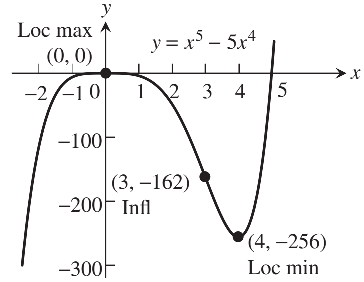

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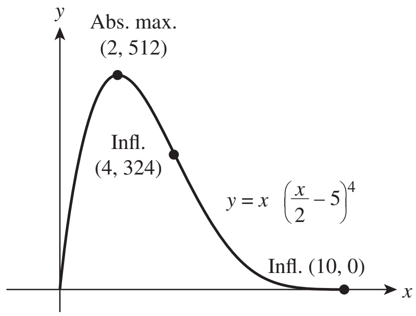

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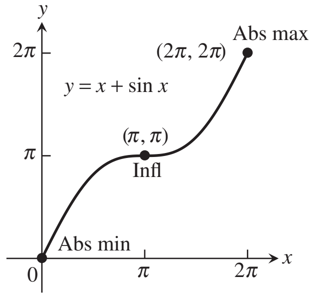

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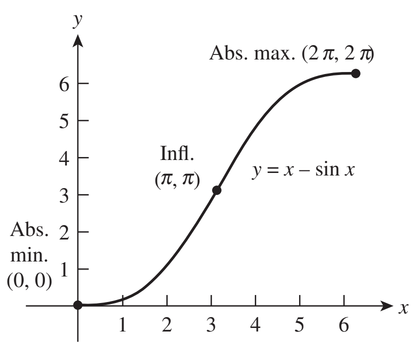

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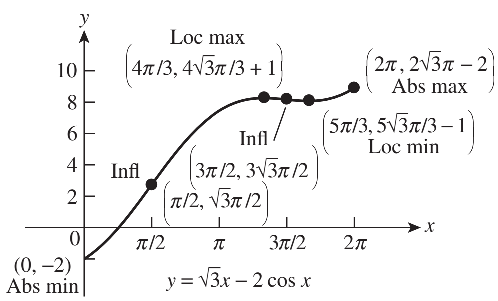

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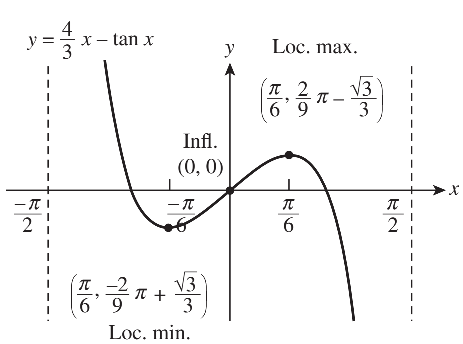

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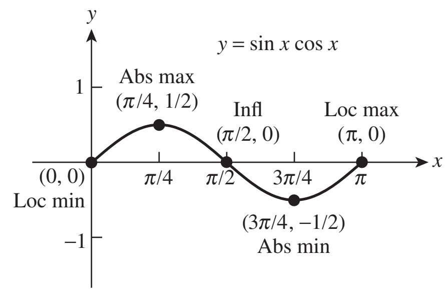

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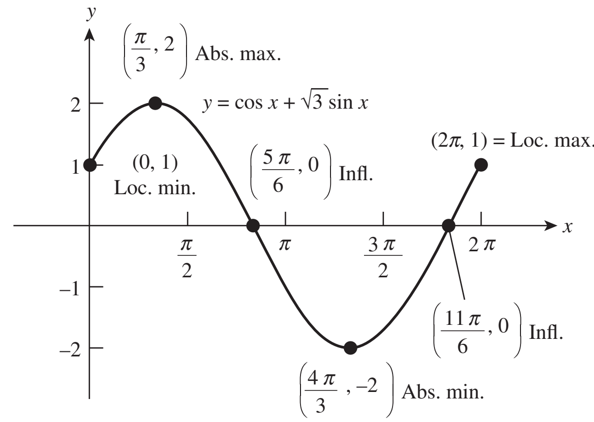

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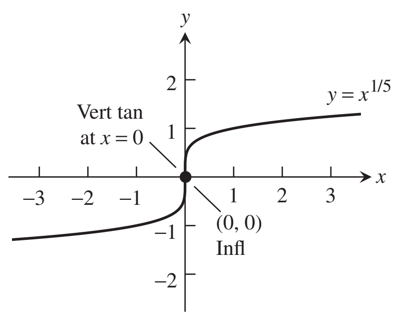

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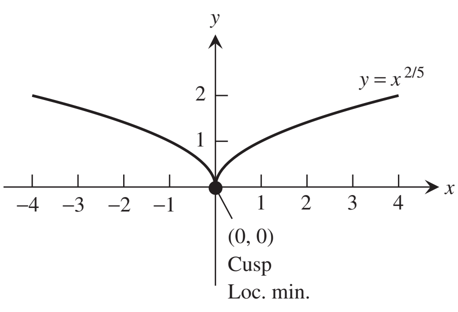

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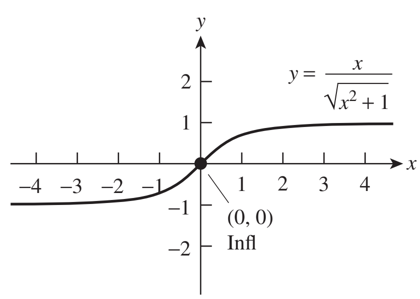

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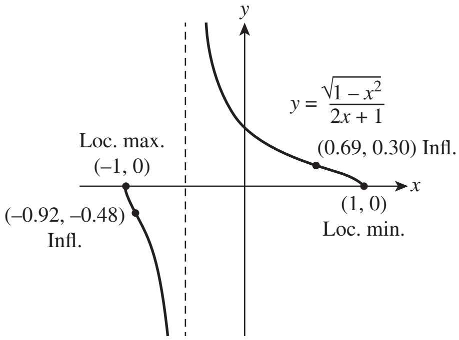

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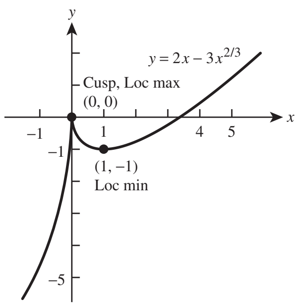

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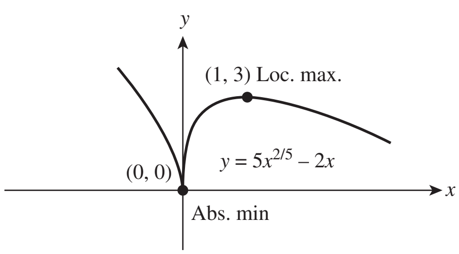

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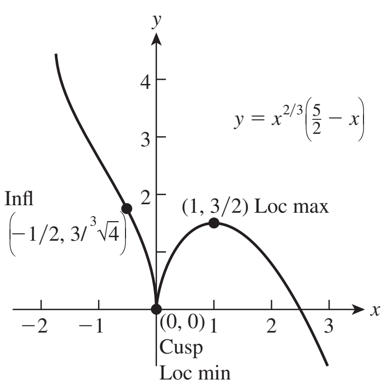

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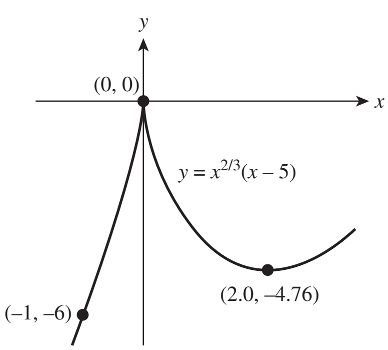

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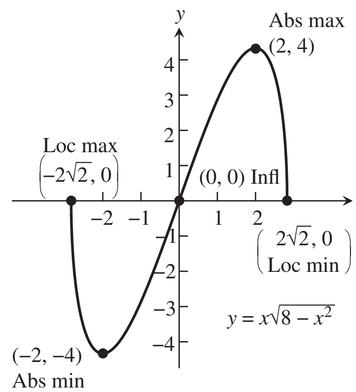

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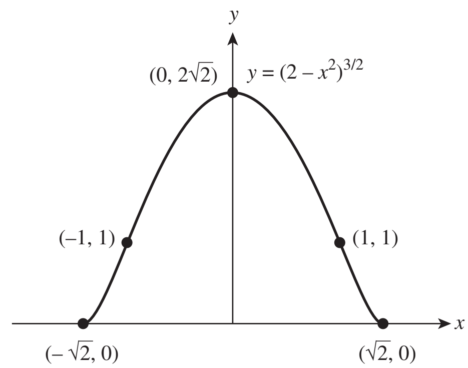

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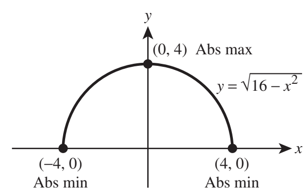

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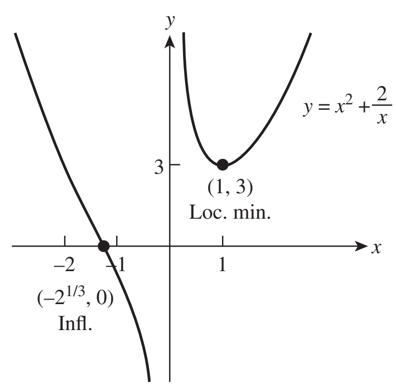

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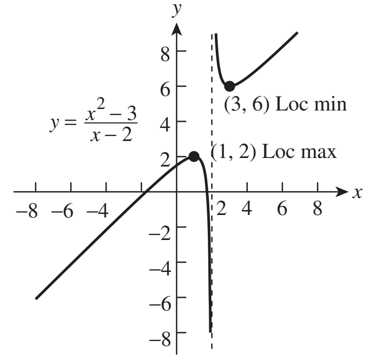

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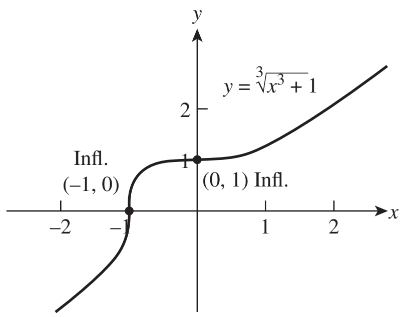

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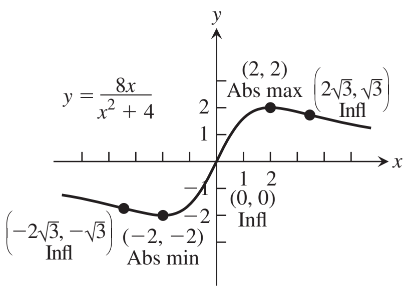

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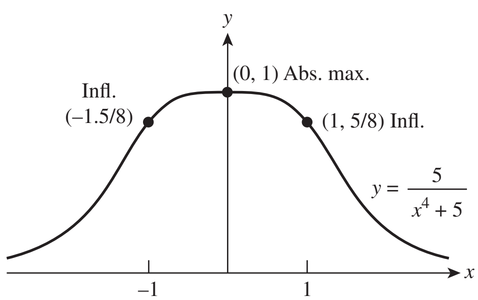

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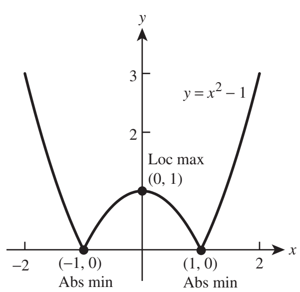

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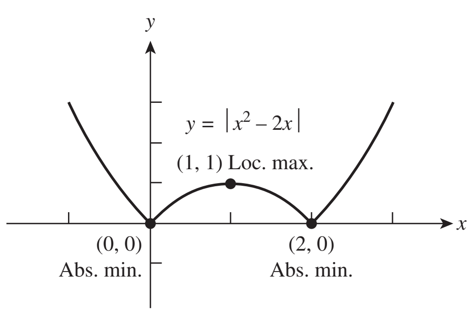

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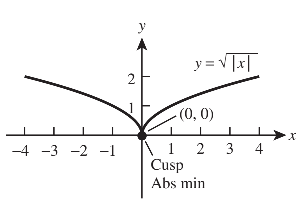

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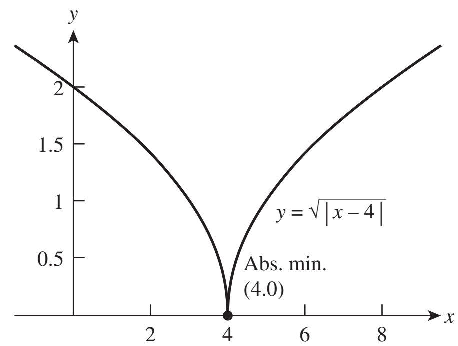
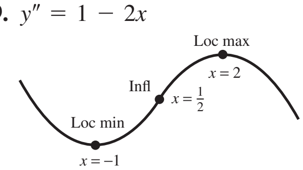
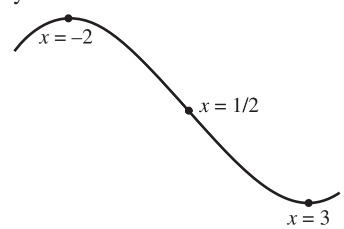


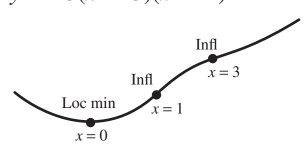
49. $y'' = 1 - 2x$



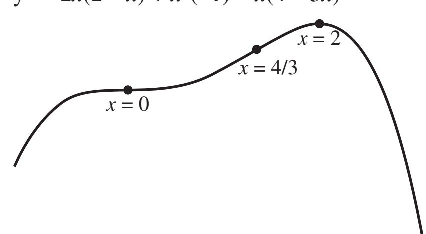
50. $y'' = 2x - 1$



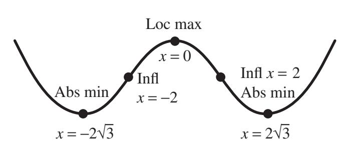
51. $y'' = 3(x - 3)(x - 1)$



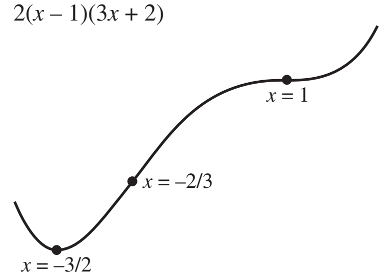
52. $y'' = 2x(2 - x) + x^2(-1) = x(4 - 3x)$



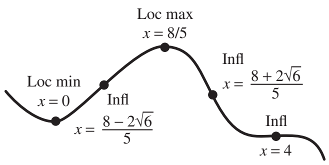
53. $y'' = 3(x - 2)(x + 2)$



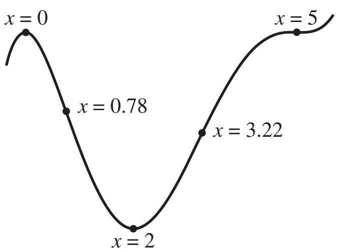
54. $2(x - 1)(3x + 2)$



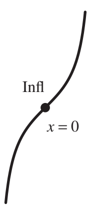
55. $y'' = 4(4 - x)(5x^2 - 16x + 8)$



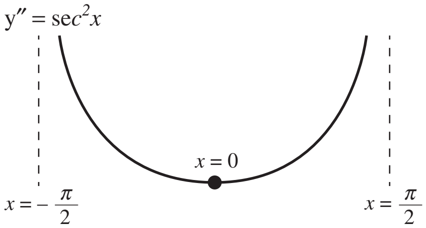
56. $2(x-5)(2x^2-8x+5)$



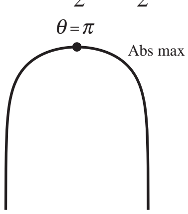
57. $y'' = 2\sec^2 x \tan x$



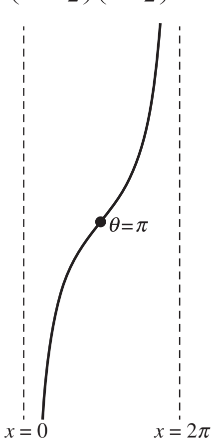
58. $y'' = \sec^2 x$



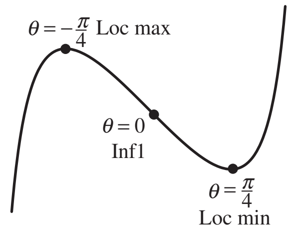
59. $y'' = -\frac{1}{2}\csc^2\frac{\theta}{2},\ 0 < \theta < 2\pi$



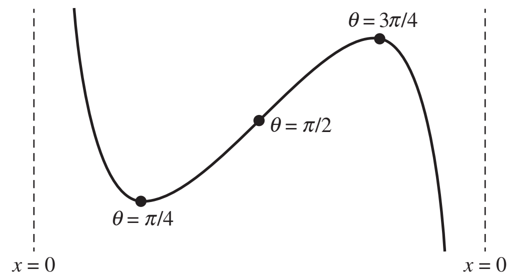
60. $\left(\csc^2\frac{\theta}{2}\right)\left(\cot\frac{\theta}{2}\right)$



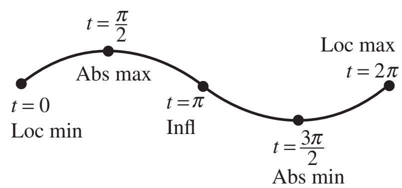
61. $y'' = 2\tan\theta\sec^2\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



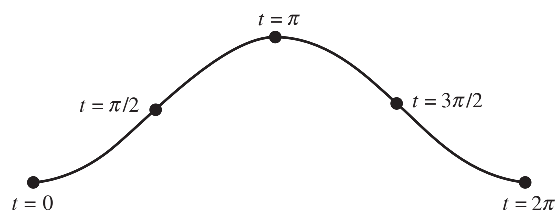
62. $y'' = -2(\cot\theta)(-\mathrm{cosec}^2\theta)$



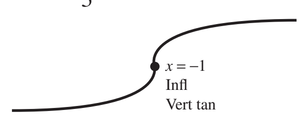
63. $y'' = -\sin t,\ 0 \le t \le 2\pi$



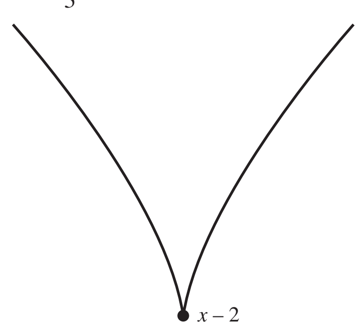
64. $y'' = \cos t$



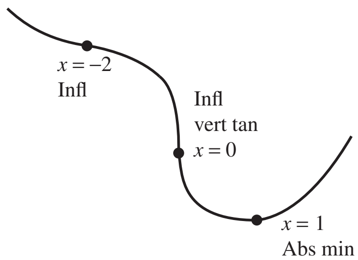
65. $y'' = -\frac{2}{3}(x + 1)^{-5/3}$



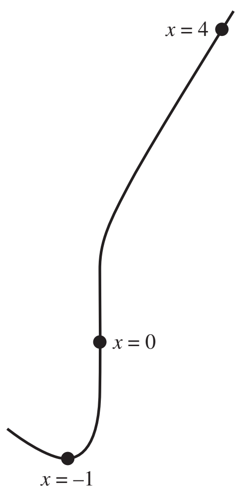
66. $y'' = \frac{1}{3}(x-2)^{-4/3}$



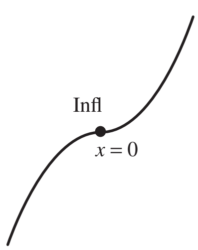
67. $y'' = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3}$



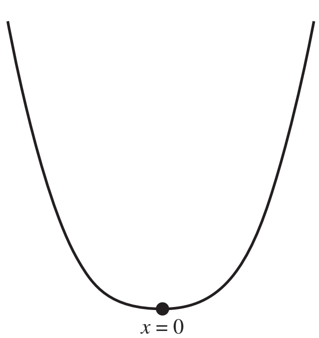
68. $\frac{1}{5}x^{\frac{-9}{5}}(x-4)$



69. $y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$



70. $y'' = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$



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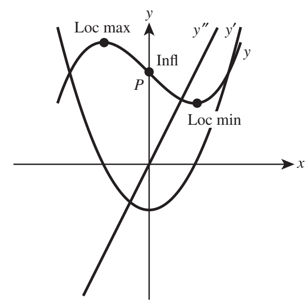

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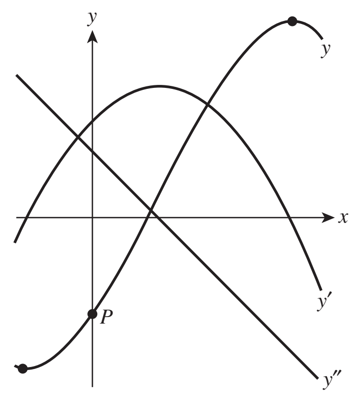

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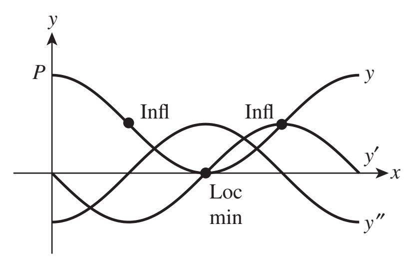

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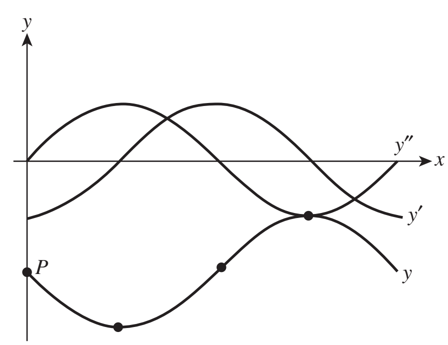

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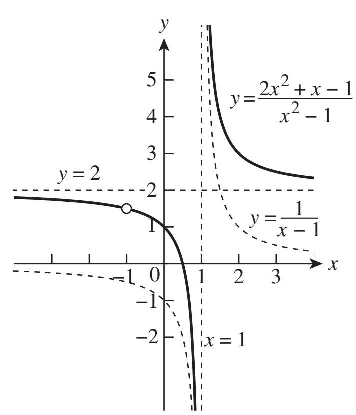

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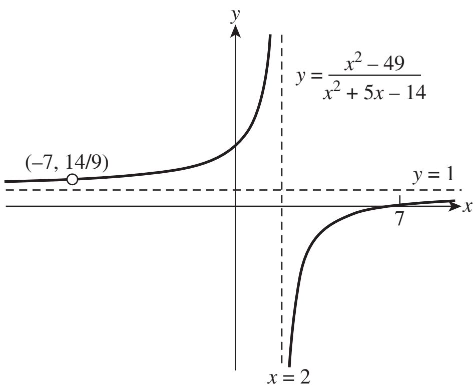

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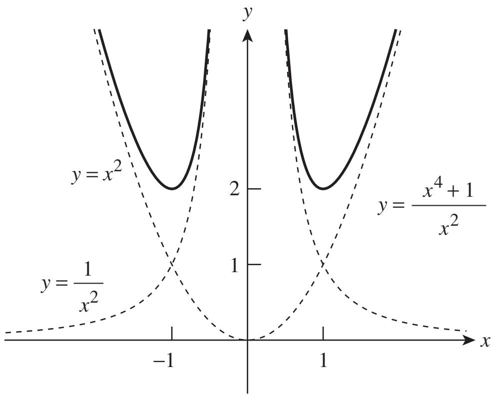

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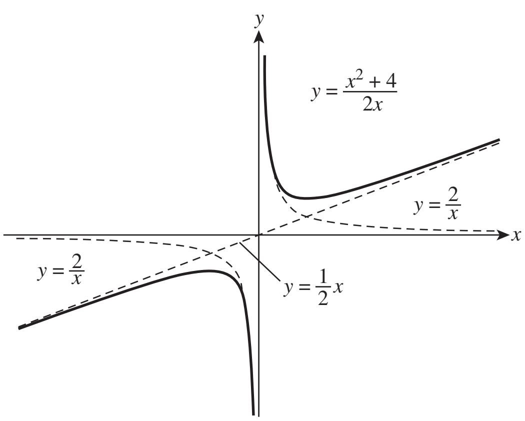

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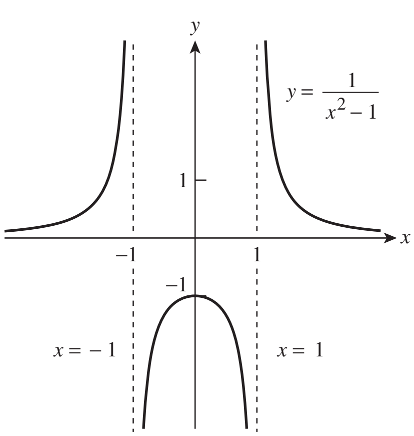

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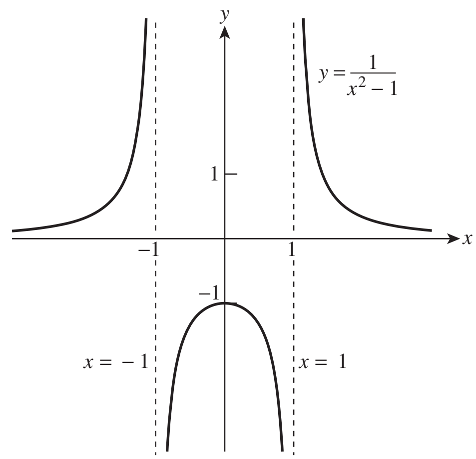

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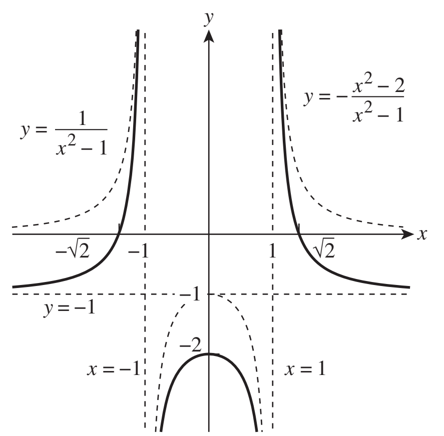

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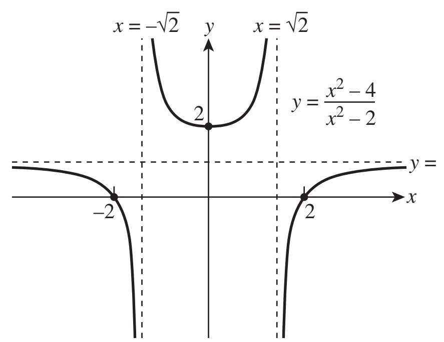

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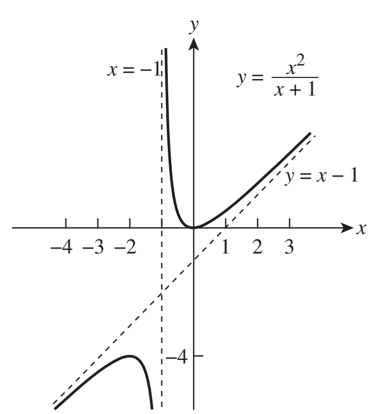

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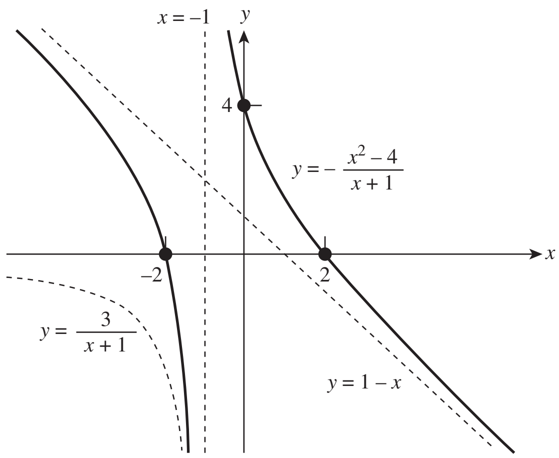

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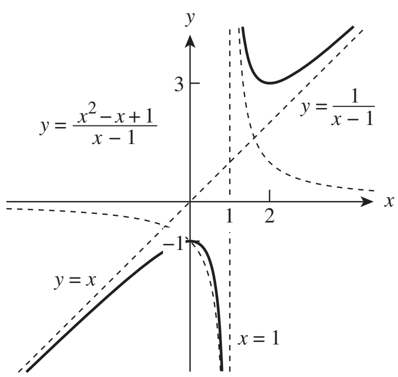

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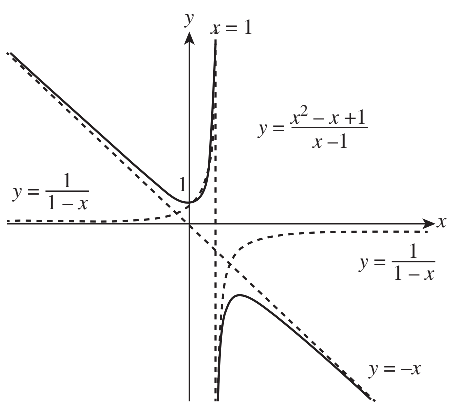

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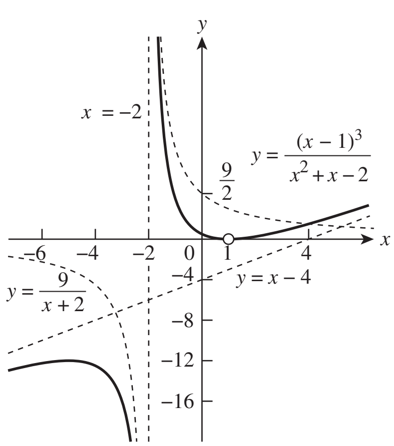

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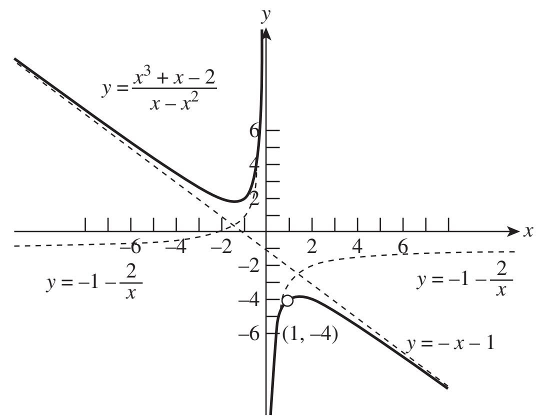

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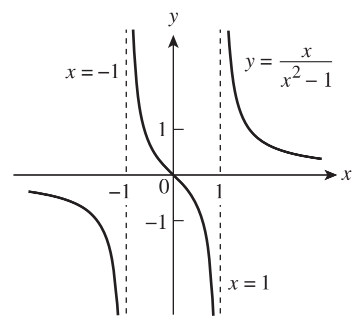

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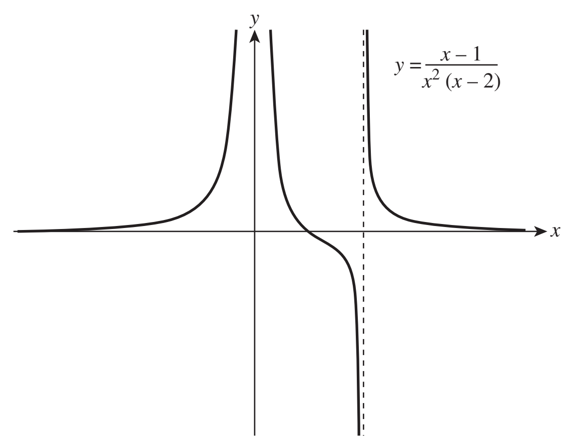

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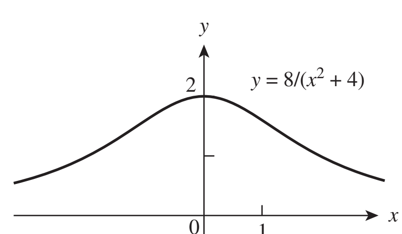

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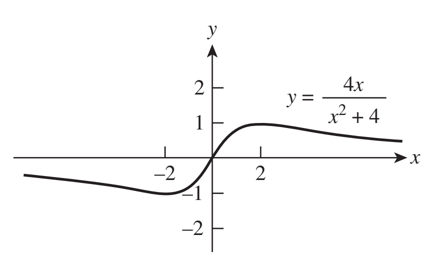
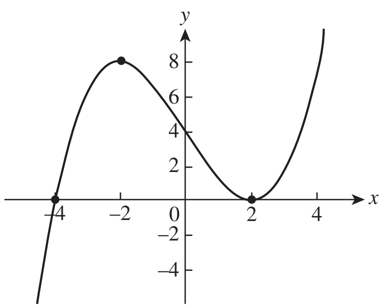


93.

Point	y'	y''
P	–	+
Q	+	0
R	+	–
S	0	–
T	–	–

94.



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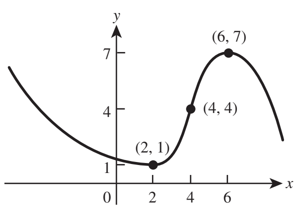
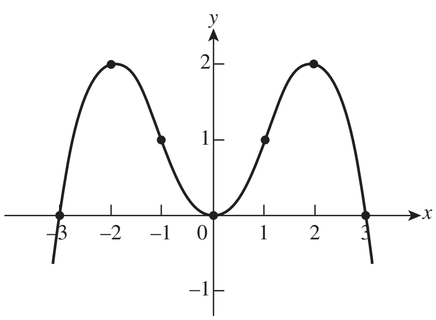


96.



97. **(a)** Towards origin: $0 \le t < 2$ and $6 \le t \le 10$; away from origin: $2 \le t \le 6$ and $10 \le t \le 15$
(b) $t = 2, t = 6, t = 10$
(c) $t = 5, t = 7, t = 13$
(d) Positive: $5 \le t \le 7$, $13 \le t \le 15$; negative: $0 \le t \le 5, 7 \le t \le 13$

98. **(a)** The body is moving away from the origin when $|\text{displacement}|$ is increasing as t increases, $1.5 < t < 4$, $10 < t < 12$ and $13.5 < t < 16$; the body is moving toward the origin when $|\text{displacement}|$ is decreasing as t increases, $0 < t < 1.5$, $4 < t < 10$ and $12 < t < 13.5$.
(b) The velocity will be zero when the slope of the tangent line for $y = s(t)$ is horizontal. The velocity is zero when t is approximately 0, 4, 12 or 16 sec.
(c) The acceleration will be zero at those values of t where the curve $y = s(t)$ has points of inflection. The acceleration is zero when t is approximately 1. 5, 6, 8, 10.5 or 13.5 sec.
(d) The acceleration is positive when the concavity is up, $0 < t < 1.5$, $6 < t < 8$ and $10 < t < 13.5$, the acceleration is negative when the concavity is down, $1.5 < t < 6$, $8 < t < 10$ and $13.5 < t < 16$.

99. $\approx$ 60 thousand units

100. The marginal revenue is $\frac{dy}{dx}$ and it is increasing when its derivative $\frac{d^2y}{dx^2}$ is positive $\Rightarrow$ the curve is concave up $\Rightarrow 0 < t < 2$ and $5 < t < 9$; marginal revenue is decreasing when $\frac{d^2y}{dx^2} < 0$ $\Rightarrow$ the curve is concave down $\Rightarrow 2 < t < 5$ and $9 < t < 12$.

101. Local minimum at $x = 2$; inflection points at $x = 1$ and $x = 5/3$

102. The curve rises on $(-\infty, 2)$ and $(4, \infty)$ and falls on $(2, 4)$. At $x = 2$ there is a local maximum and at $x = 4$ a local minimum. The curve is concave downward on $(-\infty, 1)$ and $\left(\frac{5-\sqrt{3}}{2}, \frac{5+\sqrt{3}}{2}\right)$ and concave upward on $\left(1, \frac{5-\sqrt{3}}{2}\right)$ and $\left(1, \frac{5+\sqrt{3}}{2}\right)$. At $x = 1$, $\frac{5-\sqrt{3}}{2}$ and $\frac{5+\sqrt{3}}{2}$ there are inflection points.

103. The graph must be concave down by $x > 0$.

104. The second derivative, being continuous and never zero, cannot change sign. Therefore the graph will always be concave up or concave down so it will have no inflection points and no cusps or corners.

105. $b = -3$

106. **(a)** The coordinates of the vertex are $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$.
(b) The second derivative, $f''(x) = 2a$, describes concavity $\Rightarrow$ when $a > 0$ the parabola is concave up and when $a < 0$ the parabola is concave down.

107. A quadratic curve never has an inflection point.

108. A cubic curve always has exactly one inflection point.

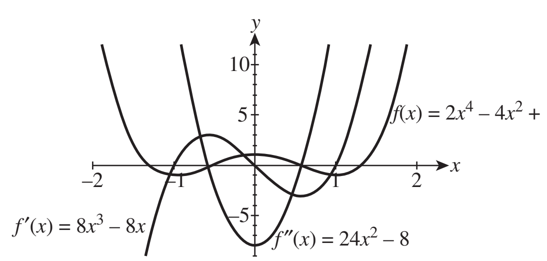
109. $-1, 2$

110. $y'' = x^2(x-2)^3(x+3)$, when $y'' = 0 \Rightarrow x = -3, x = 0$ or $x = 2$; $y'' = + + + \underset{-3}{|} - - - \underset{0}{|} - - - \underset{2}{|} + + + \Rightarrow$ points of inflection at $x = -3$ and $x = 2$.

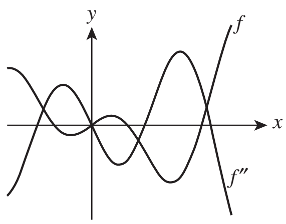
111. $a = 1, b = 3, c = 9$

112. $a = 3$, $b = 1$, and $c = -1 \Rightarrow y = \frac{x^2+3}{x-1}$.

113. The graph of f is concave down where $f'' < 0$ and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



114. The graph f is concave down where $f'' < 0$, and concave up where $f'' > 0$. It has an inflection point each time f'' changes sign, provided a tangent line exists there.



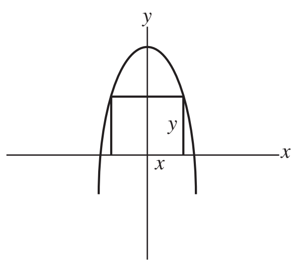
Exercises 4.8

1. 16 in., 4 in. by 4 in.

3. **(a)** $(x, 1 - x)$ **(b)** $A(x) = 2x(1 - x)$

(c) $\frac{1}{2}$ square units, 1 by $\frac{1}{2}$

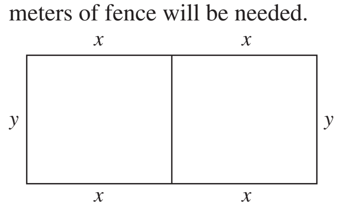
4 $A(2) = 32$ square units is the maximum area. The dimensions are 4 units by 8 units.



5. $\frac{14}{3} \times \frac{35}{3} \times \frac{5}{3}$ in., $\frac{2450}{27}$ in^3

7. 80,000 m^2; 400 m by 200 m

8. The dimensions of the outer rectangle are 18 m by 12 m $\Rightarrow$ 72 meters of fence will be needed.



9. **(a)** The optimum dimensions of the tank are 10 ft on the base edges and 5 ft deep.

(b) Minimizing the surface area of the tank minimizes its weight for a given wall thickness. The thickness of the steel walls would likely be determined by other considerations such as structural requirements.

10. **(a)** The values of $x = 15$ ft and $y = 5$ ft will minimize the cost.

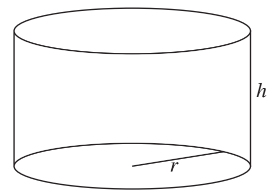
(b) The total cost of the least expensive tank is \$3375, which is the sum of \$2625 for fabrication and \$750 for the excavation.

11. 9×18 in.

12. $\frac{32\pi}{3}$ cubic units.

13. $\frac{\pi}{2}$

14. We have a minimum surface area when $r = \frac{10}{\sqrt[3]{\pi}}$ cm and $h = \frac{1000}{\pi r^2} = \frac{10}{\sqrt[3]{\pi}}$ cm. Comparing this result to the result found in Example 2, if we include both ends of the can, the n we have a minimum surface area when the can is shorter—specifically, when the height of the can is the same as its diameter.

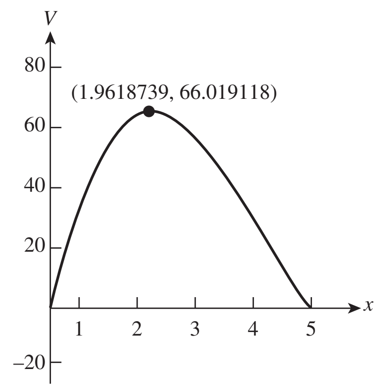


15. $h : r = 8 : \pi$

16. **(a)** The base measures of 10–$2x$ in. by $\frac{15-2x}{2}$ in., so the volume formula is $V(x) =$

$$\frac{x(10-2x)(15-2x)}{2} = 2x^3 - 25x^2 + 75x.$$

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval (0, 5).



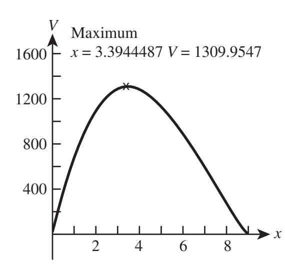
(c) The maximum volume is approximately 66.02 in.^{3} when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 – 50x + 75$. The critical point occurs when $V'(x) =$

$$0, \text{ at } x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} = \frac{25 \pm 5\sqrt{7}}{6},$$

that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x – 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$ which confirms the result in (c).

17. **(a)** $V(x) = 2x(24 - 2x)(18 - 2x)$ **(b)** Domain: (0, 9)



(c) Maximum volume $\approx$ 1309.95 in^3 when $x \approx 3.39$ in.

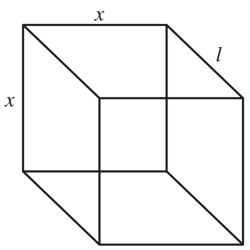
(d) $V'(x) = 24x^2 - 336x + 864$, so the critical point is at $x = 7 - \sqrt{13}$, which confirms the result in part (c).

(e) $x = 2$ in. or $x = 5$ in.

18. The dimensions of the rectangle are approximately 4.43 (width) by 1.79 (height), and the maximum area is approximately 7.923.

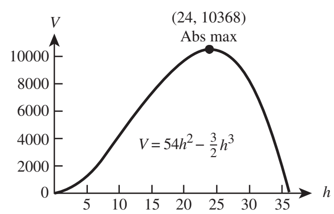
19. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16$ cm and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{\sqrt[3]{3}} \approx 2418.40 \text{ cm}^3$.

20. The dimensions of the box are $18 \times 18 \times 36$ in.

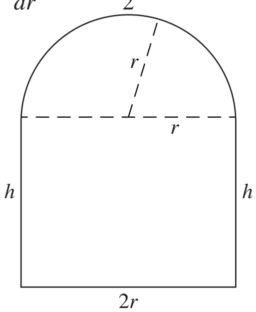


21. **(a)** $h = 24$, $w = 18$

(b)



22. $\frac{2r}{h} = \frac{8}{4+\pi}$ gives the proportions that admit the most light, since $\frac{d^2A}{dr^2} = -4 - \frac{3}{2}\pi < 0$.



23. If r is the radius of the hemisphere, h the height of the cylinder, and V the volume, then $r = \left(\frac{3V}{8\pi}\right)^{1/3}$ and $h = \left(\frac{3V}{\pi}\right)^{1/3}$.

24. At $\theta < \frac{\pi}{6}$ there is a maximum.

25. **(b)** $x = \frac{51}{8}$ **(c)** $L \approx 11$ in.

26. **(a)** The values of $x = 12$ cm and $y = 6$ cm give the largest volume.
(b) The values of $x = 12$ cm and $y = 6$ cm give the largest volume.

27. Radius $= \sqrt{2}$ m, height $= 1$ m, volume $= \frac{2\pi}{3} \text{m}^3$

28. $\left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$ is the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

29. 1

30. $x = \frac{1}{2}$.

31. $\frac{9b}{9+\sqrt{3}\pi}$ m, triangle; $\frac{b\sqrt{3}\pi}{9+\sqrt{3}\pi}$ m, circle

32. $\frac{4b}{4+\pi}$ m is the length of the square segment; $\frac{b\pi}{4+\pi}$ m is the length of the circular segment.

33. $\frac{3}{2} \times 2$

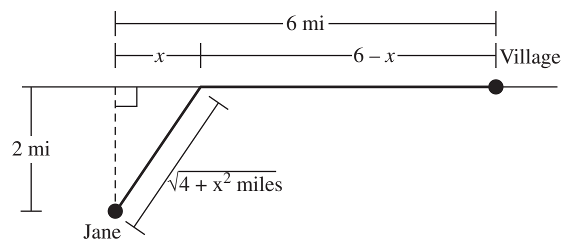
34. Base of the rectangle is $3\sqrt{2}$ and height is $3\sqrt{2}$.

35. **(a)** 16 **(b)** -1

36. **(a)** $a = -3$ and $b = -9$.
(b) $a = -3$ and $b = -24$.

37. **(a)** $v(0) = 96$ ft/sec
(b) 256 ft at $t = 3$ sec
(c) Velocity when $s = 0$ is $v(7) = -128$ ft/sec.

38.



Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

39. ≈ 46.87 ft

40. **(a)** $t = \frac{\pi}{3}$ or $\frac{4\pi}{3}$.
(b) The greatest distance between the particles is 1.
(c) At $t = \frac{\pi}{3}$ and $\frac{4\pi}{3}$ $s'(t)$ has cusps.

41. **(a)** $6 \times 6\sqrt{3}$ in.

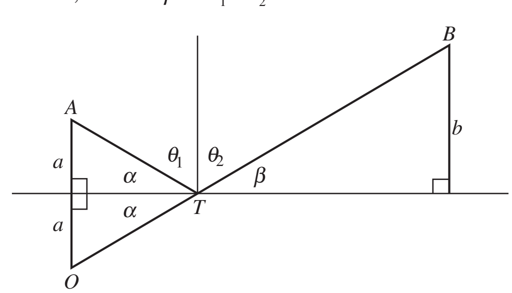
42. The firing angle of $\alpha = \frac{\pi}{4} = 45°$ will maximize the range R.

43. **(a)** $4\sqrt{3} \times 4\sqrt{6}$ in.

44. **(a)** $10\pi \approx 31.42$ cm/sec; when $t = 0.5$ sec, 1.5 sec, 2.5 sec, 3.5 sec; $s = 0$, acceleration is 0.
(b) 10 cm from rest position; speed is 0.

45. **(a)** $t = R\pi$; **(b)** $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}, \frac{\sqrt[3]{3}}{2}$

46. The distance $OT + TB$ is minimized when $\overline{OB}$ is straight line. Hence, $\angle\alpha = \angle\beta \Rightarrow \theta_1 = \theta_2$



47. **(a)** According to the graph, $y'(0) = 0$.
(b) According to the graph, $y'(-L) = 0$.
(c) $y(0) = 0$, so $d = 0$. Now, $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y'(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$, then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 + 2bL = 0$, so we have two linear equations in two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$, . Therefore, $b = 3\frac{H}{L^2}$ and the equation for y is

$$y(x) = 2\frac{H}{L^3}x^3 + 3\frac{H}{L2}x^2, \text{ or } y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$

48. At $x = \frac{c}{2} + 50$ there is a maximum profit since $p''(x) = -2b < 0$ for all x.

49. It would take 67 people to maximize the profit.

52. The average cost of producing x items is

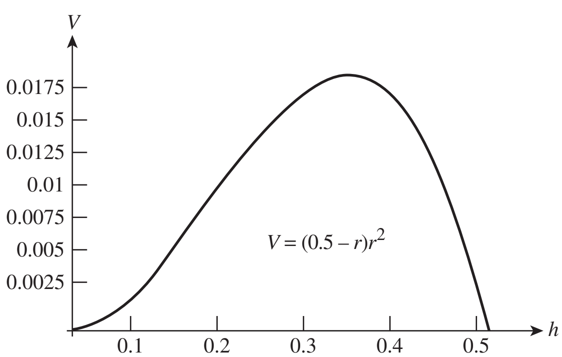
$\bar{c}(x) = \frac{c(x)}{x} = x^2 - 20x + 20000 \Rightarrow \bar{c}(x) = 2x - 20 = 0 \Rightarrow\ x = 10$, the only critical value. The average cost is $\bar{c}(10) = \$19{,}900$ per item is a minimum cost because $\bar{c}(10) = 2 > 0$.

53. The box 4 ft × 4 ft × 3 ft, with a minimum cost of \$288.

54. \$125.

55. At $M = \frac{C}{2}$ there is a maximum.

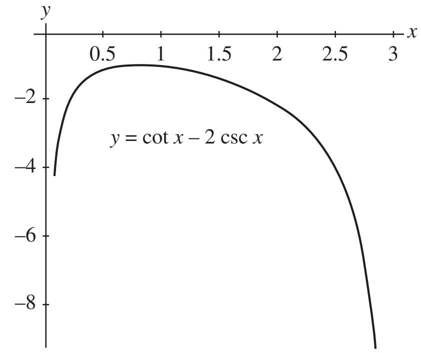
56. **(b)** The graph confirms the findings in (a).



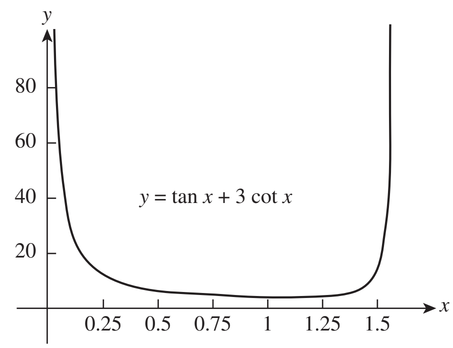
59. AT $x = c$, the tangents to the curves are parallel. Justification: The vertical distance between the curves is $D(x) = f(x) - g(x)$, so $D'(x) = f'(x) - g'(x)$. The maximum value of D will occur at a point c where $D' = 0$. At such a point, $f'(c) - g'(c) = 0$, or $f'(c) = g'(c)$.

60. **(a)** $f(x) = 3 + 4\cos x + \cos 2x$ is a periodic function with period 2π
(b) No, $f(x) = 3 + 4\cos x + \cos 2x = 3 + 4\cos x + (2\cos^2 x - 1) = 2(1 + 2\cos x + \cos^2 x) = 2(1 + \cos x)^2 \geq 0 \Rightarrow f(x)$ is never negative.

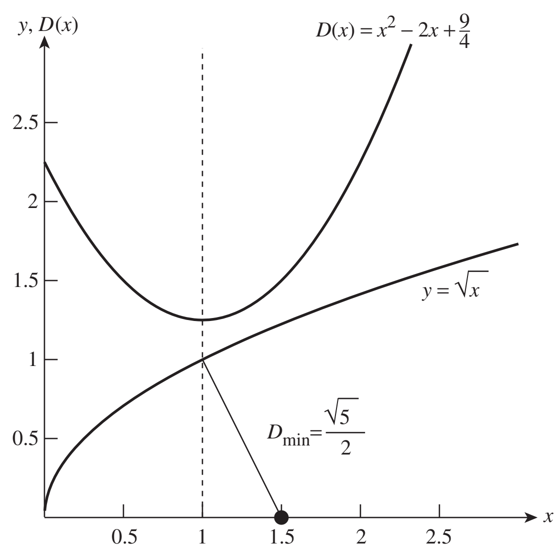
61. **(a)** At $x = \frac{\pi}{4}$ there is a maximum value of $y = -1$.
(b) The graph confirms the findings in (a).



62. **(a)** At $x = \frac{\pi}{3}$ is a minimum value of $y = 2\sqrt{3}$.
(b) The graph confirms the findings in (a).



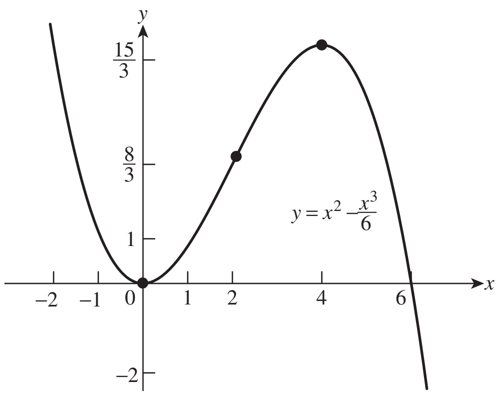
63. **(a)** The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.



64. 2

Answers to Practice Exercises

1. No.
2. No.
3. No. $x = 1$ and $x = \dfrac{11}{3}$ are critical points. $f(1) = 16$ is the absolute maximum
4. $a = 6$ and $b = -10$.
5. Yes
6. **(a)** $f(x) = x^3$
   (b) Theorem 2 says only that if f is differentiable and f has a local extreme at $x = c$ then $f'(c) = 0$. It does not assert the (false) reverse implication $f'(c) = 0 \Rightarrow f$ has a local extreme at $x = c$.
7. No
8. Absolute maximum is $|-1| = 1$, absolute maximum is $|0| = 0$, This is not inconsistent.
9. **(b)** One. The intermediate Value theorem guarantees the continuous function f has a root in $[-3, 0]$.
10. **(b)** $\tan\theta$ is undefined at $\theta = \dfrac{\pi}{2}$. Thus the tangent need not increase on this interval.
13. Yes
14. No
15. $f'(x) = g'(x) = \dfrac{2x}{(x^2+1)^2} \Rightarrow f(x) - g(x) = C$; the graphs differ by a vertical shift
16. $x = 2$
17. **(a)** $[-3, -2]$ and $[1, 2]$
    (b) $[-2, 0)$ and $(0, 1]$
    (c) The local maximum values occur only at $x = -2$ and $x = 2$; local minimum values occur at $x = -3$ and at $x = 1$ provided f is continuous at $x = 0$.
18. **(a)** $t = 0, 6, 12$
    (b) $t = 3, 9$
    (c) $6 < t < 12$
    (d) $0 < t < 6, 12 < t < 14$
19. **(a)** $t = 4$
    (b) at no time
    (c) $0 < t < 4$
    (d) $4 < t < 8$
20.



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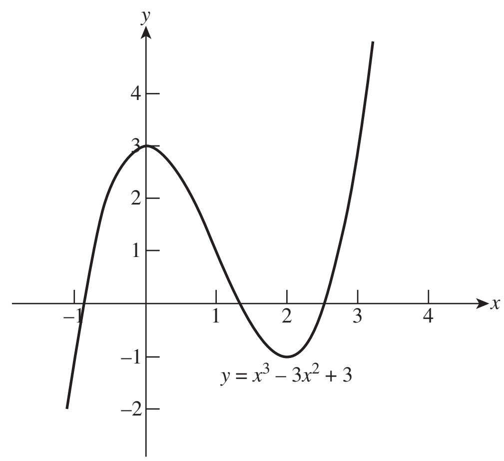

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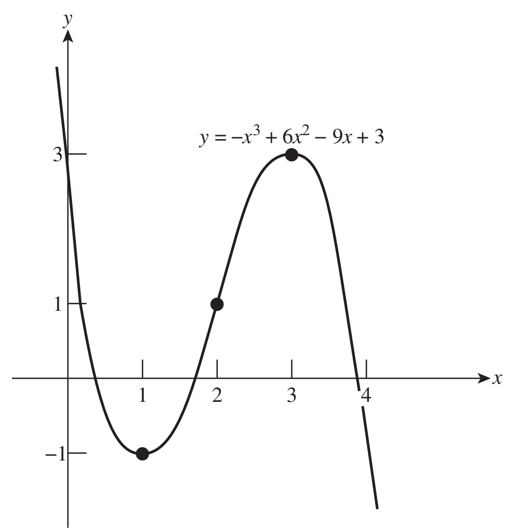

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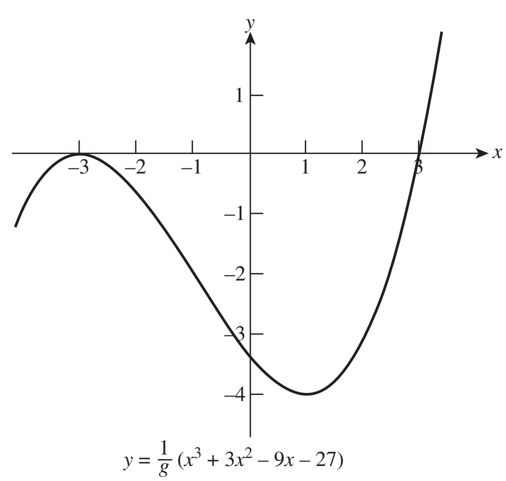

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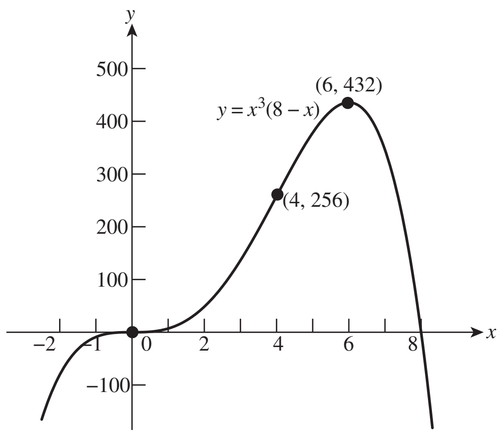

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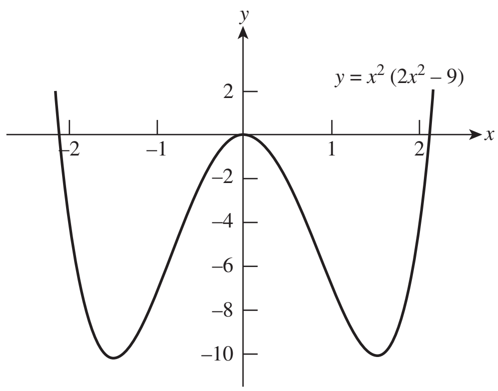

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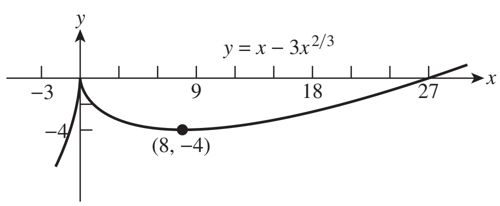
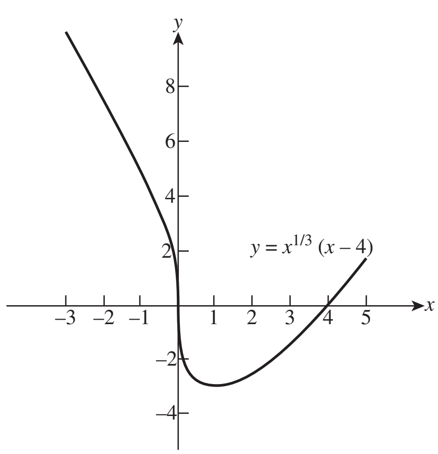


27.



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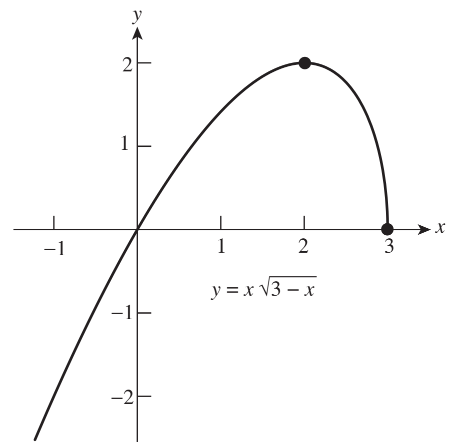

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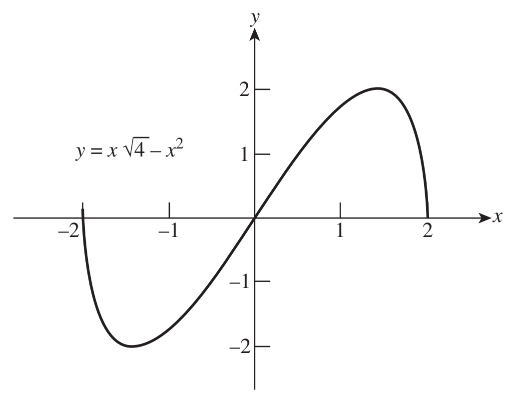
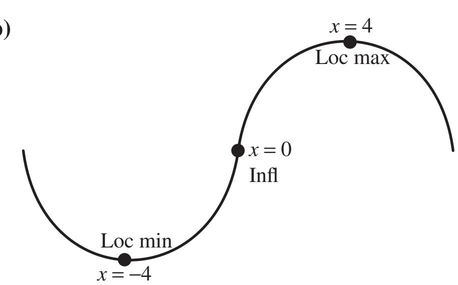


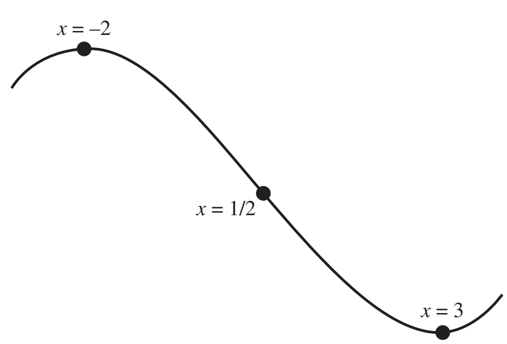
30. **(a)** a local maximum at $x = 4$, local minimum at $x = -4$, a point of inflection at $x = 0$

(b)



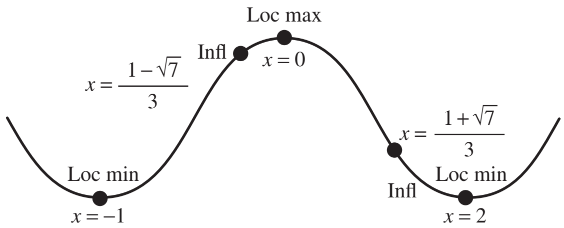
31. **(a)** local maximum at $x = -2$, local minimum at $x = 3$, a point of inflection at $x = \frac{1}{2}$

(b)



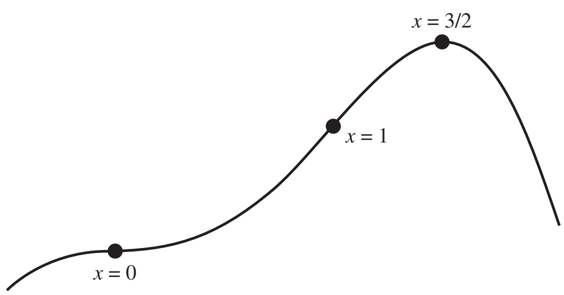
32. **(a)** a local maximum at $x = 0$, local minima at $x = -1$, inflection at $x = \dfrac{1 \pm \sqrt{7}}{3}$

(b)



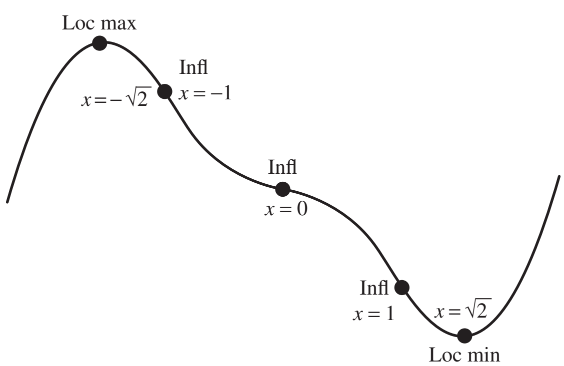
33. **(a)** a local maximum at $x = \dfrac{3}{2}$, inflection at $x = 0$ and $x = 1$

(b)



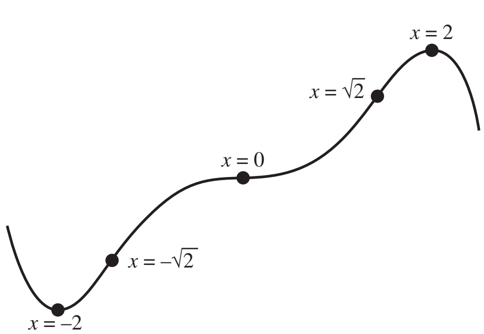
34. **(a)** a local maximum at $x = -\sqrt{2}$, local minimum at $x = \sqrt{2}$, points of inflection at $x = 0$ and $x = \pm 1$

(b)

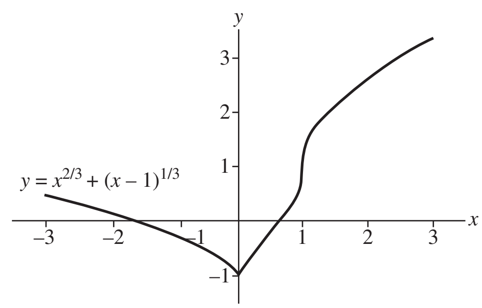


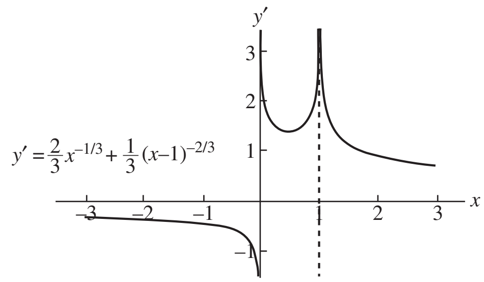
35. **(a)** Local maximum at $x = 2$, a local minimum at $x = -2$, points of inflection at $x = 0$ and $x = \pm\sqrt{2}$

(b)

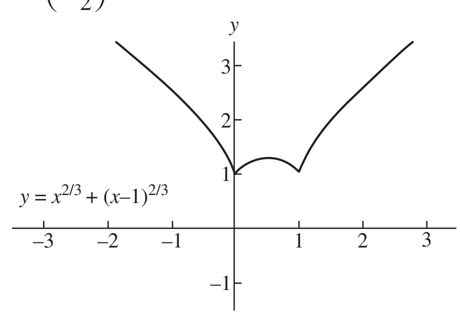


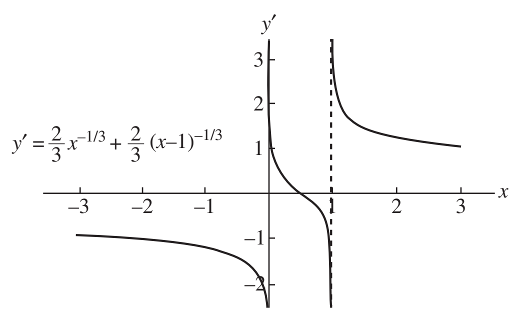
36. The values of first derivative indicate that the curve is rising on $(0, \infty)$ and falling on $(-\infty, 0)$.



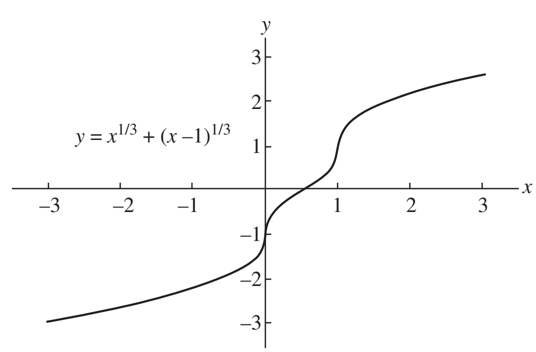


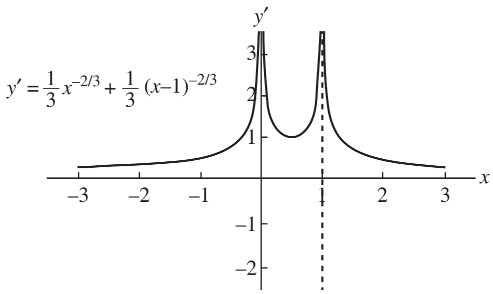
37. The values of the first derivative indicate that the curve is rising on $\left(0, \dfrac{1}{2}\right)$ and $(1, \infty)$, and falling on $(-\infty, 0)$ and $\left(\dfrac{1}{2}, 1\right)$



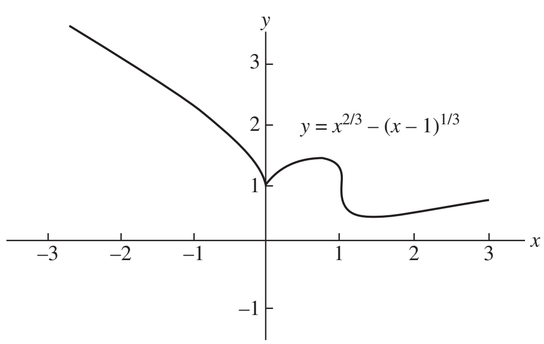


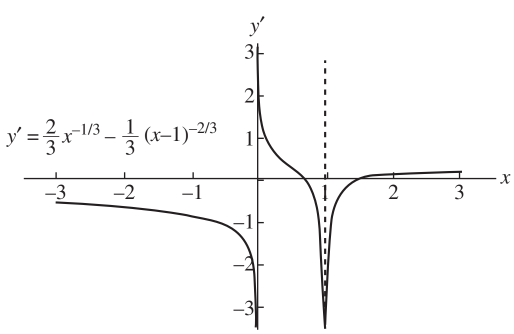
38. The values of the first derivative indicate that the curve is always rising.





39. The graph of the first derivative indicates that the curve is rising on $\left(0, \frac{17-\sqrt{33}}{16}\right)$ and $\left(\frac{17+\sqrt{33}}{16}, \infty\right)$, falling on $(-\infty, 0)$ and $\left(\frac{17-\sqrt{33}}{16}, \frac{17+\sqrt{33}}{16}\right) \Rightarrow$ a local maximum at $x = \frac{17-\sqrt{33}}{16}$, a local minimum at $x = \frac{17+\sqrt{33}}{16}$.





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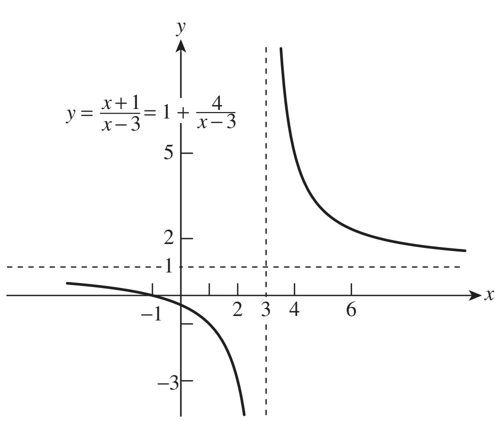

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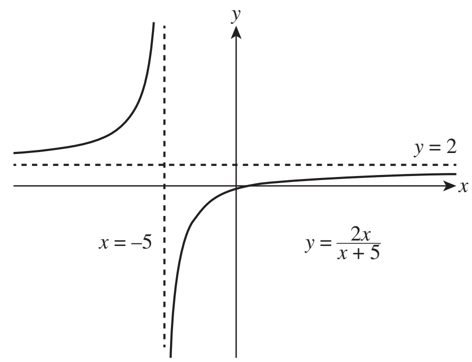

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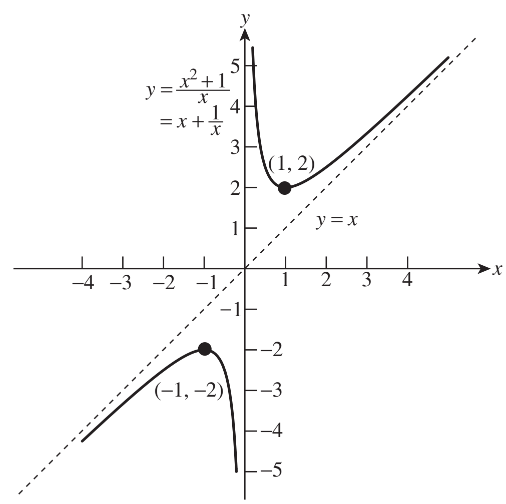

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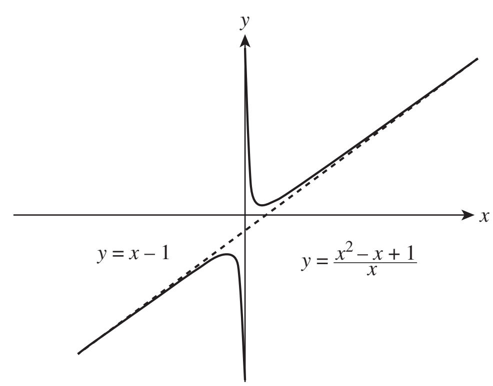

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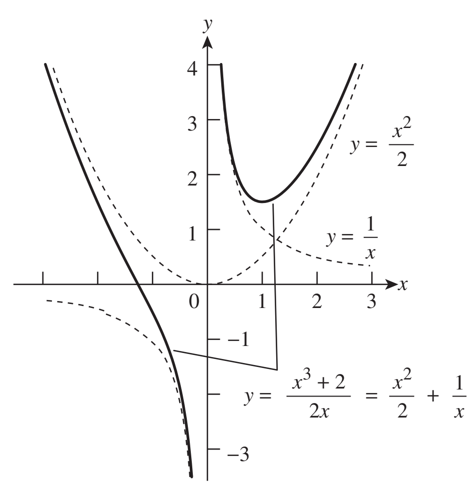

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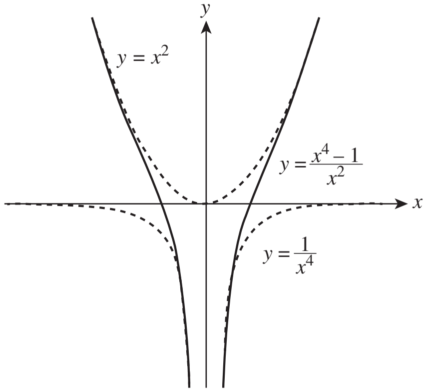

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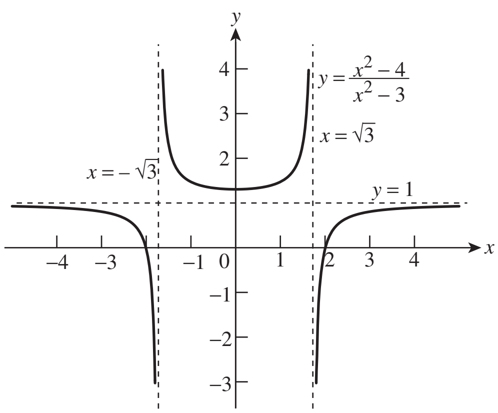

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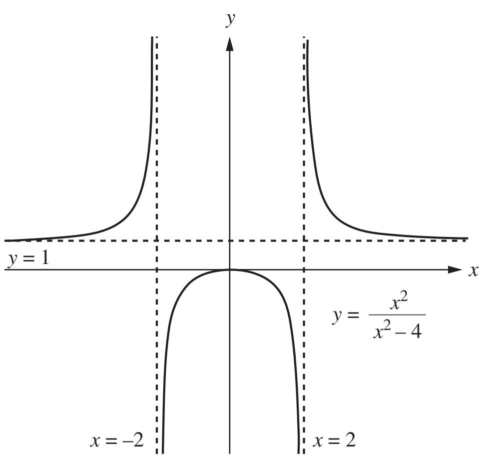


48. **(a)** the numbers are 0 and 36
(b) the numbers are 18 and 18.

49. **(a)** the numbers are $\frac{20}{3}$ and $\frac{40}{3}$.
(b) the numbers must be $\frac{79}{4}$ and $\frac{1}{4}$

50. $A(3) = 54$ sq units

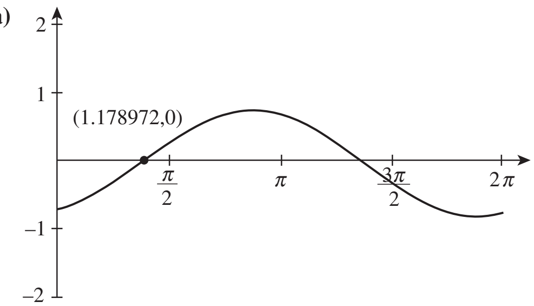
51. The dimensions 4 ft by 4 ft by 2 ft minimize the surface area.

52. The dimensions of the largest cylinder are radius $\sqrt{2}$ and height = 2

53. $r = 4$ and $h = 4$

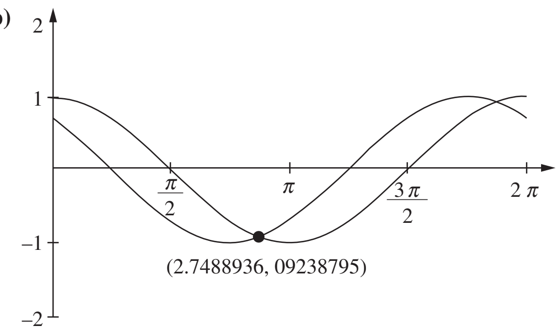
54. $x \approx 276$ tires and $y \approx 553$ tires

55. **(a)**



0.765 units

(b)



Particles collide when $t = \frac{7\pi}{8} \approx 2.749$

56. 19.7 ft

57. **(a)** $(4\pi r + 2\pi h)\frac{dr}{dt}$ **(b)** $+2\pi r\frac{dh}{dt}$

(c) $(4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dr}{dt}$ **(d)** $\frac{-r}{2r+h}\frac{dh}{dt}$

58. **(a)** $\left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt}$

(b) $\frac{\pi rh}{\sqrt{r^2+h^2}}\frac{dh}{dt}$

(c) $\left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2+h^2}}\frac{dh}{dt}$

59. $-40\ \text{m}^2/\text{sec}$

60. **(a)** $r = \frac{2}{5}h$ **(b)** $\frac{dh}{dt} = -\frac{125}{144\pi}$ ft min.

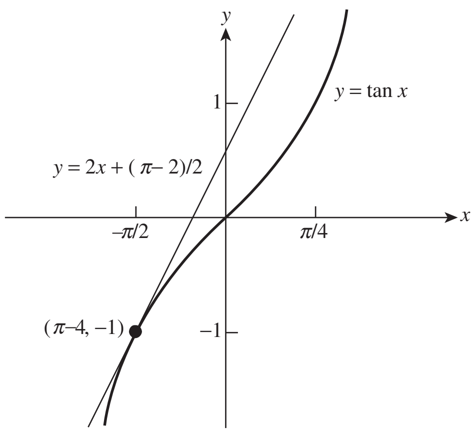
61. $\frac{d\theta}{dt} = 5$ rad/sec

62. **(a)** Speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point *A*.

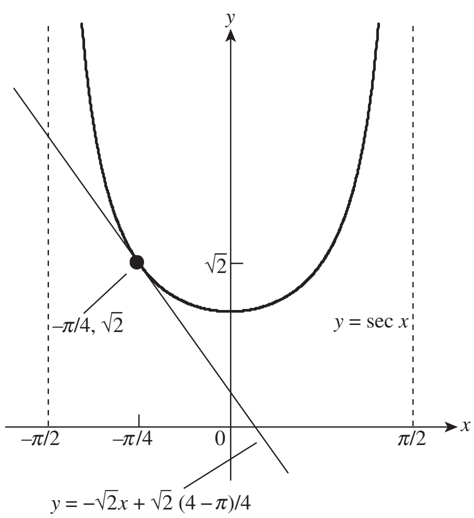
(b) $\frac{18}{\pi}$ revs/min

63. $\frac{r}{10\sqrt{3}}$ m/sec since $\frac{da}{dt}$ is positive, the distance *OA* is increasing when $OB = 2r$ and *B* is moving toward *O* at the rate of $0.3r$ m/sec.

64. **(a)** $2x + \dfrac{\pi - 2}{2}$



(b) $-\sqrt{2}x + \dfrac{\sqrt{2}(4-\pi)}{4}$



65. $f(x) = \dfrac{1}{1+\tan x} \Rightarrow f'(x) = \dfrac{-\sec^2 x}{(1+\tan x)^2}$. The linearization at $x = 0$ is $L(x) = f'(0)(x-0) + f(0) = 1 - x$

66. $L(x) = 1.5x + 0.5$, the linearization of $f(x)$

67. $2.5x - 0.1$, the linearization of $f(x)$

68. $dS = \dfrac{\pi\, r\, h_0 (dh)}{\sqrt{r^2 + h_0^2}}$

69. **(a)** The measurement of the edge r must have an error less than 1%
(b) 3%

Answers to Single Choice Questions

1. (c)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (c)	**8.** (d)	**9.** (d)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (d)	**26.** (c)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (a)
31. (a)	**32.** (d)	**33.** (d)	**34.** (d)	**35.** (c)	**36.** (a)	**37.** (a)	**38.** (c)	**39.** (a)	**40.** (b)
41. (d)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (b)	**47.** (a)	**48.** (b)	**49.** (a)	**50.** (b)
51. (d)	**52.** (c)	**53.** (a)	**54.** (a)	**55.** (d)	**56.** (d)	**57.** (b)	**58.** (c)	**59.** (a)	**60.** (c)

Answers to Multiple Choice Questions

1. (b), (c) and (d) **2.** (a) and (d) **3.** (a), (b), (c) and (d) **4.** (b), (c) and (d) **5.** (b) and (c)
6. (a), (c) and (d) **7.** (b), (c) and (d) **8.** (a), (c) and (d) **9.** (b), (c) and (d) **10.** (b) and (c)
11. (a), (c) and (d) **12.** (a) and (c) **13.** (a), (b) and (d) **14.** (a), (b) and (d) **15.** (b) and (c)
16. (a), (b), (c) and (d) **17.** (b), (c) and (d) **18.** (b) and (c) **19.** (a), (b), (c) and (d) **20.** (b) and (c)
21. (a), (b), (c) and (d) **22.** (a), (b) and (d) **23.** (b) and (d) **24.** (a), (c) and (d) **25.** (c) and (d)

Answers to Passage Type Questions

Passage 1
1. (c) **2.** (a) **3.** (b)
Passage 2
1. (c) **2.** (a) **3.** (b)
Passage 3
1. (b) **2.** (d) **3.** (d)
Passage 4
1. (d) **2.** (d) **3.** (c)
Passage 5
1. (a), (b) and (c) **2.** (a) and (d)
Passage 6
1. (a) **2.** (a) **3.** (c)
Passage 7
1. (b) **2.** (a) **3.** (a)

Answers to Matrix Match Type Questions

1. (a) → (r) (b) → (q), (r) and (s) (c) → (p) (d) → (s)
2. (a) → (s) (b) → (p) (c) → (q) (d) → (r)
3. (a) → (p), (q) and (r) (b) → (p) and (s) (c) → (r) and (s) (d) → (p) and (q)
4. (a) → (q) (b) → (t) (c) → (p), (q), (r), (s) and (t) (d) → (q)
5. (a) → (r) (b) → (p) (c) → (s) (d) → (q)

Answers to Integer Type Questions

1. 1 **2.** 9 **3.** 503 **4.** 729 **5.** 12 **6.** 7 **7.** 0 **8.** 2 **9.** 2 **10.** 2
11. 2 **12.** 8 **13.** 1 **14.** 35 **15.** 166 **16.** 100 **17.** 101 **18.** 13 **19.** 6 **20.** 100
21. 32 **22.** 49 **23.** 4 **24.** 4 **25.** 2

Answers to Additional and Advanced Exercises

1. If M and m are the maximum and minimum values, respectively, then $m \le f(x) \le M$ for all $x \in I$. If $m = M$ then f is constant on I.

2. No.

3. On an open interval the extreme values of a continuous function (if any) must occur at an interior critical point. On a half-open interval the extreme values of a continuous function may be at a critical point or at the closed endpoint. Extreme values occur only where $f' = 0$, f' does not exist, or at the endpoints of the interval. Thus the extreme points will not be at the ends of an open interval.

4. The pattern $f' = +++\underset{1}{|}----\underset{2}{|}----\underset{3}{|}++++\underset{4}{|}+++$ indicates a local maximum at $x = 1$ and a local minimum at $x = 3$.

5. **(a)** Local minimum at $x = -1$. f has points of inflection at $x = 0$ and $x = 2$. There is no local maximum.

(b) If $y' = 6x(x + 1)(x - 2)$, then $y' < 0$ for $x < -1$ and $0 < x < 2$; $y' > 0$ for $-1 < x < 0$ and $x > 2$. The sign pattern is

$y' = ---\underset{-1}{|}+++\underset{0}{|}---+\underset{2}{|}+++$. Therefore f has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 2$.

Also,

$$y'' = 18\left[x-\left(\frac{1-\sqrt{7}}{3}\right)\right]\left[x-\left(\frac{1+\sqrt{7}}{3}\right)\right],$$

so $y'' < 0$ for $\dfrac{1-\sqrt{7}}{3} < x < \dfrac{1+\sqrt{7}}{3}$ and $y'' > 0$ for all other $x \Rightarrow f$ has points of inflection at $x = \dfrac{1\pm\sqrt{7}}{3}$.

6. $f(6) - f(0) \le 12$ indicates the most that f can increase is 12.

8. **(b)** There exists $c \in (a, b)$ such that

$$\frac{c}{1+c^2} = \frac{f(b)-f(a)}{b-a} \Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| = \left|\frac{c}{1+c^2}\right| \le \frac{1}{2},$$

from part (a) $\Rightarrow |f(b)-f(a)| \le \frac{1}{2}|b-a|$.

9. No.

10. **(a)** $h(x)$ does have a local maximum at $x = a$. **(b)** No.

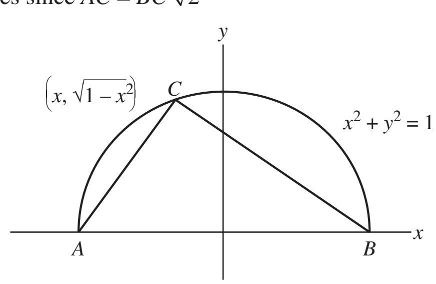
11. $a = 1$, $b = 0$, and $c = 1$

12. $k^2 = 9$ or $k = \pm 3$.

13. The area of the ΔABC is

$A(x) = \frac{1}{2}(2)\sqrt{1-x^2} = (1-x^2)^{1/2}$, where $0 \le x \le 1$.

Thus $A'(x) = \dfrac{-x}{\sqrt{l-x^2}} \Rightarrow 0$ and ± 1 are critical points. Also $A(\pm 1) = 0$ so $A(0) = 1$ is the maximum. When $x = 0$ the ΔABC is isosceles since $AC = BC\sqrt{2}$



15. The best place to drill the hole is at $y = \dfrac{h}{2}$.

16. $h = \sqrt{a(a+b)}$.

17. If H $\in$ (0, 2R], then the maximum surface area is at $r = R$.

If $H \in (2R, \infty)$, then the maximum is at $r = r^* = \dfrac{RH}{2(H-R)}$.

18. The smallest acceptable value for m is $\dfrac{1}{4}$.

19. **(a)** The profit function is maximized at $x = \dfrac{c-b}{2e}$.

(b) $\dfrac{c+b}{2}$ dollars.

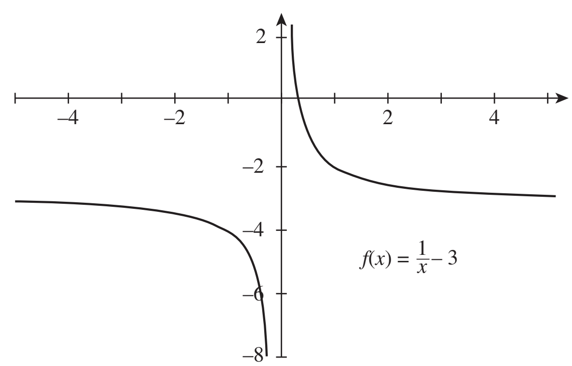
(c) The weekly profit at this production level is

$$P(x) = -e\left(\frac{c-b}{2e}\right)^2 + (c-b)\left(\frac{c-b}{2e}\right) - a\frac{(c-b)^2}{4e} - a.$$

(d) $\dfrac{c+b+t}{2}$ dollars.

Thus, such a tax increases the cost per unit by $\dfrac{c+b+t}{2} - \dfrac{c+b}{2} = \dfrac{t}{2}$ dollars if units are priced to maximize profit.

20.



(a) $x = \dfrac{1}{3}$.

21. $m_0 = \dfrac{q-1}{q}$ and $m_1 = \dfrac{1}{9}$.

In the case where $x_0 = \dfrac{a}{x_0^{q-1}}$ we have $x_0^q = a$; $x_1 = \dfrac{a}{x_0^q - 1}$.

22. $1 + \dfrac{dy}{dx} + y\dfrac{d^2y}{dx^2} - \left(\dfrac{x + y\dfrac{dy}{dx}}{1 + \dfrac{dy}{dx}}\right) = 0$.

23. **(a)** 38.72 ft/sec^2.

(b) 25 feet.

24. $h(10) = 5$.

25. Yes the curve $y = x$ satisfies all three conditions

26. $y = x^3 + 2x - 4$.

27. $v_0 = s'(0) = \dfrac{(4b)^{3/4}}{3} = \dfrac{2\sqrt{2}}{3}b^{3/4}$.

28. **(a)** $v(t) = \dfrac{2}{3}t^{3/2} - 2t^{1/2} + \dfrac{4}{3}$.

(b) $s(t) = \dfrac{4}{15}t^{5/2} - \dfrac{4}{3}t^{3/2} + \dfrac{4}{3}t - \dfrac{4}{15}$.

31. 40, $ 4.00

32. **(a)** 9% **(b)** Increasing at 10%

33. **(a)** $\dfrac{10}{52}$ **(b)** left: –10, Right: 10
(c) 0, 0 and 10, –10 **(d)** $t = \dfrac{\pi}{4}$, –10, 10, 0

34. **(a)** 2 sec, 64 ft/sec **(b)** 12.31 sec, 393.85 ft

35. $t = 1$ sec and $t = 2.5$ sec
(a) 0.8156 ft **(b)** 0.00613 sec

38. **(c)** Loses, 8.83 min/day

39. 11 hr

Answers to Exercises

Chapter 5

Exercises 5.1

1. **(a)** x^2 **(b)** $\frac{x^3}{3}$ **(c)** $\frac{x^3}{3} - x^2 + x$

2. **(a)** $3x^2$ **(b)** $\frac{x^8}{8}$ **(c)** $\frac{x^8}{8} - 3x^2 + 8x$

3. **(a)** x^{-3} **(b)** $-\frac{1}{3}x^{-3}$ **(c)** $-\frac{1}{3}x^{-3} + x^2 + 3x$

4. **(a)** $-x^2$ **(b)** $\frac{-x^{-2}}{4} + \frac{x^3}{3}$ **(c)** $\frac{x^{-2}}{2} + \frac{x^2}{2} - x$

5. **(a)** $-\frac{1}{x}$ **(b)** $-\frac{5}{x}$ **(c)** $2x + \frac{5}{x}$

6. **(a)** $\frac{1}{x^2}$ **(b)** $-\frac{1}{4x^2}$ **(c)** $\frac{x^4}{4} + \frac{1}{2x^2}$

7. **(a)** $\sqrt{x^3}$ **(b)** $\sqrt{x}$ **(c)** $\frac{2\sqrt{x^3}}{3} + 2\sqrt{x}$

8. **(a)** $x^{\frac{4}{3}}$ **(b)** $\frac{1}{2}x^{\frac{2}{3}}$ **(c)** $\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}}$

9. **(a)** $x^{2/3}$ **(b)** $x^{1/3}$ **(c)** $x^{-1/3}$

10. **(a)** $\frac{1}{\sqrt{3}+1}x^{\sqrt{3}+1}$ **(b)** $\frac{1}{\pi+1}x^{\pi+1}$ **(c)** $\frac{1}{\sqrt{2}}x^{\sqrt{2}}$

11. **(a)** $\cos(\pi x)$ **(b)** $-3\cos x$ **(c)** $-\frac{1}{\pi}\cos(\pi x) + \cos(3x)$

12. **(a)** $\sin(\pi x)$ **(b)** $\sin\left(\frac{\pi x}{2}\right)$ **(c)** $\left(\frac{2}{\pi}\right)\sin\left(\frac{\pi x}{2}\right) + \pi \sin x$

13. **(a)** $\frac{1}{2}\tan x$ **(b)** $2\tan\left(\frac{x}{3}\right)$ **(c)** $-\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

14. **(a)** $-\cot x$ **(b)** $\cot\left(\frac{3x}{2}\right)$ **(c)** $x + 4\cot(2x)$

15. **(a)** $-\csc x$ **(b)** $\frac{1}{5}\csc(5x)$ **(c)** $2\csc\left(\frac{\pi x}{2}\right)$

16. **(a)** $\sec x$ **(b)** $\frac{4}{3}\sec(3x)$ **(c)** $\frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

17. $\frac{x^2}{2} + x + C$ **18.** $5x - 3x^2 + C$

19. $t^3 + \frac{t^2}{4} + C$ **20.** $\frac{t^3}{6} + t^4 + C$

21. $\frac{x^4}{2} - \frac{5x^2}{2} + 7x + C$ **22.** $x - \frac{1}{x}x^3 - \frac{1}{2}x^6 + C$

23. $-\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$ **24.** $\frac{x}{5} + \frac{1}{x^2} + x^2 + C$

25. $\frac{3}{2}x^{2/3} + C$ **26.** $\frac{-4}{\sqrt[4]{x}} + C$

27. $\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$ **28.** $\frac{1}{3}x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + C$

29. $4y^2 - \frac{8}{3}y^{3/4} + C$ **30.** $\frac{y}{7} + \frac{4}{y^{\frac{1}{4}}} + C$

31. $x^2 + \frac{2}{x} + C$ **32.** $-\frac{1}{x} - \frac{1}{2x^2} + C$

33. $2\sqrt{t} - \frac{2}{\sqrt{t}} + C$ **34.** $\frac{-2}{t^2} - \frac{2}{3t^{\frac{3}{2}}} + C$

35. $-2\sin t + C$ **36.** $5\cos t + C$

37. $-21\cos\frac{\theta}{3} + C$ **38.** $\frac{3}{5}\sin 5\theta + C$

39. $3\cot x + C$ **40.** $-\frac{\tan x}{3} + C$

41. $-\frac{1}{2}\csc\theta + C$ **42.** $\frac{2}{5}\sec\theta + C$

43. $4\sec x - 2\tan x + C$

44. $-\frac{1}{2}\cot x + \frac{1}{2}\csc x + C$

45. $-\frac{1}{2}\cos 2x + \cot x + C$

46. $\sin 2x + \cos 3x + C$

47. $\frac{t}{2} + \frac{\sin 4t}{8} + C$ **48.** $\frac{t}{2} - \frac{\sin 6t}{12} + C$

49. $\frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$ **50.** $\frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

51. $\tan\theta + C$ **52.** $\theta + \tan\theta + C$

53. $-\cot x - x + C$ **54.** $2x + \cot x + C$

55. $-\cos\theta + \theta + C$ **56.** $\tan\theta + C$

63. **(a)** Wrong: $\frac{d}{dx}\left(\frac{x^2}{2}\sin x + C\right) = \frac{2x}{2}\sin x + \frac{x^2}{2}\cos x = x\sin x + \frac{x^2}{2}\cos x$

(b) Wrong: $\frac{d}{dx}(-x\cos x + C) = -\cos x + x\sin x$

(c) Right: $\frac{d}{dx}(-x\cos x + \sin x + C) = -\cos x + x\sin x + \cos x = x\sin x$

64. **(a)** Wrong **(b)** Right **(c)** Right

65. **(a)** Wrong: $\frac{d}{dx}\left(\frac{(2x+1)^3}{3} + C\right) = \frac{3(2x+1)^2(2)}{3} = 2(2x+1)^2$

(b) Wrong: $\frac{d}{dx}((2x+1)^3 + C) = 3(2x+1)^2(2) = 6(2x+1)^2$

(c) Right: $\frac{d}{dx}((2x+1)^3 + C) = 6(2x+1)^2$

66. **(a)** Wrong **(b)** Wrong **(c)** Right
67. Right
68. Wrong
69. (b)
70. Graph (b)
71. $y = x^2 - 7x + 10$
72. $y = 10x - \frac{x^2}{2} - 1$
73. $y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$
74. $y = 3x^3 - 2x^2 + 5x + 10$
75. $y = 9x^{1/3} + 4$
76. $y = x^{1/2} - 2$
77. $s = t + \sin t + 4$
78. $S = \sin t - \cos t$
79. $r = \cos(\pi\theta) - 1$
80. $r = \frac{1}{\pi}\sin(\pi\theta) + 1$
81. $v = \frac{1}{2}\sec t + \frac{1}{2}$
82. $v = 4t^2 - \cot t - 7 - \pi^2$
83. $y = x^2 - x^3 + 4x + 1$
84. $y = 2x$
85. $r = \frac{1}{t} + 2t - 2$
86. $s = \frac{t^3}{16}$
87. $y = x^3 - 4x^2 + 5$
88. $\theta = -t^2 - \frac{1}{2}t + \sqrt{2}$
89. $y = -\sin t + \cos t + t^3 - 1$
90. $y = -\cos x + \frac{1}{2}\sin(2x) + \frac{2}{-3}x^3 + 4$
91. $y = 2x^{3/2} - 50$
92. **(a)** $y = x^3 + 1$ **(b)** One
93. $y = x - x^{4/3} + \frac{1}{2}$
94. $y = \frac{x^2}{2} - x - \frac{1}{2}$
95. $y = -\sin x - \cos x - 2$
96. $y = \sqrt{x} - \cos \pi x$
97. **(a)** **(i)** 33.2 units, **(ii)** 33.2 units, **(iii)** 33.2 units **(b)** True
98. 1200 m/sec.
99. $t = 88/k, k = 16$
100. 21.5 ft/sec^2
101. **(a)** $v = 10t^{3/2} - 6t^{1/2}$ **(b)** $s = 4t^{5/2} - 4t^{3/2}$
102. 1.24 Sec.
104. $S = -\frac{1}{2}gt^2 + v_0 t + s_0$
105. **(a)** $-\sqrt{x} + C$ **(b)** $x + C$ **(c)** $\sqrt{x} + C$
(d) $-x + C$ **(e)** $x - \sqrt{x} + C$ **(f)** $-x - \sqrt{x} + C$
106. Yes

Exercises 5.2

1. **(a)** 0.125 **(b)** 0.21875 **(c)** 0.625 **(d)** 0.46875
2. **(a)** $\frac{1}{16}$ **(b)** $\frac{9}{64}$ **(c)** $\frac{9}{16}$ **(d)** $\frac{25}{64}$
3. **(a)** 1.066667 **(b)** 1.283333
(c) 2.666667 **(d)** 2.083333
4. **(a)** 0 **(b)** 6 **(c)** 16 **(d)** 14
5. 0.3125, 0.328125
6. $\frac{31}{128}$
7. 1.5, 1.574603
8. 11
9. **(a)** 87 in. **(b)** 87 in.
10. 4920 meters.
11. **(a)** 3490 ft. **(b)** 3840 ft.
12. **(a)** 0.967 miles **(b)** 120 mi/hr.
13. [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]
15. [0, 0.5], [0.5, 1], [1, 1.5] and [1.5, 2]
17. **(a)** upper estimate = 758 gal
lower estimate = 543 gal
(b) upper estimate = 2363 gal
lower estimate = 1693 gal
(c) worst case $t \approx 31.4$ hrs.
best case $t \approx 32.4$ hrs.
18. **(a)** upper = 60.9 tons.
lower = 46.8 tons
(b) lower (best case) = 126.6 tons.
19. **(a)** 2 **(b)** 2.828 **(c)** $\frac{\pi}{16}$

Exercises 5.3

1. 7
2. $\frac{7}{6}$
3. 0
4. 0
5. $\left(\sqrt{3} - \frac{2}{2}\right)$
6. 4
7. (a), (b) and (c)
8. (a) and (b)
9. (b)
10. (a) and (c)
11. $\sum_{K=1}^{6} K$
12. $\sum_{K=1}^{4} K^2$
13. $\sum_{K=1}^{4} \frac{1}{2^K}$
14. $\sum_{K=1}^{5} 2K$
15. $\sum_{K=1}^{5} \frac{(-1)^{K+1}}{K}$
16. $\sum_{K=1}^{5} (-1)^k \frac{K}{5}$
17. **(a)** –15 **(b)** 1 **(c)** 1
(d) –11 **(e)** 16
18. **(a)** 0 **(b)** 250 **(c)** n **(d)** $1–n$
19. **(a)** 55 **(b)** 385 **(c)** 3025
20. **(a)** 91 **(b)** 819 **(c)** 8281
21. –56
22. π
23. –73
24. 61
25. 240
26. 308
27. 3376
28. 588
29. **(a)** 21 **(b)** 3500 **(c)** 2610
30. **(a)** 630 **(b)** 1780 **(c)** 117648
31. **(a)** $4n$ **(b)** cn **(c)** $\frac{n^2 - n}{2}$
32. **(a)** $1 + 2n^2$ **(b)** c **(c)** $\frac{n+1}{2n}$

Exercises 5.5

1. $\frac{1}{6}(2x + 4)^6 + C$
2. $\frac{2}{3}(7x - 1)^{\frac{3}{2}} + C$
3. $-\frac{1}{3}(x^2 + 5)^{-3} + C$
4. $\frac{-1}{x^4 + 1} + C$
5. $\frac{1}{10}(3x^2 + 4x)^5 + C$
6. $\frac{3}{2}\left(1 + \sqrt{x}\right)^{\frac{4}{3}} + C$

7. $-\frac{1}{3}\cos 3x + C$ **8.** $-\frac{1}{4}\cos 2x^2 + C$

9. $\frac{1}{2}\sec 2t + C$ **10.** $\frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3 + C$

11. $-6(1 - r^3)^{1/2} + C$ **12.** $\left(y^4+4y^2+1\right)^3 + C$

13. $\frac{1}{3}(x^{3/2} - 1) - \frac{1}{6}\sin(2x^{3/2} - 2) + C$

14. $-\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$

15. **(a)** $-\frac{1}{4}(\cot^2 2\theta) + C$ **(b)** $-\frac{1}{4}(\csc^2 2\theta) + C$

16. **(a)** $\frac{2}{5}\sqrt{5x+8} + C$ **(b)** $\frac{2}{5}\sqrt{5x+8} + C$

17. $-\frac{1}{3}(3 - 2s)^{3/2} + C$ **18.** $\frac{2}{5}\sqrt{5s+4} + C$

19. $-\frac{2}{5}(1 - \theta^2)^{5/4} + C$ **20.** $-\frac{1}{3}\left(7-3y^2\right)^{\frac{3}{2}} + C$

21. $\left(-2/\left(1 + \sqrt{x}\right)\right) + C$ **22.** $\frac{2}{3}\sin^{\frac{3}{2}} x - \frac{2}{7}\sin^{\frac{7}{2}} x + C$

23. $\frac{1}{3}\tan(3x + 2) + C$ **24.** $\frac{1}{3}\tan^3 x + C$

25. $\frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$ **26.** $\frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$

27. $\left(\frac{r^3}{18} - 1\right)^6 + C$ **28.** $-\frac{1}{2}\left(7-\frac{r^5}{10}\right)^4 + C$

29. $-\frac{2}{3}\cos(x^{3/2} + 1) + C$ **30.** $-2\csc\left(\frac{v-\pi}{2}\right) + C$

31. $\frac{1}{2\cos(2t + 1)} + C$ **32.** $2\sqrt{\sec z} + C$

33. $-\sin\left(\frac{1}{t} - 1\right) + C$ **34.** $2\sin\left(\sqrt{t}+3\right) + C$

35. $-\frac{\sin^2(1/\theta)}{2} + C$ **36.** $\frac{-2}{\sin\sqrt{\theta}} + C$

37. $\frac{2}{3}(1 + x)^{3/2} - 2(1 + x)^{1/2} + C$

38. $\frac{2}{3}\left(1-\frac{1}{x}\right)^{\frac{3}{2}} + C$ **39.** $\frac{2}{3}\left(2 - \frac{1}{x}\right)^{3/2} + C$

40. $\frac{1}{3}\left(1-\frac{1}{x^2}\right)^{\frac{3}{2}} + C$ **41.** $\frac{2}{27}\left(1 - \frac{3}{x^3}\right)^{3/2} + C$

42. $\frac{2}{3}\left(x^3-1\right)^{\frac{3}{2}} + C$

43. $\frac{1}{12}(x - 1)^{12} + \frac{1}{11}(x - 1)^{11} + C$

44. $\frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{8}{3}(4-x)^{\frac{3}{2}} + C$

45. $-\frac{1}{8}(1 - x)^8 + \frac{4}{7}(1 - x)^7 - \frac{2}{3}(1 - x)^6 + C$

46. $\frac{3}{7}(x-5)^{\frac{7}{3}} + \frac{15}{2}(x-5)^{\frac{4}{3}} + C$

47. $\frac{1}{5}(x^2 + 1)^{5/2} - \frac{1}{3}(x^2 + 1)^{3/2} + C$

48. $\frac{2}{5}(x^3+1)^{\frac{5}{2}} - \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} + C$ **49.** $\frac{-1}{4(x^2 - 4)^2} + C$

50. $\frac{3}{16}(2x-1)^{\frac{4}{3}} + \frac{3}{4}(2x-1)^{\frac{1}{3}} + C$

51. **(a)** $-\frac{6}{2 + \tan^3 x} + C$ **(b)** $-\frac{6}{2 + \tan^3 x} + C$

(c) $-\frac{6}{2 + \tan^3 x} + C$

52. **(a)** $\frac{1}{3}\left(1+\sin^2(x-1)\right)^{\frac{3}{2}} + C$ **(b)** $\frac{1}{3}\left(1+\sin^2(x-1)\right)^{\frac{3}{2}} + C$

(c) $\frac{1}{3}\left(1+\sin^2(x-1)\right)^{\frac{3}{2}} + C$

53. $\frac{1}{6}\sin\sqrt{3(2r - 1)^2 + 6} + C$ **54.** $\frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$

55. $s = \frac{1}{2}(3t^2 - 1)^4 - 5$ **56.** $3\left(x^2+8\right)^{\frac{2}{3}} - 12$

57. $s = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

58. $r = \frac{3}{2}\theta - \frac{3}{4}\cos 2\theta + \frac{\pi}{8} + \frac{3}{4}$

59. $s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$

60. $y = \frac{1}{2}\tan 2x + 3x - 1$

61. 6 m **62.** 10 m.

Exercises 5.6

1. $\ln 5$ **2.** $x - \tan^{-1} x + C$

3. $2\tan x - 2\sec x - x + C$ **4.** $\frac{1}{2}\ln 3$

5. $\sin^{-1} x + \sqrt{1-x^2} + C$ **6.** $2\ln\left|\sqrt{x}-1\right| + C$

7. $e^{-\cot z} + C$ **8.** $\frac{2^{\ln z^3}}{48\ln 2} + C$

9. $\tan^{-1}\left(e^z\right) + C$ **10.** 2π

11. π **12.** $-4 + 2\ln 3$

13. $t + \cot t + \operatorname{cosec} t + C$ **14.** $\sin 2t + t + C$

15. $\sqrt{2}$ **16.** $\sin^{-1}(\theta - 1) + C$

17. $\frac{1}{8}\ln\left(1+4\ln^2 y\right) + C$ **18.** $\frac{1}{\ln 2}2^{\sqrt{y}} + c$

19. $\ln\left|1+\sin\theta\right| + C$ **20.** $\frac{-1}{\sqrt{3}}\operatorname{csc}h^{-1}\left|\frac{t}{\sqrt{3}}\right| + c$

21. $2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C$ **22.** $\sqrt{x-1} + \ln|x| + c$

23. $2\left(\sqrt{2}-1\right) \simeq 0.82843$

24. $\tan t - 2\ln\left|\operatorname{cosec} t + \cot t\right|\left| - \cot t - t + c\right|$

25. $\sec^{-1}\left(e^y\right) + C$ **26.** $12\tan^{-1}\sqrt{y} + C$

27. $\sin^{-1}\left(2\ln x\right) + C$ **28.** $\sec^{-1}|x-2| + C$

29. $\ln|\sin x| + \ln|\cos x| + C$ **30.** $6\cosh\left(\frac{x}{2} + \ln 5\right) + C$

31. $7 + \ln 8$ **32.** 0

33. $\left(\sin^{-1} y - \sqrt{1-y^2}\right)_{-1}^{0} = \frac{\pi}{2} - 1$ **34.** $e^{e^z} + C$

35. $\sec^{-1}\left|\frac{x-1}{7}\right|+C$

36. $\frac{1}{2}\sec^{-1}|2x+1|+C$

37. $\frac{\theta^3}{3}-\frac{\theta^2}{2}+\theta+\frac{5}{2}\ln|2\theta-5|+C$

38. $\cot\theta+\csc\theta+C$

39. $x-\ln(1+e^x)+C$

40. $\frac{2}{3}\tan^{-1}x^{\frac{3}{2}}+C$

41. $xe^{x^3}+C$

42. $\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+C$

43. $\frac{1}{30}\left(x^4+1\right)^{\frac{3}{2}}\left(3x^4-2\right)+C$

44. $\frac{3}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}+C$

Exercises 5.7

1. $-2x\cos(x/2)+4\sin(x/2)+C$

2. $\frac{\theta}{\pi}\sin\pi\theta+\frac{1}{\pi^2}\cos\pi\theta+c$

3. $t^2\sin t+2t\cos t-2\sin t+C$

4. $-x^2\cos x+2x\sin x+2\cos x+C$

5. $\ln 4-\frac{3}{4}$

6. $\frac{3e^4+1}{16}$

7. xe^x-e^x+C

8. $\frac{x}{3}e^{3x}-\frac{1}{9}e^{3x}+c$

9. $-(x^2+2x+2)e^{-x}+C$

10. $\left(\frac{1}{2}x^2-\frac{3}{2}x+\frac{5}{4}\right)e^{2x}+c$

11. $y\tan^{-1}(y)-\ln\sqrt{1+y^2}+C$

12. $y\sin^{-1}y+\sqrt{1-y^2}+C$

13. $x\tan x+\ln|\cos x|+C$

14. $2x\tan 2x-\ln|\sec 2x|+C$

15. $\left(x^3-3x^2+6x-6\right)e^x+C$

16. $(-P^4-4P^3-12P^2-24P-24)e^{-p}+C$

17. $\left(x^2-7x+7\right)e^x+C$

18. $(r^2-r+2)e^r+c$

19. $\left(x^5-5x^4+20x^3-60x^2+120x-120\right)e^x+C$

20. $\left(\frac{t^2}{4}-\frac{t}{8}+\frac{1}{32}\right)e^{4t}+c$

21. $\frac{1}{2}\left(-e^\theta\cos\theta+e^\theta\sin\theta\right)+C$

22. $\frac{1}{2}(e^{-y}\sin y-e^{-y}\cos y)+C$ $\left(\text{where } c=\frac{c'}{2}\right)$

23. $\frac{e^{2x}}{13}\left(3\sin 3x+2\cos 3x\right)+C$

24. $\frac{-e^{-2x}}{4}(\sin 2x+\cos 2x)+C$ $\left(\text{where } c=\frac{c'}{2}\right)$

25. $\frac{2}{3}\left(\sqrt{3s+9}e^{\sqrt{3s+9}}-e^{\sqrt{3s+9}}\right)+C$

26. $\frac{4}{15}$

27. $\frac{\pi\sqrt{3}}{3}-\ln(2)-\frac{\pi^2}{18}$

28. $x\ln(x+x^2)-2x+\ln|x+1|+C$

29. $\frac{1}{2}\left[-x\cos(\ln x)+x\sin(\ln x)\right]+C$

30. $\frac{2^2}{4}[2(\ln 2)^2-2\ln z+1]+c$

31. $\frac{1}{2}\ln\left|\sec x^2+\tan x^2\right|+C$

32. $2\sin\sqrt{x}+C$

33. $\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C$

34. $\frac{-1}{\ln x}+C$

35. $-\frac{1}{x}\ln x-\frac{1}{x}+C$

36. $\frac{1}{4}(\ln x)^4+C$

37. $\frac{1}{4}e^{x^4}+C$

38. $\frac{1}{3}x^3e^{x^3}-\frac{1}{3}e^{x^3}+C$

39. $\frac{1}{3}x^2\left(x^2+1\right)^{\frac{3}{2}}-\frac{2}{15}\left(x^2+1\right)^{\frac{5}{2}}+C$

40. $-\frac{1}{3}\cos x^3+C$

41. $-\frac{2}{5}\sin 3x\sin 2x-\frac{3}{5}\cos 3x\cos 2x+C$

42. $\frac{1}{3}\sin 2x\sin 4x+\frac{1}{6}\cos 2x\cos 4x+C$

43. $\frac{2}{9}x^{\frac{3}{2}}(3\ln x-2)+C$

44. $2e^{\sqrt{x}}+C$

45. $2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}+C$

46. $2xe^{\sqrt{x}}-4\sqrt{x}e^{\sqrt{x}}+4e^{\sqrt{x}}+C$

47. $\frac{\pi^2-4}{8}$

48. $\frac{3\left(4-\pi^2\right)}{16}$

49. $\frac{5\pi-3\sqrt{3}}{9}$

50. $\frac{\pi+6\sqrt{3}-12}{12}$

51. $\frac{1}{2}\left(x^2+1\right)\tan^{-1}x-\frac{x}{2}+C$

52. $\frac{x^3}{3}\tan^{-1}\frac{x}{2}-\frac{1}{3}x^2+\frac{4}{3}In\left(1+\frac{x^2}{4}\right)+C$

53. $u=x^n, dv=\cos x dx$

54. $-x^n\cos x+\int nx^{n-1}\cos x dx$

55. $u=x^n,\ dv=e^{ax}dx$

56. $I=x(Inx)^n-\int n(Inx)^{n-1}dx$

57. $x\sin^{-1}x+\cos\left(\sin^{-1}x\right)+C$

60. $\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx+C$

61. $x\sin^{-1}x+\cos(\sin^{-1}x)+C$

62. $x\tan^{-1}x+\ln\left|\cos\left(\tan^{-1}x\right)\right|+C$

63. $x\sec^{-1}x-\ln\left|x+\sqrt{x^2-1}\right|+C$

64. $x\log_2 x-\frac{x}{\ln 2}+C$

65. Yes

66. Yes

Exercises 5.8

1. $\frac{1}{2}\sin 2x+C$

2. $\frac{9}{2}$

3. $-\frac{1}{4}\cos^4 x+C$

4. $\frac{1}{10}\sin^5 2x+C$

5. $\frac{1}{3}\cos^3 x-\cos x+C$

6. $\frac{1}{4}\sin 4x-\frac{1}{12}\sin^3 4x+C$

7. $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$

8. $\frac{16}{15}$

9. $\sin x-\frac{1}{3}\sin^3 x+C$

10. $\frac{8}{15}$

11. $\frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+C$

12. $\frac{1}{12}\sin^6 2x-\frac{1}{16}\sin^8 2x+C$

13. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

14. $\frac{\pi}{4}$

15. $\frac{16}{35}$

16. $7\sin t-7\sin^3 t+\frac{21}{5}\sin^5 t-\sin^7 t+C$

17. 3π

18. $3x+\frac{1}{\pi}\sin 4\pi x+\frac{1}{8\pi}\sin 8\pi x+C$

19. $-4\sin x\cos^3 x+2\cos x\sin x+2x+C$

20. $\frac{\pi}{2}$

21. $-\cos^4 2\theta+C$

22. 0

23. 4

24. $2\sqrt{2}$

25. 2

26. 2

27. $\sqrt{\frac{3}{2}}-\frac{2}{3}$

28. $2-\sqrt{2}$

29. $\frac{4}{5}\left(\frac{3}{2}\right)^{\frac{5}{2}}-\frac{18}{35}-\frac{2}{7}\left(\frac{3}{2}\right)^{\frac{7}{2}}$

30. $\frac{\sqrt{2}-1}{\sqrt{2}}$

31. $\sqrt{2}$

32. $\frac{8}{3}$

33. $\frac{1}{2}\tan^2 x+C$

34. $\frac{1}{2}\sec x\tan x-\frac{1}{2}\ln|\sec x+\tan x|+C$

35. $\frac{1}{3}\sec^3 x+C$

36. $\frac{1}{5}\sec^5 x-\frac{1}{3}\sec^3 x+C$

37. $\frac{1}{3}\tan^3 x+C$

38. $\frac{1}{5}\tan^5 x+\frac{1}{3}\tan^3 x+C$

39. $2\sqrt{3}+\ln\left(2+\sqrt{3}\right)$

40. $\frac{1}{2}\left[\sec\left(e^x\right)\tan\left(e^x\right)+In\left|\sec\left(e^x\right)+\tan\left(e^x\right)\right|\right]+C$

41. $\frac{2}{3}\tan\theta+\frac{1}{3}\sec^2\theta\tan\theta+C$

42. $\tan(3x)+\frac{1}{3}\tan^3(3x)+C$

43. $\frac{4}{3}$

44. $\frac{1}{5}\tan^5 x+\frac{2}{3}\tan^3 x+\tan x+C$

45. $2\tan^2 x-2\ln\left(1+\tan^2 x\right)+C$

46. $3\pi-8$

47. $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+C$

48. $-\frac{1}{10}\cot^5 2x+\frac{1}{6}\cot^3 2x-\frac{1}{2}\cot 2x-x+C$

49. $\frac{4}{3}-\ln\sqrt{3}$

50. $-\frac{8}{3}\cot^3 t+8\cot t+8t+C$

51. $-\frac{1}{10}\cos 5x-\frac{1}{2}\cos x+C$

52. $\frac{1}{2}\cos x-\frac{1}{10}\cos 5x+C$

53. π

54. $\frac{1}{2}$

55. $\frac{1}{2}\sin x+\frac{1}{14}\sin 7x+C$

56. 0

57. $\frac{1}{6}\sin 3\theta-\frac{1}{4}\sin\theta-\frac{1}{20}\sin 5\theta+C$

58. $-\frac{4}{5}\cos^5\theta+\frac{4}{3}\cos^3\theta-\cos\theta+C$

59. $-\frac{2}{5}\cos^5\theta+C$

60. $\frac{2}{5}\cos^5\theta-\cos^3\theta+\cos\theta+C$

61. $\frac{1}{4}\cos\theta-\frac{1}{20}\cos 5\theta+C$

62. $-\frac{1}{8}\cos 2\theta-\frac{1}{16}\cos 4\theta+\frac{1}{24}\cos 6\theta+C$

63. $\sec x-\ln|\csc x+\cot x|+C$

64. $\frac{1}{3}\sec^3 x-\sec x+C$

65. $\cos x+\sec x+C$

66. $-\ln|\csc 2x+\cot 2x|+C$

67. $\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$

68. $x\sin x-\frac{1}{3}x\sin^3 x+\frac{2}{3}\cos x+\frac{1}{9}\cos^3 x+C$

Exercises 5.9

1. $\ln\left|\sqrt{9+x^2}+x\right|+C$

2. $\ln\left|\sqrt{1+9x^2}+3x\right|+C$

3. $\frac{\pi}{4}$

4. $\frac{\pi}{16}$

5. $\frac{\pi}{6}$

6. $\frac{\pi}{4}$

7. $\frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right)+\frac{t\sqrt{25-t^2}}{2}+C$

8. $\frac{1}{6}\left[\sin^{-1}(3t)+3t\sqrt{1-9t^2}\right]+C$

9. $\frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right|+C$

10. $\ln\left|\frac{5x}{3}+\frac{\sqrt{25x^2-9}}{3}\right|+C$

11. $7\left[\frac{\sqrt{y^2-49}}{7}-\sec^{-1}\left(\frac{y}{7}\right)\right]+C$

12. $\left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10}-\frac{\sqrt{y^2-25}}{2y^2}\right]+C$

13. $\frac{\sqrt{x^2-1}}{x}+C$

14. $\sec^{-1} x+\frac{\sqrt{x^2-1}}{x^2}+C$

15. $-\sqrt{9-x^2}+C$

16. $x-2\tan^{-1}\left(\frac{x}{2}\right)+C$

17. $\frac{1}{3}\left(x^2+4\right)^{\frac{3}{2}}-4\sqrt{x^2+4}+C$

18. $\frac{-\sqrt{x^2+1}}{x}+C$

19. $\frac{-2\sqrt{4-w^2}}{w}+C$

20. $\frac{-\sqrt{9-w^2}}{w}-\sin^{-1}\left(\frac{w}{3}\right)+C$

21. $\sin^{-1} x-\sqrt{1-x^2}+C$

22. $\frac{1}{3}\left(x^2-4\right)^{\frac{3}{2}}+C$

23. $4\sqrt{3}-\frac{4\pi}{3}$

24. $\frac{1}{4\sqrt{3}}$

25. $-\dfrac{x}{\sqrt{x^2-1}}+C$

26. $-\dfrac{x^3}{3\left(x^2-1\right)^{\frac{3}{2}}}+C$

27. $-\dfrac{1}{5}\left(\dfrac{\sqrt{1-x^2}}{x}\right)^5+C$

28. $-\dfrac{1}{3}\left(\dfrac{\sqrt{1-x^2}}{x}\right)^3+C$

29. $2\tan^{-1}2x+\dfrac{4x}{\left(4x^2+1\right)}+C$

30. $\tan^{-1}3t+\dfrac{3t}{9t^2+1}+C$

31. $\dfrac{1}{2}x^2+\dfrac{1}{2}\ln\left|x^2-1\right|+C$

32. $\dfrac{1}{8}In\left(25+4x^2\right)+C$

33. $\dfrac{1}{3}\left(\dfrac{v}{\sqrt{1-v^2}}\right)^3+C$

34. $-\dfrac{1}{7}\left[\dfrac{\sqrt{1-r^2}}{r}\right]^7+C$

35. $\ln 9-\ln\left(1+\sqrt{10}\right)$

36. $\dfrac{1}{5}$

37. $\dfrac{\pi}{6}$

38. $In\left(1+\sqrt{2}\right)$

39. $\sec^{-1}|x|+C$

40. $\tan^{-1}x+C$

41. $\sqrt{x^2-1}+C$

42. $\sin^{-1}x+C$

43. $\dfrac{1}{2}\ln\left|\sqrt{1+x^3}+x^2\right|+C$

44. $-\ln\left|\dfrac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right|+\sqrt{1-(\ln x)^2}+C$

45. $4\sin^{-1}\dfrac{\sqrt{x}}{2}+\sqrt{x}\sqrt{4-x}+C$

46. $\dfrac{2}{3}\sin^{-1}\left(x^{3/2}\right)+C$

47. $\dfrac{1}{4}\sin^{-1}\sqrt{x}-\dfrac{1}{4}\sqrt{x}\sqrt{1-x}\,(1-2x)+C$

48. $\sqrt{x-1}\sqrt{x-2}-\ln\left|\sqrt{x-1}+\sqrt{x-2}\right|+C$

49. $y=2\left[\dfrac{\sqrt{x^2-4}}{2}-\sec^{-1}\left(\dfrac{x}{2}\right)\right]$

50. $\ln\left|\dfrac{x}{3}+\dfrac{\sqrt{x^2-9}}{3}\right|$

51. $y=\dfrac{3}{2}\tan^{-1}\left(\dfrac{x}{2}\right)-\dfrac{3\pi}{8}$

52. $y=\dfrac{x}{\sqrt{x^2+1}}+1$

53. **(a)** $-\dfrac{1}{3}x^2\left(1-x^2\right)^{\frac{3}{2}}-\dfrac{2}{15}\left(1-x^2\right)^{\frac{5}{2}}+C$

(b) $-\dfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}-\dfrac{1}{5}\left(1-x^2\right)^{\frac{5}{2}}+C$

(c) $\dfrac{1}{5}\left(1-x^2\right)^{\frac{5}{2}}-\dfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C$

54. $30\ln\left|\dfrac{30}{x}+\dfrac{\sqrt{900-x^2}}{x}\right|-\sqrt{900-x^2}$

Exercises 5.10

1. $\dfrac{2}{x-3}+\dfrac{3}{x-2}$

2. $\dfrac{3}{x-2}+\dfrac{2}{x-1}$

3. $\dfrac{1}{x+1}+\dfrac{3}{(x+1)^2}$

4. $\dfrac{2}{x-1}+\dfrac{4}{(x-1)^2}$

5. $\dfrac{-2}{z}+\dfrac{-1}{z^2}+\dfrac{2}{z-1}$

6. $\dfrac{\frac{1}{5}}{z-3}+\dfrac{-\frac{1}{5}}{z+2}$

7. $1+\dfrac{17}{t-3}+\dfrac{-12}{t-2}$

8. $1+\dfrac{1}{t^2}+\dfrac{-10}{t^2+9}$

9. $\dfrac{1}{2}\left[\ln|1+x|-\ln|1-x|\right]+C$

10. $\dfrac{1}{2}\left[\ln|x|-\ln|x+2|\right]+C$

11. $\dfrac{1}{2}\ln\left|(x+6)^2(x-1)^5\right|+C$

12. $\ln\left|\dfrac{(x-4)^9}{(x-3)^7}\right|+C$

13. $\dfrac{(\ln 15)}{2}$

14. $\ln\dfrac{27}{4}$

15. $-\dfrac{1}{2}\ln|t|+\dfrac{1}{6}\ln|t+2|+\dfrac{1}{3}\ln|t-1|+C$

16. $\dfrac{1}{16}\ln\left|\dfrac{(x-2)^5(x+2)}{x^6}\right|+C$

17. $3\ln 2-2$

18. $2-3\ln 2$

19. $\dfrac{1}{4}\ln\left|\dfrac{x+1}{x-1}\right|-\dfrac{x}{2(x^2-1)}+C$

20. $\dfrac{\ln\left|(x-1)(x+1)^3\right|}{4}+\dfrac{1}{2(x+1)}+C$

21. $\dfrac{(\pi+2\ln 2)}{8}$

22. $\ln\left(\dfrac{9}{\sqrt{2}}\right)+\dfrac{\pi}{12}$

23. $\tan^{-1}y-\dfrac{1}{Y^2+1}+C$

24. $\tan^{-1}2x-\dfrac{1}{4x^2+1}+C$

25. $-(s-1)^{-2}+(s-1)^{-1}+\tan^{-1}s+C$

26. $\ln|s|+\dfrac{9}{S^2+9}+C$

27. $\dfrac{2}{3}In|x-1|+\dfrac{1}{6}\ln\left|x^2+x+1\right|-\sqrt{3}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+C$

28. $\ln|x|-\dfrac{1}{3}\ln|x+1|-\dfrac{1}{3}\ln\left|x^2-x+1\right|+C$

29. $\dfrac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right|+\dfrac{1}{2}\tan^{-1}x+C$

30. $\dfrac{3}{10}\ln|x-2|-\dfrac{1}{10}\ln|x+2|-\dfrac{1}{10}\ln\left|x^2+1\right|+\dfrac{1}{5}\tan^{-1}x+C$

31. $\dfrac{-1}{\theta^2+2\theta+2}+\ln\left(\theta^2+2\theta+2\right)-\tan^{-1}(\theta+1)+C$

32. $\tan^{-1}\theta+2\left(\theta^2+1\right)^{-1}-\dfrac{1}{4}\left(\theta^2+1\right)^{-2}+C$

33. $x^2+\ln\left|\dfrac{x-1}{x}\right|+C$

34. $\dfrac{x^3}{3}+x+\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|+C$

35. $9x+2\ln|x|+\dfrac{1}{x}+7\ln|x-1|+C$

36. $2x^2+4x+3\ln|2x-1|-(2x-1)^{-1}+C$ (Where $=2+C_1$)

37. $\dfrac{y^2}{2}-\ln|y|+\dfrac{1}{2}\ln\left(1+y^2\right)+C$

38. $y^2+2y+\ln|y-1|-\dfrac{1}{2}\ln\left(y^2+1\right)-\tan^{-1}y+C$ (Where $C=C_1+1$)

39. $\ln\left(\dfrac{e^t+1}{e^t+2}\right)+C$

40. $\frac{1}{2}e^{2t}+\frac{1}{2}\ln\left|e^{2t}+1\right|-\tan^{-1}\left(e^{t}\right)+C$

41. $\frac{1}{5}\ln\left|\frac{\sin y-2}{\sin y+3}\right|+C$

42. $-\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right|+C$

43. $\frac{\left(\tan^{-1}2x\right)^2}{4}-3\ln|x-2|+\frac{6}{x-2}+C$

44. $\frac{1}{6}\left(\tan^{-1}3x\right)^2+\ln|x+1|+\frac{1}{x+1}+C$

45. $\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|+C$

46. $6x^{1/6}+3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right|+C$

47. $2\sqrt{1+x}+\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$

48. $\frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right|+C$

49. $\frac{1}{4}\ln\left|\frac{x^4}{x^4+1}\right|+C$

50. $\frac{1}{80}\ln\left|\frac{x^5+4}{x^5}\right|-\frac{1}{20x^5}+C$

51. $x=\ln|t-2|-\ln|t-1|+\ln 2$

52. $3\tan^{-1}\left(\sqrt{3t}\right)-\sqrt{3}\tan^{-1}t-\pi$

53. $x=\frac{6t}{t+2}-1$

54. $x=\tan\left(\ln(t+1)\right), t>-1$

Exercises 5.11

1. $\frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{\frac{x-3}{3}}\right)+C$

2. $\frac{1}{2}\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+C$

3. $\sqrt{x-2}\left(\frac{2(x-2)}{3}+4\right)+C$

4. $\frac{(x+3)}{\sqrt{2x+3}}+C$

5. $\frac{(2x-3)^{\frac{3}{2}}(x+1)}{5}+C$

6. $\frac{(7x+5)^{\frac{5}{2}}}{49}\left(\frac{14x-4}{7}\right)+C$

7. $\frac{-\sqrt{9-4x}}{x}-\frac{2}{3}\ln\left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right|+C$

8. $\frac{\sqrt{4x-9}}{9x}+\frac{4}{27}\tan^{-1}\sqrt{\frac{4x-9}{9}}+C$

9. $\frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$

10. $\sqrt{x-x^2}+\frac{1}{2}\sin^{-1}(2x-1)+C$

11. $-\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right|+C$

12. $\frac{-1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right|+C$

13. $\sqrt{4-x^2}-2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right|+C$

14. $\sqrt{x^2-4}-\sec^{-1}\left|\frac{x}{2}\right|+C$

15. $\frac{e^{2t}}{13}(2\cos 3t+3\sin 3t)+C$

16. $\frac{e^{-3t}}{25}(-3\sin 4t-4\cos 4t)+C$

17. $\frac{x^2}{2}\cos^{-1}x+\frac{1}{4}\sin^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+C$

18. $\frac{1}{2}\left(\left(x^2+1\right)\tan^{-1}x-x\right)+C$

19. $\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln\left(1+x^2\right)+C$

20. $\frac{-1}{x}\tan^{-1}x+\ln|x|-\frac{1}{2}\ln\left(1+x^2\right)+C$

21. $-\frac{\cos 5x}{10}-\frac{\cos x}{2}+C$

22. $-\frac{\cos 5x}{10}+\frac{\cos x}{2}+C$

23. $8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7}-\frac{\sin\left(\frac{9t}{2}\right)}{9}\right]+C$

24. $3\sin\left(\frac{t}{6}\right)-\sin\left(\frac{t}{2}\right)+C$

25. $6\sin\left(\frac{\theta}{12}\right)+\frac{6}{7}\sin\left(\frac{7\theta}{12}\right)+12$

26. $\frac{\sin\left(\frac{13\theta}{2}\right)}{13}+\frac{\sin\left(\frac{15\theta}{2}\right)}{15}+C$

27. $\frac{1}{2}\ln\left(x^2+1\right)+\frac{x}{2\left(1+x^2\right)}+\frac{1}{2}\tan^{-1}x+C$

28. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{3}{x^2+3}-\frac{x}{2\left(x^2+3\right)}+C$

29. $\left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+C$

30. $2\left(\sqrt{x}\cos^{-1}\sqrt{x}-\sqrt{1-x}\right)+C$

31. $\sin^{-1}\sqrt{x}-\sqrt{x-x^2}+C$

32. $\sqrt{2x-x^2}+2\sin^{-1}\sqrt{\frac{x}{2}+C}$

33. $\sqrt{1-\sin^2 t}-\ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right|+C$

34. $-\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right|+C$

35. $\ln\left|\ln y+\sqrt{3+(\ln y)^2}\right|+C$

36. $y\tan^{-1}\sqrt{y}+\tan^{-1}\sqrt{y}-\sqrt{y}+C$

37. $\ln\left|x+1+\sqrt{x^2+2x+5}\right|+C$

38. $\frac{7}{2}\ln\left|(x-2)+\sqrt{x^2-4x+5}\right|+\frac{(x+6)\sqrt{x^2-4x+5}}{2}+C$

39. $\frac{x+2}{2}\sqrt{5-4x-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right)+C$

40. $\frac{5}{8}\sin^{-1}(x-1)-\frac{2}{3}(2x-x^2)^{\frac{3}{2}}+\frac{x-1}{8}\sqrt{2x-x^2}\left(2x^2-4x+5\right)+C$

41. $-\frac{\sin^4 2x\cos 2x}{10}-\frac{2\sin^2 2x\cos 2x}{15}-\frac{4\cos 2x}{15}+C$

42. $\frac{\cos^3 2\pi t\sin 2\pi t}{\pi}+\frac{3\cos 2\pi t\sin 2\pi t}{2\pi}+3t+C$

43. $\frac{\sin^3 2\theta\cos^2 2\theta}{10}+\frac{\sin^3 2\theta}{15}+C$

44. $\frac{2}{3}\tan^3 t+C$

45. $\tan^2 2x-2\ln|\sec 2x|+C$

46. $8\left(-\frac{1}{3}\cot^3 t+\cot t+t\right)+C$

47. $\frac{(\sec\pi x)(\tan\pi x)}{\pi}+\frac{1}{\pi}\ln|\sec\pi x+\tan\pi x|+C$

48. $\frac{\sec^2 3x\tan^3 x}{3}+\frac{2}{3}\tan 3x+C$

49. $\frac{-\csc^3 x\cot x}{4}-\frac{3\csc x\cot x}{8}-\frac{3}{8}\ln|\csc x+\cot x|+C$

50. $4x^4(\ln x)^2-2x^4\ln x+\frac{x^4}{2}+C$

51. $\frac{1}{2}\left[\sec\left(e^t-1\right)\tan\left(e^t-1\right)+\ln\left|\sec\left(e^t-1\right)+\left(\tan e^t-1\right)\right|\right]+C$

52. $-\csc\sqrt{\theta}\cot\sqrt{\theta}-\ln\left|\csc\sqrt{\theta}+\cot\sqrt{\theta}\right|+C$

53. $\sqrt{2}+\ln\left(\sqrt{2}+1\right)$

54. $2\sqrt{3}$

55. $\frac{\pi}{3}$

56. $\frac{203}{480}$

57. $\frac{\pi}{8}$

58. $\frac{\pi}{2}$

Exercises 5.13

1. **(a)** 0 **(b)** –8 **(c)** –12 **(d)** 10 **(e)** –2 **(f)** 16
2. **(a)** 2 **(b)** 9 **(c)** –2 **(d)** 1 **(e)** –6 **(f)** 1
3. **(a)** 5 **(b)** $5\sqrt{3}$ **(c)** –5 **(d)** –5
4. **(a)** $-\sqrt{2}$ **(b)** $\sqrt{2}$ **(c)** $-\sqrt{2}$ **(d)** 1
5. **(a)** 4 **(b)** –4
6. **(a)** 6 **(b)** 6
7. Area = 21 square units.
8. 2 square units
9. Area = $\frac{9\pi}{2}$ square units.
10. 4π square units
11. Area = 2.5 square units.
12. 1 square units
13. Area = 3 square units
14. $2+\frac{\pi}{2}$ square units
15. $\frac{b^2}{4}$
16. $2b^2$
17. b^2-a^2
18. $\frac{3}{2}\left(b^2-a^2\right)$
19. **(a)** 2π **(b)** π
20. **(a)** $\frac{\pi}{4}-\frac{3}{2}$ **(b)** $\frac{\pi}{2}$
21. $\frac{1}{2}$
22. 3
23. $\frac{3\pi^2}{2}$
24. 24
25. $\frac{7}{3}$
26. 0.009
27. $\frac{1}{24}$
28. $\frac{\pi^3}{24}$
29. $\frac{3a^2}{2}$
30. a^2
31. $\frac{b}{3}$
32. $9b^3$
33. –14
34. 10
35. –2
36. –1
37. $-\frac{7}{4}$
38. 0
39. 7
40. 7
41. 0
42. $\frac{7}{2}$
43. $c(b-a)$
44. 6
45. $\frac{b^3}{3}-\frac{a^3}{3}$
46. $-\frac{5}{6}$
47. 9
48. 0
49. $\frac{b^4}{4}-\frac{a^4}{4}$
50. $\frac{5}{4}$
51. $a=0$ and $b=1$ maximize the integral.
52. $x^4-2x^2\le 0\ln\left[-\sqrt{2},\sqrt{2}\right]\Rightarrow a=-\sqrt{2}$ and $b=\sqrt{2}$ minimize the integral.
53. upper bound = 1, lower bound = $\frac{1}{2}$
54. $\frac{13}{20}\le\int_0^1\frac{1}{1+x^2}dx\le\frac{9}{10}$

55. For Example, $\int_0^1 \sin\left(x^2\right)dx \le \int_0^1 dx = 1$

56. $2\sqrt{2} \le \int_0^1 \sqrt{x+8}\ dx \le 3$

57. $\int_a^b f(x)dx \ge \int_a^b 0\,dx = 0$

58. $\int_a^b f(x)dx \le 0$

59. Upper Bound = $\frac{1}{2}$

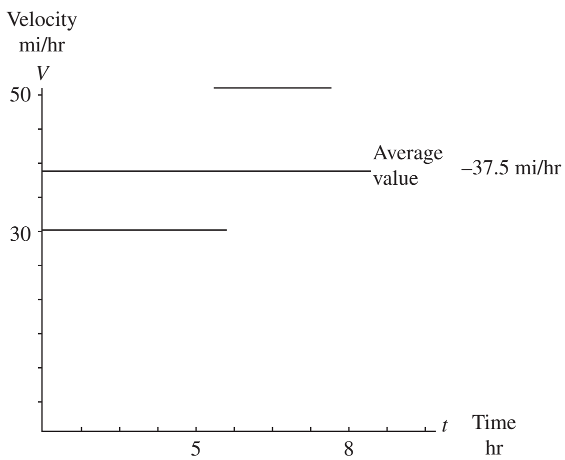
60. $\frac{7}{6}$

66. **(a)** The area of the shaded region is $\sum_{i=1}^{n} x_i \cdot m_i$ which is equal to L.

(b) The area of the shaded region is $\sum_{i=1}^{n} x_i \cdot M_i$ which is equal to U.

(c) The area of the shaded region is the difference in the areas of the shaded regions shown in the second part of the figure and the first part of the figure. Thus this area is $U - L$.

68. 37.5 mi/hr



Exercises 5.14

1. $-\frac{10}{3}$

2. $\frac{20}{3}$

3. $\frac{124}{125}$

4. 0

5. $\frac{753}{16}$

6. $\frac{105}{4}$

7. 1

8. $\frac{5}{2}$

9. $2\sqrt{3}$

10. π

11. 0

12. 4

13. $-\frac{\pi}{4}$

14. $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

15. $1-\frac{\pi}{4}$

16. $2\sqrt{3}-\frac{\pi}{6}-2$

17. $\frac{2-\sqrt{2}}{4}$

18. $4\sqrt{3}-3$

19. $-\frac{8}{3}$

20. $10\sqrt{3}$

21. $-\frac{3}{4}$

22. $\frac{22}{3}$

23. $\sqrt{2}-\sqrt[4]{8}+1$

24. $-\frac{137}{20}$

25. –1

26. $\frac{5\pi}{6}+\frac{9\sqrt{3}}{8}$

27. 16

28. 1

29. $\frac{1}{2}$

30. $2+2\cos 1$

31. $\sqrt{26}-\sqrt{5}$

32. $\frac{\sqrt{3}}{8}$

33. (cos $\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$

34. $3\sin^2 x\cos x$

35. $4t^5$

36. **(a)** $\left(\sec^2\left(\tan\theta\right)\right)\sec^2\theta$ **(b)** $\left(\sec^2\left(\tan\theta\right)\right)\sec^2\theta$

37. 3

38. **(a)** $\frac{1}{2}t^{3/2}+\frac{3}{2}t^{-2}$ **(b)** $\frac{1}{2}t^{\frac{3}{2}}+\frac{3}{2}t^{-2}$

39. $\sqrt{1+x^2}$

40. $\frac{dy}{dx}=\frac{1}{x}, x>0$

41. $-\frac{1}{2}x^{-\frac{1}{2}}\sin x$

42. $2x^2\sin x^6+\int_2^{x^2}\sin t^3 dt$

43. 0

44. $3\left(x^3+1\right)^{10}\left(\int_0^x\left(t^3+1\right)^{10}dt\right)^2$

45. 1

46. 1

47. **d**, since $y' = \frac{1}{x} and y(\pi) = \int_\pi^\pi \frac{l}{t}\,dt - 3 = -3$

48. (c)

49. b, since $y' = \sec x$ and $y(0) = \int_0^0 \sec t\,dt + 4 = 4$

50. (a)

51. $y = \int_2^x \sec t\,dt + 3$

52. $y=\int_1^x \sqrt{1+t^2dt}-2$

53. **(a)** T(0) = 70°F, T(16) = 76°F, T(25) = 85°F
(b) av(T) = 75°F

54. **(a)** 13 ft **(b)** 9.67 ft

55. $2x-2$

56. 1

57. $-3x+5$

58. $-2x+1$

59. **(a)** True. Since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
(b) True: g is continuous because it is differentiable.
(c) True, since $g'(1) = f(1) = 0$
(d) False, since $g''(1) = f'(1) > 0$
(e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$
(f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign
(g) True, Since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$

61. **(a)** $V = \frac{ds}{dt} = \frac{d}{dt}\int_0^1 f(x)dx = f(t) \Rightarrow v(5) = f(5) = 2m/\text{sec}$

(b) $a = df/dt$ is negative, since the slope of the tangent line at $t = 5$ is negative.

(c) $s = \int_0^3 f(x) = \frac{1}{2}(3)(3) = \frac{9}{2}$ m, since the integral is the area of the triangle formed by $y = f(x)$, the x–axis, and $x = 3$.

(d) $t = 6$, since after $t = 6$ to $t = 9$, the region lies below the x–axis.

(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there.

(f) Toward the origin between $t = 6$ and $t = 9$, since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$, since the velocity is positive there.

(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x–axis than below.

62. 85

Exercises 5.15

1. **(a)** $\frac{14}{3}$ **(b)** $\frac{2}{3}$

2. **(a)** $\frac{1}{3}$ **(b)** 0

3. **(a)** $\frac{1}{2}$ **(b)** $-\frac{1}{2}$

4. **(a)** 2 **(b)** 2

5. **(a)** $\frac{15}{16}$ **(b)** 0

6. **(a)** $\frac{45}{8}$ **(b)** $-\frac{45}{8}$

7. **(a)** 0 **(b)** $\frac{1}{8}$

8. **(a)** $\frac{10}{3}$ **(b)** $\frac{70}{27}$

9. **(a)** 4 **(b)** 0

10. **(a)** $\frac{\sqrt{10}-3}{2}$ **(b)** $\frac{3-\sqrt{10}}{2}$

11. **(a)** $\frac{506}{375}$ **(b)** $\frac{86744}{375}$

12. **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{2}$

13 **(a)** 0 **(b)** 0

14. **(a)** 3 **(b)** 8

15. $2\sqrt{3}$ **16.** $\frac{1}{6}$

17. $\frac{3}{4}$ **18.** 12

19. $3^{\frac{5}{2}} - 1$ **20.** $\frac{1}{5}$

21. 3 **22.** $\frac{-2}{3}$

23. $\frac{\pi}{3}$ **24.** $\frac{1}{2} - \frac{1}{4}\sin 2$

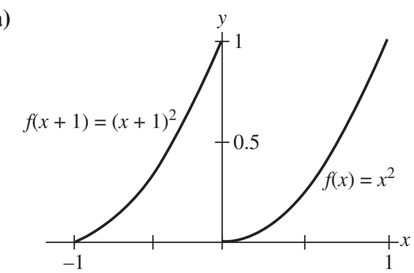
25. $F(6) - F(2)$

27. **(a)** –3

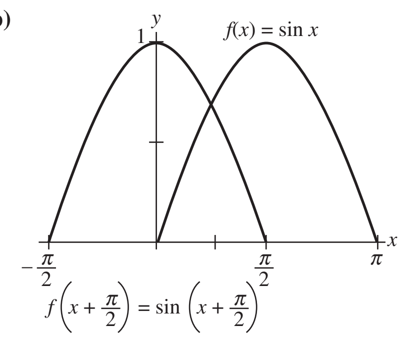
(b) 3

29. $I = \frac{a}{2}$

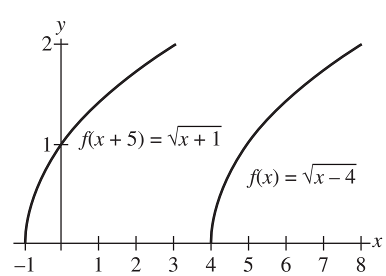
32. **(a)**



(b)



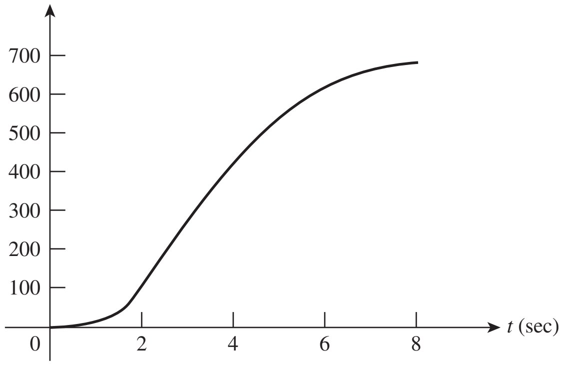
(c)



Answers to Practice Exercises

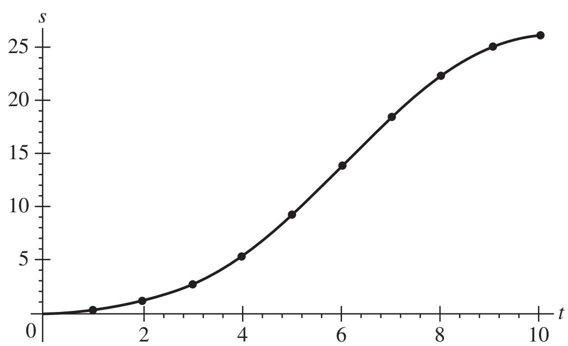
1. **(a)** 680 ft

(b)



2. **(a)** approximately 26 m

(b)



3. **(a)** $-\frac{1}{2}$ **(b)** 31 **(c)** 13 **(d)** 0

4 **(a)** 0 **(b)** 7 **(c)** 8 **(d)** –40

5. **(a)** 4 **(b)** 2 **(c)** –2 **(d)** -2π **(e)** $\frac{8}{5}$

6. **(a)** 1 **(b)** –1 **(c)** $-\pi$ **(d)** $\pi\sqrt{2}$ **(e)** $1 - 3\pi$

7. 3 **8.** 3

9. –3 **10.** 2

11. $-4(\cos x)^{1/2} + C$ **12.** $\frac{-2}{(\tan x)^{1/2}} + C$

13. $\theta^2 + \theta + \sin(2\theta + 1) + C$ **14.** $(2\theta - \pi)^{1/2} + \tan(2\theta + \pi) + C$

15. $\frac{t^3}{3} + \frac{4}{t} + C$ **16.** $-\frac{1}{t} - \frac{1}{t^2} + C$

17. $-\frac{1}{3}\cos(2t^{3/2}) + C$ **18.** $\frac{2}{3}(1 + \sec\theta)^{3/2} + C$

19. 16 **20.** 3

21. 2 **22.** 2

23. 1 **24.** $\frac{4}{3}(3\sqrt{3} - 2\sqrt{2})$

25. 8 **26.** $\frac{3}{5}\left(\sqrt[3]{7} - \sqrt[3]{2}\right)$

27. $\frac{27\sqrt{3}}{160}$ **28.** $\frac{1}{90}$

29. $\frac{\pi}{2}$ **30.** $\frac{\pi}{8}$

31. $\sqrt{3}$ **32.** 2

33. $6\sqrt{3} - 2\pi$ **34.** $3\sqrt{3} - \pi$

35. –1 **36.** 0

37. 2 **38.** –2

39. 1 **40.** $\frac{3}{7}$

41. $\frac{1}{2k}(2bk) = b$

42. **(a)** 2 **(b)** $\frac{2}{3}a$

43. $f'_{av} = \frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{1}{b-a}\left[f(x)\right]_a^b = \frac{1}{b-a}[f(b) - f(a)]$
$= \frac{f(b) - f(a)}{b-a}$ so that average value of f' over $[a, b]$ is the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$, which is the average rate of change of f over $[a, b]$

44. Yes **45.** $\sqrt{2 + \cos^3 x}$

46. $14x\sqrt{2 + \cos^3(7x^2)}$ **47.** $-\frac{6}{3 + x^4}$

48. $\frac{-\sec x \tan x}{1 + \sec^2 x}$ **49.** Yes

50. $\int_0^1 \sqrt{1 + x^4}\,dx = F(1) - F(0)$ **51.** $-1\sqrt{1 + x^2}$

52. cosec x **53.** $\frac{x^4}{4} + \frac{5x^2}{2} = -7x + C$

54. $2t^4 - \frac{t^3}{6} + \frac{t^2}{x} + C$ **55.** $2t^{3/2} - \frac{4}{t} + C$

56. $\sqrt{t} + \frac{1}{t^2} + C$ **57.** $\int \frac{dr}{(r+5)^2} = -\frac{1}{(r+5)} + C$

58. $\int \frac{6dr}{\left(r - \sqrt{2}\right)^3} = -\frac{3}{\left(r - \sqrt{2}\right)^2} + C$ **59.** $(\theta^2 + 1)^{3/2} + C$

60. $\sqrt{7 + \theta^2} + C$ **61.** $\frac{1}{3}(1 + x^4)^{3/4} + C$

62. $-\frac{5}{8}(2 - x)^{8/5} + C$ **63.** $10\tan\frac{s}{10} + C$

64. $-\frac{1}{\pi}\cot \pi s + C$ **65.** $-\frac{1}{\sqrt{2}}\csc\sqrt{2\theta} + C$

66. $3\sec\frac{\theta}{3} + C$ **67.** $\frac{x}{2} - \sin\frac{x}{2} + C$

68. $\frac{x}{2} + \frac{1}{2}\sin x + C$ **69.** $x - \frac{1}{x} - 1$

70. $\frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$ **71.** $4t^{5/2} + 4t^{3/2} - 8t$

72. $r = \sin t - t - 1$

73. $(x+1)\ln(x+1)-(x+1)+C$

74. $\frac{x^2}{3}\ln x-\frac{x^3}{9}+C$

75. $x\tan^{-1}(3x)-\frac{1}{6}\ln(1+9x^2)+C$

76. $x\cos^{-1}\left(\frac{x}{2}\right)-2\sqrt{1-\left(\frac{x}{2}\right)^2}+C$

77. $\left[(x+1)^2-2(x+1)+2\right]e^x+C$

78. $x^2\cos(1-x)+2x\sin(1-x)-2\cos(1+x)+C$

79. $I=\frac{e^x\cos 2x}{5}+\frac{2e^x\sin 2x}{5}+C$

80. $\frac{x}{2}\sin^2 x-\frac{1}{4}x+\frac{1}{8}\sin 2x+C$

81. $2\ln|x-2|-\ln|x-1|+C$

82. $\frac{3}{2}\ln|x+3|-\frac{1}{2}\ln|x+1|+C$

83. $\ln|x|-\ln|x+1|+\frac{1}{x+1}+C$

84. $-2\ln|x|+\frac{1}{x}+2\ln|x-1|+C$

85. $-\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right|+C$

86. $\frac{1}{5}\ln\left|\frac{\sin\theta-2}{\sin\theta+3}\right|+C$

87. $4\ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+4\tan^{-1}x+C$

88. $2\tan^{-1}\left(\frac{x}{2}\right)+C$

89. $\frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right|+C$

90. $\ln\left|\frac{(v-2)(v-3)}{(v-1)^2}\right|+C$

91. $\frac{1}{2}\tan^{-1}t-\frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}}+C$

92. $\frac{1}{6}\ln\left|t^2-2\right|-\frac{1}{6}\left(t^2+1\right)+C$

93. $\frac{x^2}{2}+\frac{4}{3}\ln|x+2|+\frac{2}{3}\ln|x-1|+C$

94. $x+\ln|x-1|-\ln|x|+C$

95. $\frac{x^2}{2}-\frac{9}{2}\ln|x+3|+\frac{3}{2}\ln|x+1|+C$

96. $x^2-3x+\frac{2}{3}\ln|x+4|+\frac{1}{3}\ln|x-2|+C$

97. $\frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$

98. $3\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+C$

99. $\ln|1-e^{-s}|+C$

100. $\ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right|+C$

101. $-\sqrt{16-y^2}+C$

102. $\sqrt{4+x^2}+C$

103. $-\frac{1}{2}\ln\left|4-x^2\right|+C$

104. $\frac{1}{4}\sqrt{4t^2-1}+C$

105. $\ln\frac{1}{\sqrt{9-x^2}}+C$

106. $\frac{1}{9}\ln|x|-\frac{1}{18}\ln|9-x^2|+C$

107. $\frac{1}{6}\ln\left|\frac{x+3}{x-3}\right|+C$

108. $\sin^{-1}\frac{x}{3}+C$

109. $-\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+C$

110. $\frac{\sin^6 x}{6}-\frac{2\sin^8 x}{8}+\frac{\sin^{10} x}{10}+C$

111. $\frac{\tan^5 x}{5}+C$

112. $\frac{\sec^5 x}{5}-\frac{\sec^3 x}{3}+C$

113. $\frac{1}{2}\cos\theta-\frac{1}{22}\cos 11\theta+C$

114. $\sec\theta+\cos\theta+C$

115. $4\sqrt{2}\left|\sin\frac{t}{4}\right|+C$

116. $\ln|\sec e^t+\tan e^t|+C$

117. $\frac{x}{2}$

118. -1

119. 6

120. $\int_{-2}^{\infty}\frac{d\theta}{(\theta+1)^{3/5}}$ diverges

121. ln3

122. $1+\ln\frac{3}{4}$

123. 2

124. $-\frac{1}{9}$

125. $\frac{\pi}{6}$

126. π

127. $\int_{6}^{\infty}\frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

128. $\frac{1}{2}$ converges

129. $\infty \Rightarrow$ diverges

130. $\int_1^{\infty}\frac{e^{-t}}{\sqrt{t}}$ converges

131. $\int_{-\infty}^{\infty}\frac{2dx}{e^x+e^{-x}}$ converges

132. $\int_{-\infty}^{\infty}\frac{dx}{x^2\left(1+e^x\right)}$ diverges

133. $\frac{2x^{3/2}}{3}-x+2\sqrt{x}-2\ln\left(1+\sqrt{x}\right)+C$

134. $-\frac{x^2}{2}-\frac{3}{2}\ln|x+2|-\frac{5}{2}\ln|x-2|+C$

135. $\frac{1}{2}\sin^{-1}(x-1)+\frac{1}{2}(x-1)\sqrt{2x-x^2}+C$

136. $\sin^{-1}(x+1)+C$

137. $-2\cot x+\csc x-\ln|\csc x+\cot x|+C$

138. $\frac{\sin^3\theta}{3}-\frac{2}{5}\sin^5+\frac{1}{7}\sin^7\theta+C$

139. $\frac{1}{12}\ln\left|\frac{3+v}{3-v}\right|+\frac{1}{6}\tan^{-1}\frac{v}{3}+C$

140. 1

141. $\frac{\theta}{2}\sin(2\theta+1)+\frac{1}{4}\cos(2\theta+1)+C$

142. $\frac{x^2}{2}+2x+3\ln|x-1|-\frac{1}{x-1}+C$

143. $\frac{1}{4}\sec^2\theta+C$

144. $\frac{\sqrt{2}}{2}$

145. $2\left[\frac{\left(\sqrt{2-x}\right)^3}{3}-2\sqrt{2-x}\right]+C$

146. $-\sin^{-1}v-\frac{\sqrt{1-v^2}}{v}+C$

147. $\tan^{-1}(y-1)+C$

148. $\frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right)+C$

149. $\frac{1}{4}\ln|z|-\frac{1}{4z}-\frac{1}{8}\ln\left(z^2+4\right)-\frac{1}{8}\tan^{-1}\frac{z}{2}+C$

150. $\frac{3}{10}(x-1)^{10/3}+\frac{6}{7}(x-1)^{7/3}+\frac{3}{4}(x-1)^{4/3}+C$

151. $-\frac{1}{4}\sqrt{9-4t^2}+C$

152. $-\frac{\tan^{-1}x}{x}+\ln|x|-\ln\sqrt{1+x^2}+C$

153. $\ln\left(\frac{e^t+1}{e^t+2}\right)+C$

154. $\frac{\tan^2 t}{2}-\ln|\sec t|+C$

155. $\frac{1}{4}$

156. $y^{5/2}\left(\frac{2}{5}(\ln y)^2-\frac{8}{25}\ln y+\frac{16}{125}\right)+C$

157. $\frac{2}{3}x^{3/2}+C$

158. $\frac{1}{6}\left(3+4e^{\theta}\right)^{3/2}+C$

159. $-\frac{1}{5}\tan^{-1}(\cos 5t)+C$

160. $\sec^{-1}(e^v)+C$

161. $2\sqrt{r}-2\ln\left(1+\sqrt{r}\right)+C$

162. $\ln\left|x^4-10x^2+9\right|+C$

163. $\frac{1}{2}x^2-\frac{1}{2}\ln(1+x^2)+C$

164. $\frac{1}{3}\ln|1+x^3|+C$

165. $\frac{2}{3}\ln|1+x|+\frac{1}{6}\ln|1-x+x^2|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

166. $\ln|1+x|+\frac{2}{1+x}-\frac{1}{(1+x)^2}+C$

167. $\frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2}-\frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2}+\frac{32}{105}\left(1+\sqrt{x}\right)^{7/2}+C$

168. $\frac{4}{3}\sqrt{1+x}\left(1+\sqrt{1+x}\right)^{3/2}-\frac{8}{15}\left(1+\sqrt{1+x}\right)^{5/2}+C$

169. $2\ln\left|\sqrt{1+x}+\sqrt{x}\right|+C$

170. $\frac{\left(4+\sqrt{3}\right)\sqrt{2-\sqrt{3}}}{3\sqrt{2}}$

171. $\ln x-\ln|1+\ln x|+C$

172. $\ln|\ln(\ln x)|+C$

173. $\frac{1}{2}x^{\ln x}+C$

174. $(\ln x)^{\ln x}+C$

175. $-\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^4}}{x^2}\right|+C$

176. $2\sqrt{1-x}+\frac{1}{2}\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right|+C$

177. $\frac{x}{2}-\frac{1}{2}\ln|\sin x+\cos x|+C$

178. $x-\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+C$

179. $-2\cot x+2\csc x-x+C$

Answers to Single Choice Questions (Indefinite)

1. (b)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (d)
11. (a)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (b)
21. (a)	**22.** (d)	**23.** (b)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (a)	**28.** (a)	**29.** (d)	**30.** (d)
31. (c)	**32.** (d)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (a)	**37.** (a)	**38.** (a)	**39.** (b)	**40.** (b)
41. (a)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (a)	**49.** (d)	**50.** (a)
51. (b)	**52.** (d)	**53.** (a)	**54.** (b)	**55.** (d)	**56.** (a)	**57.** (d)	**58.** (c)	**59.** (c)	

Answers to Multiple Choice Questions (Indefinite)

1. (a) and (d) **2.** (a), (b) and (d) **3.** (b) and (d) **4.** (a), (b), (c) and (d) **5.** (a) and (b)
6. (a), (b), (c) and (d) **7.** (a), (b), (c) and (d) **8.** (a), (b), (c) and (d) **9.** (b), (c) and (d) **10.** (c) and (d)
11. (b) and (d) **12.** (a), (b) and (c) **13.** (a), (b), (c) and (d) **14.** (a) and (b) **15.** (a), (b) and (c)
16. (a) and (c) **17.** (a) and (d) **18.** (c) and (d) **19.** (a) and (d) **20.** (a) and (d)

Answers to Passage Type Questions (Indefinite)

Passage 1
1. (c) **2.** (b) **3.** (a)
Passage 2
1. (a) **2.** (b)
Passage 3
1. (b) **2.** (c)
Passage 4
1. (a) **2.** (d) **3.** (b)
Passage 5
1. (d) **2.** (c) **3.** (d)

Answers to Matrix Match Type Questions (Indefinite)

1. (a) → (s) and (t) (b) → (r) (c) → (p) and (t) (d) → (q) and (t)
2. (a) → (s) (b) → (p) and (s) (c) → (r) and (s) (d) → (q) and (s)
3. (a) → (s) (b) → (r) (c) → (p) (d) → (q)
4. (a) → (q) (b) → (p) (c) → (s) (d) → (r)

Answers to Integer Type Questions (Indefinite)

1. 6 **2.** 6 **3.** 3 **4.** 5 **5.** 8 **6.** 4 **7.** 20 **8.** 4 **9.** 0
10. 75 **11.** 4 **12.** 628 **13.** 2 **14.** 2 **15.** 403 **16.** 3 **17.** 3 **18.** 8
19. 5 **20.** 5 **21.** 2 **22.** 9 **23.** 20

Answers to Single Choice Questions (Definite)

1. (a) **2.** (a) **3.** (c) **4.** (d) **5.** (b) **6.** (b) **7.** (b) **8.** (d) **9.** (b) **10.** (b)
11. (b) **12.** (c) **13.** (c) **14.** (c) **15.** (a) **16.** (d) **17.** (d) **18.** (d) **19.** (a) **20.** (d)
21. (b) **22.** (a) **23.** (b) **24.** (c) **25.** (a) **26.** (a) **27.** (b) **28.** (d) **29.** (b) **30.** (d)
31. (d) **32.** (d) **33.** (d) **34.** (c) **35.** (c) **36.** (d) **37.** (b) **38.** (b) **39.** (b) **40.** (c)
41. (c) **42.** (a) **43.** (c) **44.** (b) **45.** (a) **46.** (b) **47.** (a) **48.** (b) **49.** (a) **50.** (b)

Answers to Multiple Choice Questions (Definite)

1. (a), (b) and (c) **2.** (b), (c) and (d) **3.** (b), (c) and (d) **4.** (c) and (d) **5.** (b), (c) and (d)
6. (b), (c) and (d) **7.** (a), (b) and (d) **8.** (a) and (c) **9.** (a), (b), (c) and (d) **10.** (a), (c) and (d)
11. (a), (c) and (d) **12.** (a) and (c) **13.** (a), (b) and (c) **14.** (a) and (d) **15.** (a), (b) and (c)
16. (a), (b) and (d) **17.** (b) **18.** (a) and (c) **19.** (a), (b), (c) and (d) **20.** (a) and (b)
21. (a) and (d) **22.** (a), (b), (c) and (d) **23.** (a) and (c) **24.** (b) and (d) **25.** (a), (b), (c) and (d)

Answers to Passage Type Questions (Definite)

Passage 1
1. (b) **2.** (d) **3.** (b)
Passage 2
1. (d) **2.** (c)
Passage 3
1. (b) **2.** (c)
Passage 4
1. (c) **2.** (d) **3.** (a)
Passage 5
1. (a) **2.** (d)
Passage 6
1. (d) **2.** (b) **3.** (a)
Passage 7
1. (c) **2.** (b) **3.** (b)

Answers to Matrix Match Type Questions (Definite)

1. (a) $\to$ (q) and (t) (b) $\to$ (r) and (t) (c) $\to$ (s) and (t) (d) $\to$ (p)
2. (a) $\to$ (r) (b) $\to$ (s) (c) $\to$ (q) (d) $\to$ (p)
3. (a) $\to$ (r) (b) $\to$ (s) (c) $\to$ (p) (d) $\to$ (q)
4. (a) $\to$ (s) (b) $\to$ (s) (c) $\to$ (p) (d) $\to$ (r)
5. (a) $\to$ (q) (b) $\to$ (r) (c) $\to$ (p) (d) $\to$ (s)
6. (a) $\to$ (q) (b) $\to$ (p) (c) $\to$ (p) (d) $\to$ (r)

Answers to Integer Type Questions (Definite)

1. 5 **2.** 2 **3.** 3 **4.** 9 **5.** 6 **6.** 500 **7.** 9 **8.** 2 **9.** 1 **10.** 4
11. 5 **12.** 2 **13.** 4 **14.** 2 **15.** 0 **16.** 7 **17.** 3 **18.** 20 **19.** 4 **20.** 5051
21. 0 **22.** 4 **23.** 2 **24.** 252 **25.** 5 **26.** 8 **27.** 24 **28.** 16 **29.** 17 **30.** 1

Answers to Additional and Advanced Exercises

1. **(a)** Yes, because $\int_0^1 f(x)dx = \frac{1}{7}\int_0^1 7f(x)dx = \frac{1}{7}(7) = 1$
(b) No

2. **(a)** True **(b)** True **(c)** False

4. $\frac{d^2y}{dx^2} = 4y$, and the constant of proportionality is 4

5. **(a)** $\frac{1}{4}$ **(b)** $\sqrt[3]{12}$

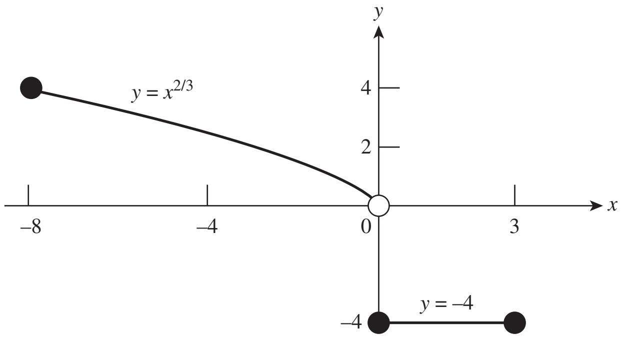
6. $\frac{1}{2}$

7. $\frac{x}{\sqrt{x^2+1}}$

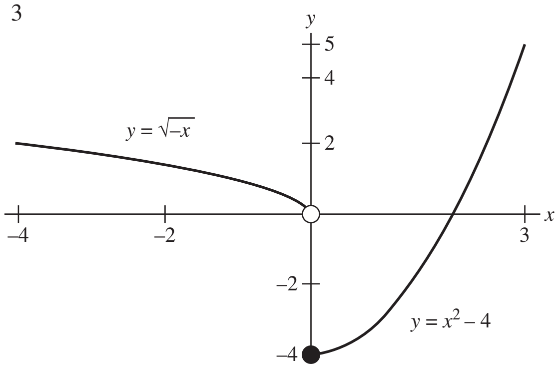
9. $y = x^3 + 2x - 4$

10. You had better duck

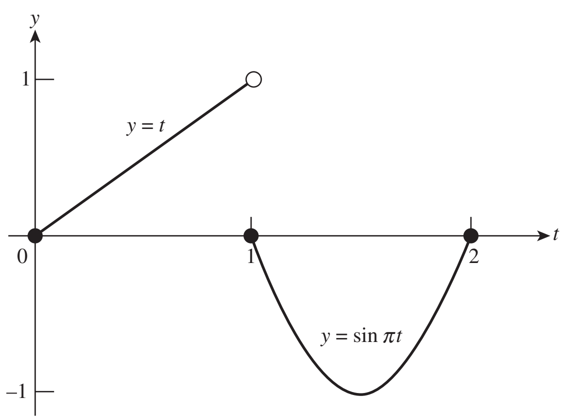
11. $\frac{36}{5}$



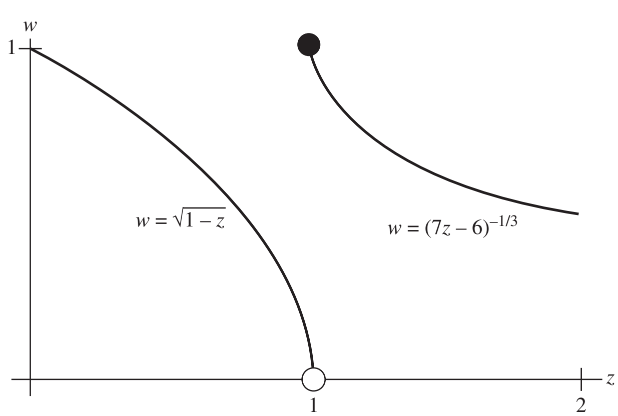
12. $\frac{7}{3}$



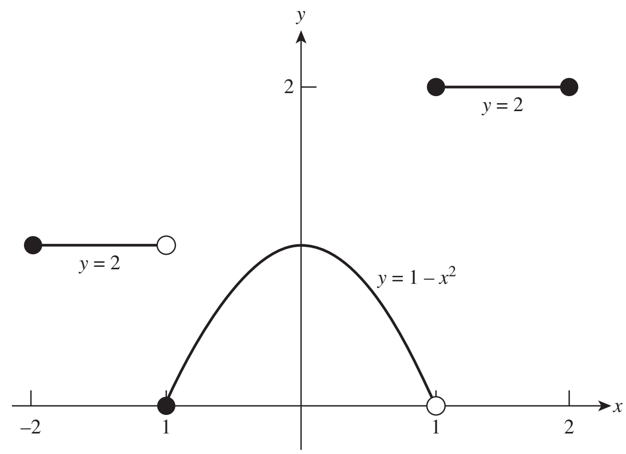
13. $\frac{1}{2} - \frac{2}{\pi}$



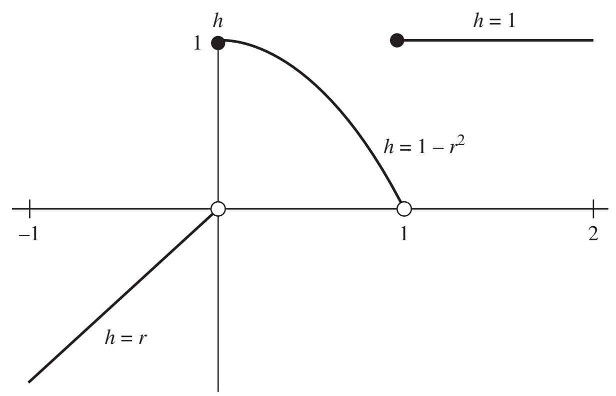
14. $\frac{55}{42}$



15. $\frac{13}{3}$



16. $\frac{7}{6}$



17. $\frac{1}{2}$ **18.** $\frac{2}{3}$

19. $\frac{1}{6}$ **20.** $\frac{1}{4}$

21. $\int_0^1 f(x)dx$

22. **(a)** 1 **(b)** $\frac{1}{16}$ **(c)** $\frac{2}{\pi}$
(d) 0 **(e)** ∞

23. **(b)** πr^2 **24.** $\frac{1}{3}$

25. **(a)** 0 **(b)** -1 **(c)** $-\pi$ **(d)** $x = 1$
(e) $y = 2x + 2 - \pi$ **(f)** $x = 2$ **(g)** $[-2\pi, 0]$

27. $\frac{2}{x}$

28. $\frac{1}{\cos x} + \frac{1}{\sin x}$ **29.** $\frac{\sin 4y}{\sqrt{y}} - \frac{\sin y}{2\sqrt{y}}$

30. $x = 1$

31. $x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1 - x^2} - 2x + C$

32. $\sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln |x + k| \right] + C$

33. $\frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2} - \sin^{-1}x}{4} + C$

34. $y\sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$

35. $\frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1}t + C$

36. $\frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$

37. 0 **38.** 1

39. $\ln 4 - 1$ **40.** $\frac{\pi}{2}$

41. $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

45. $I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$

46. $I = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + C$

47. $I = \frac{\sin 3x\cos x - 3\cos 3x\sin x}{8} + C$

48. $I = \frac{1}{9}(4\cos 5x\cos 4x + 5\sin 5x\sin 4x) + C$

49. $I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + \mathrm{C}$

50. $I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

51. $x\ln(ax) - x + C$ **52.** $\frac{1}{3}x^3\ln(ax) - \frac{1}{9}x^3 + C$

53. $\frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$ **54.** $\ln\left|\tan\left(\frac{x}{2}\right) + 1\right| + C$

55. 1 **56.** $\sqrt{3} - 1$

57. $\frac{\sqrt{3}\pi}{9}$ **58.** $\frac{1}{2}\left(\ln\sqrt{3} - 1\right)$

59. $\frac{1}{\sqrt{2}}\ln\left|\frac{1}{2}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right|\right| + C$

60. $-\cot\left(\frac{t}{2}\right) - t + C$

61. $\ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$ **62.** $\ln\left|\tan\frac{\theta}{2}\right| + C$

64 **(b)**

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	**Calculator**
10	3598695.619	3628800
20	2.4227868 × 1018	2.432902 × 1018
30	2.6451710 × 1032	2.652528 × 1032
40	8.1421726 × 1047	8.1591528 × 1047
50	3.0363446 × 1064	3.0414093 × 1064
60	8.3094383 × 1081	8.3209871 × 1081

(c)

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}e^{1/12n}$	**Calculator**
10	3598695.619	3628810.051	3628800

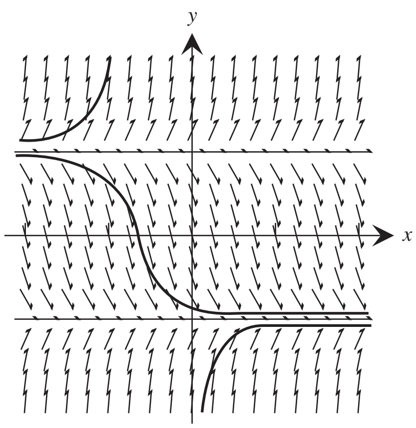
Answers to Exercises

Chapter 6

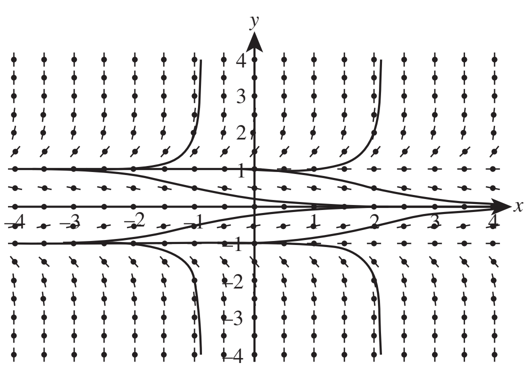
Exercises 6.1

1. (d) **2.** (c)
3. (a) **4.** (b)

5.



6.



7. $\frac{dy}{dx} = x - y,\ y(1) = -1$

8. $\frac{dy}{dx} = \frac{1}{x},\ y(1) = 0$

9. $\frac{dy}{dx} = -(1 + y)\sin x,\ y(0) = 2$

10. $\frac{dy}{dx} = y,\ y(0) = 1$

11. $\frac{dy}{dx} = x + y$

12. $y = \int_{x_0}^{x} f(t)dt + y_0$

25. $\frac{2}{3}y^{3/2} - x^{1/2} = C$ (where $C = \frac{1}{2}C_1$)

26. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2y^{1/2}dx \Rightarrow y^{-1/2}dy = x^2dx$

$\Rightarrow 2y^{1/2} = \frac{x^3}{3} + C$

27. $e^y - e^x = c$

28. $\frac{dy}{dx} = 3x^2e^{-y} \Rightarrow dy = 3x^2e^{-y}dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

29. $-x + 2\tan\sqrt{y} = C$

30. $\sqrt{2xy}\frac{dy}{dx} = 1 \Rightarrow dy = \frac{1}{\sqrt{2xy}}dx \Rightarrow \sqrt{2}\sqrt{y}dy = \frac{1}{\sqrt{x}}dx$

$\Rightarrow \sqrt{2}\int y^{1/2}dy = \int x^{-1/2}\,dx \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C$ where $C = \frac{3}{2}C_1$

31. $e^{-y} + 2e^{\sqrt{x}} = C$

32. $(\sec x)\frac{dy}{dx} = e^{y+\sin x} \Rightarrow e^{-y}dy = e^{\sin x}\cos x\,dx \Rightarrow e^{-y} + e^{\sin x} = C$
where $C = -C_1$

33. $y = \sin(x^2 + C)$

34. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}}dx \Rightarrow e^{2y}dy = e^xdx \Rightarrow \int e^{2y}dy = \int e^xdx$

$\Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$

35. $\frac{1}{3}\ln\left|y^3 - 2\right| = x^3 + C$

36. $\frac{dy}{dx} = xy + 3x - 2y - 6 \Rightarrow \frac{1}{y+3}\,dy = (x-2)\,dx$

$\Rightarrow \int \frac{1}{y+3}dy = \int (x-2)\,dx \Rightarrow \ln(y+3) = \frac{1}{2}x^2 - 2x + C$

37. $4\ln(\sqrt{y} + 2) = e^{x^2} + C.$

38. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1 = (e^{-y} + 1)(e^x + 1)$

$\Rightarrow \int \frac{1}{e^{-y}+1}dy = \int (e^x + 1)dx \Rightarrow \ln\left|1 + e^y\right| = e^x + x + C$

$\Rightarrow \ln(1 + e^y) = e^x + x + C$

39. **(a)** –0.00001
(b) 10,536 years
(c) 82%

40. **(a)** ≈ -0.121
(b) $\approx$ 2.389 millibars
(c) $\approx$ 0.9777 km

41. 54.88g

42. $\approx$ 585.35 kg

43. 59.8 ft

44. $\approx$ 92.1 sec

45. 2.8147498×10^{14}

46. 1250

47. **(a)** 8 years **(b)** 32.02 years

48. **(a)** $y = \left(y_0 - \frac{r}{k}\right)e^{-kt} + \frac{r}{k}$

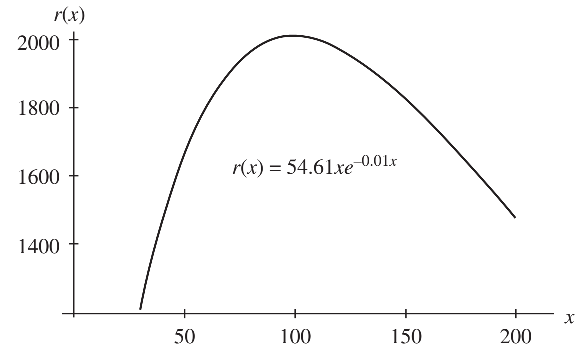
49. yes, $y(20) < 1$

50. **(a)** ≈ 0.0083583
(b) $\approx$ 333,664,000

51. 15.28 years

52. **(a)** $P(x) = 54.61e^{-0.01x}$ (in dollars)
(b) \$22.20

(c) $r(x)$ must be maximum at $x = 100$



53. 56,562 years **54.** ≈ 600 days

56. ≈0.262 **(b)** ≈3.816 years **(c)** ≈11.431 years

57. **(a)** 17.5 min **(b)** 13.26 min

58. 5° **59.** –3°C

60. **(a)** ≈ 53.44°C **(b)** ≈ 23.79°C
(c) ≈ 232 minutes from now the silver will be 10°C above room temperature.

61. About 6693 years

62. **(a)** 12,571 BC **(b)** 12,101 BC
(c) 13,070 BC

63. 54.62%

64. ≈ 41 years old **65.** ≈15.683 years

66. **(a)** about 94% **(b)** ≈ 588 years

Exercises 6.2

1. $y = \dfrac{e^x + c}{x}$, $x > 0$ **2.** $e^{-x} + Ce^{-2x}$

3. $y = \dfrac{c - \cos x}{x^3}$, $x > 0$ **4.** $\sin x \cos x + C \cos x$

5. $y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{C}{x^2}$, $x > 0$ **6.** $\dfrac{2x^{3/2}}{3(1+x)} + \dfrac{C}{1+x}$

7. $y = \dfrac{1}{2}xe^{x/2} + Ce^{x/2}$ **8.** $x^2e^{-2x} + \mathrm{C}e^{-2x}$

9. $y = x(\ln x)^2 + \mathrm{C}$ **10.** $\dfrac{\sin x + C}{x^2}$

11. $\mathrm{S} = \dfrac{t^3}{3(t-1)^4} - \dfrac{t}{(t-1)^4} + \dfrac{C}{(t-1)^4}$

12. $(t + 1) + (t + 1)^{-2} \ln(t + 1) + \dfrac{C}{(t+1)^2}$; $t > -1$

13. $r = (\csc \theta) (\ln | \sec \theta | + C)$, $0 < \theta < \pi / 2$

14. $\dfrac{\sin^2\theta}{3} + \dfrac{C}{\sin\theta}$ **15.** $y = \dfrac{3}{2} - \dfrac{1}{2}e^{-2t}$

16. $y = \dfrac{t^3}{5} - \dfrac{12}{5t^2}$ **17.** $y^2 = -\dfrac{-1}{\theta}\cos\theta + \dfrac{\pi}{2\theta}$

18. $y = \theta^2 \sec\theta + \left(\dfrac{18}{\pi^2} - 2\right)\theta^2$ **19.** $y = 6e^{x^2} - \dfrac{e^{x^2}}{x+1}$

20. $y = 1 - \dfrac{7}{e^{x^2/2}}$ **21.** $y = y_0e^{kt}$

22. **(a)** $u = u_0e^{-(k/m)t}$ **(b)** $u = u_0e^{-(k/m)t}$

23. (b) is correct **24.** $y = \dfrac{e^x}{e^x + C}$

25. $y = \dfrac{e^x}{e^x - xe^x + c}$ **26.** $y = (1 + \dfrac{C}{x^3})^{1/3}$

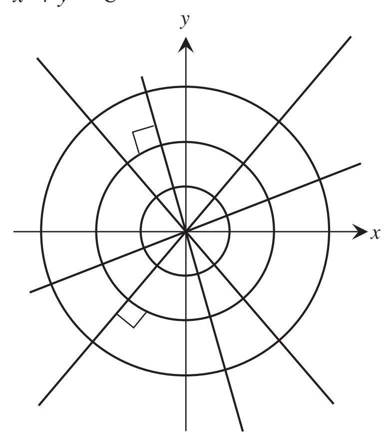
27. $y = \left(\dfrac{2}{5x} + Cx^4\right)^{-1/2}$

Exercises 6.3

1. **(a)** 168.5 m **(b)** 41.13 sec

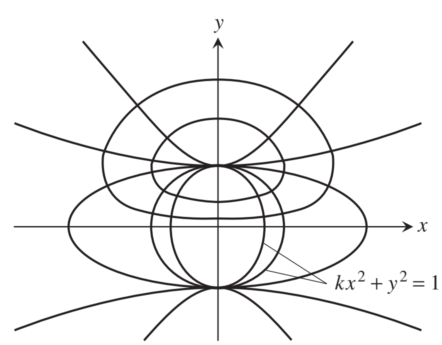
2. **(a)** 7780 m , or 7.78 m

3. $x^2 + y^2 = C$

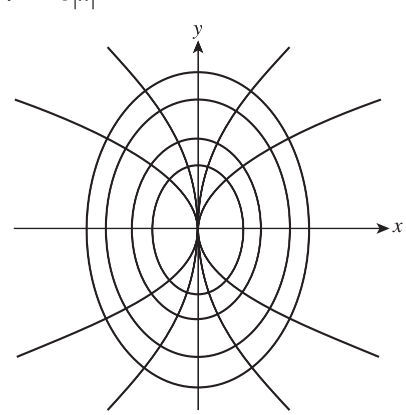


4. $y = \pm\sqrt{\dfrac{-x^2 + c}{2}}$

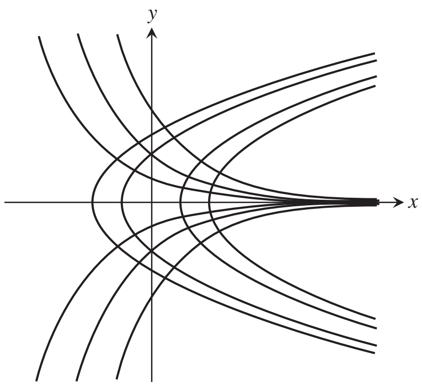
5. $\ln | y | - \dfrac{1}{2}y^2 = \dfrac{x^2}{2} + C$



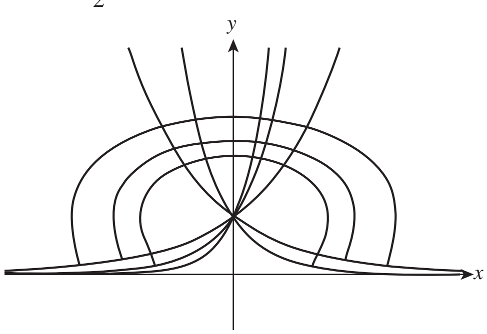
6. $y = C_1\left|x\right|^{1/2}$



7. $y = \pm\sqrt{2x + c_1}$

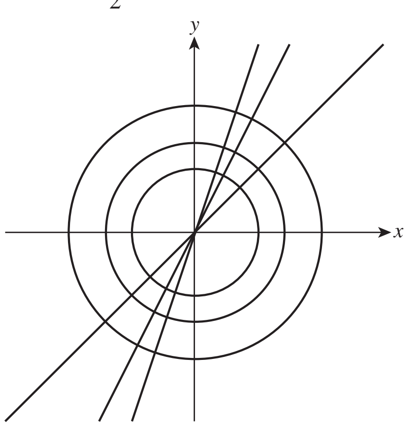


8. $y^2 \ln y - \dfrac{y^2}{2} = -x^2 + C_1$



10. **(a)** $\ln y = \ln x + \mathrm{C}$ or $y = \mathrm{C}_1 x$ (where $\mathrm{C}_1 = e^{C}$)

(b) $y^2 = \dfrac{-x^2}{2} + \mathrm{C}$ is general equation of orthogonal



11. **(a)** 10 lb/min **(b)** $(100 + t)$ gal **(c)** $4\left(\dfrac{y}{100+t}\right)$ lb/min

(d) $\dfrac{dy}{dt} = 10 - \dfrac{4y}{100+t}$, $y(0) = 50$, $y = 2(100 + t) - \dfrac{150}{(1+\frac{t}{100})^4}$

(e) Concentration $\approx \dfrac{y(25)}{\text{amount of brine in tank}} = \dfrac{188.6}{125} \approx 1.5$ lb/gal

12. **(a)** $t = 50$ min
(b) ≈ 83.22 pounds of concentrate

13. $Y(27.8) \approx 14.8$ lb, $t \approx 27.8$ min

14. ≈ 37.55

Exercises 6.4

1. b^3

2. $\dfrac{\pi b^3}{3}$

3. b^2

4. $\dfrac{1}{4}b^2 + b$

5. 0

6. $\dfrac{-3}{2}$

7. -2

8. -2

9. 1

10. $\dfrac{3}{2}$

11. **(a)** $-\dfrac{1}{2}$ **(b)** 1 **(c)** $\dfrac{1}{4}$

12. **(a)** $\dfrac{-1}{2}$ **(b)** $\dfrac{-1}{2}$ **(c)** $\dfrac{-1}{2}$

13. $\dfrac{28}{3}$

14. 12

15. $\dfrac{1}{2}$

16. $\dfrac{83}{4}$

17. Area of shaded region is $2\pi - \pi = \pi$

18. $\sqrt{3} - \dfrac{\pi}{3}$

19. $\dfrac{\pi\sqrt{2}}{2}$

20. $\dfrac{1}{3} + \dfrac{\pi}{2}$

21. $\dfrac{16}{3}$

22. 2

23. $2^{5/2}$

24. 2

25. $\dfrac{\pi}{2}$

26. $\dfrac{4\pi}{3}$

27. $\dfrac{128}{15}$

28. $\dfrac{1}{12}$

29. $\dfrac{4}{3}$

30. $\dfrac{22}{15}$

31. $\dfrac{5}{6}$

32. $\dfrac{5}{6}$

33. $\dfrac{38}{3}$

34. 16

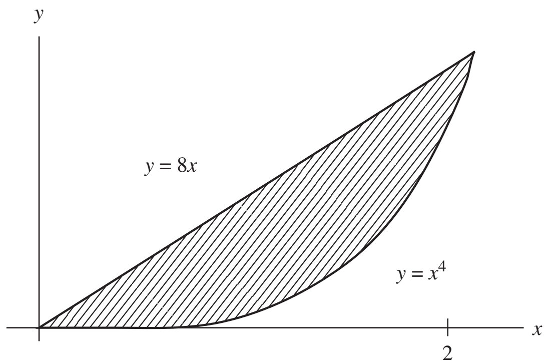
35. $\dfrac{49}{6}$

36. $\dfrac{19}{4}$

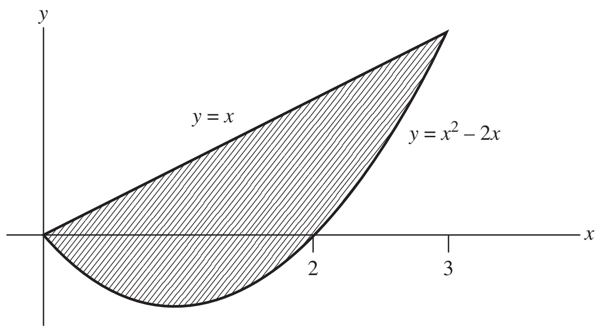
37. $\dfrac{32}{3}$

38. $\dfrac{32}{3}$

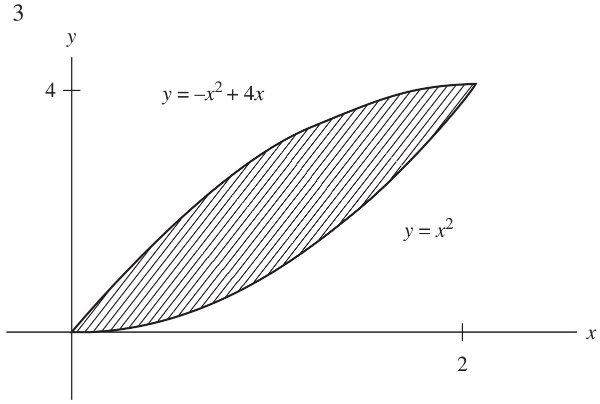
39. $\frac{48}{5}$



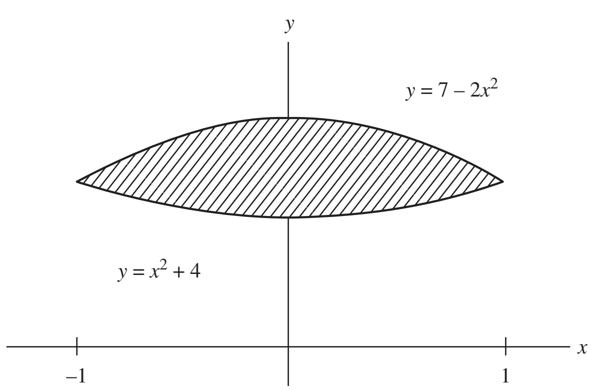
40. $\frac{9}{2}$



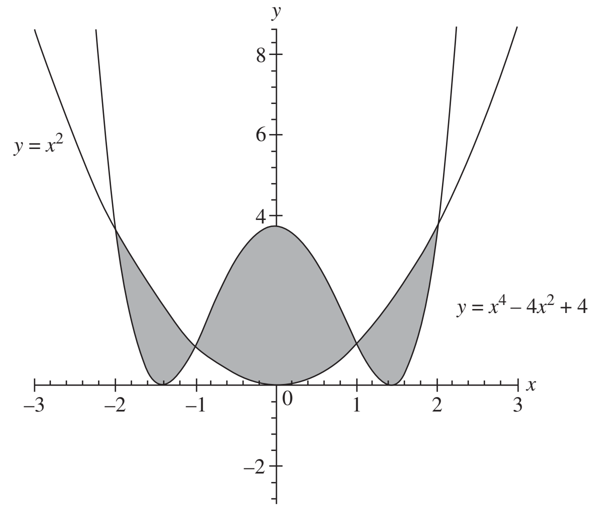
41. $\frac{8}{3}$



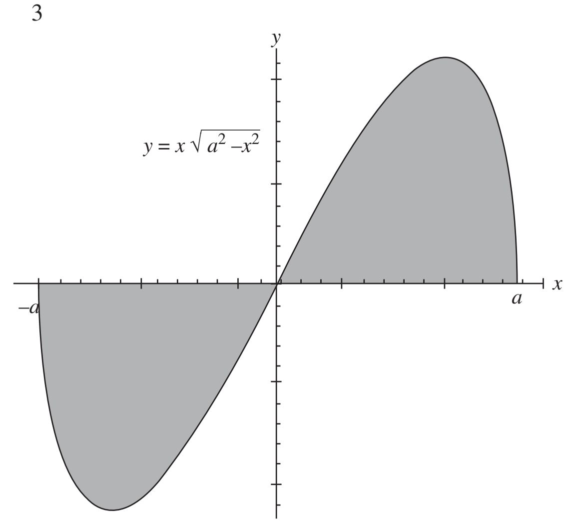
42. 4



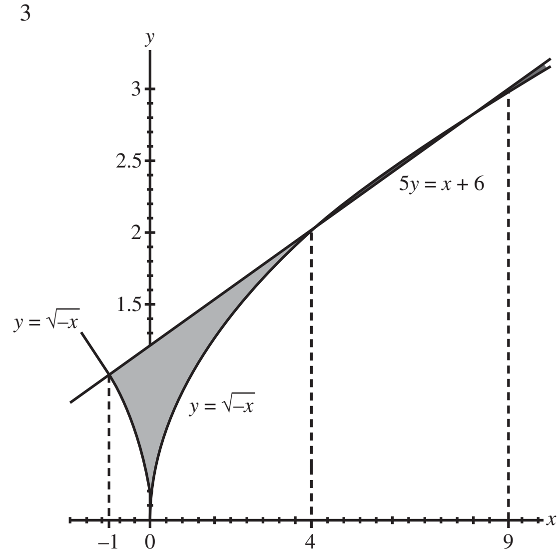
43. 8



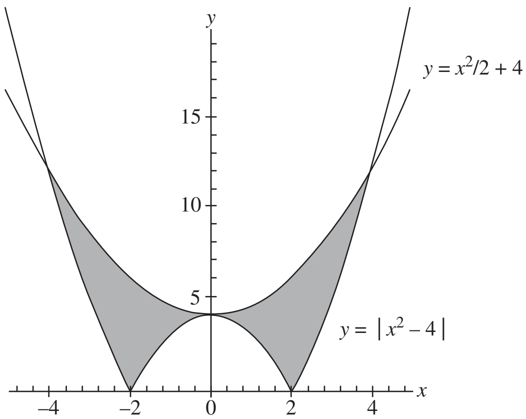
44. $\frac{2a^3}{3}$



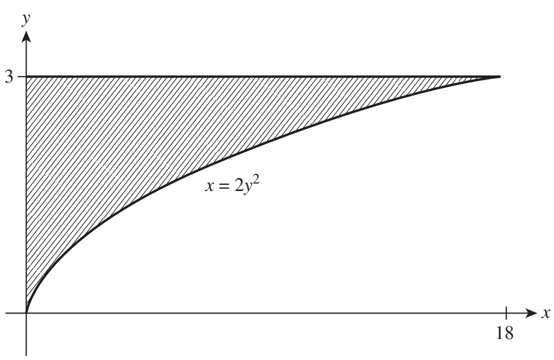
45. $\frac{5}{3}$



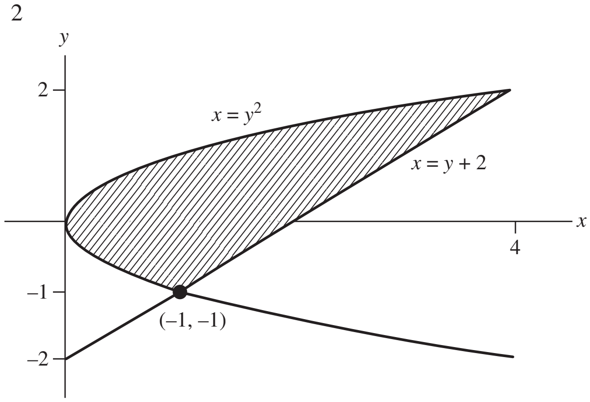
46. $\frac{64}{3}$



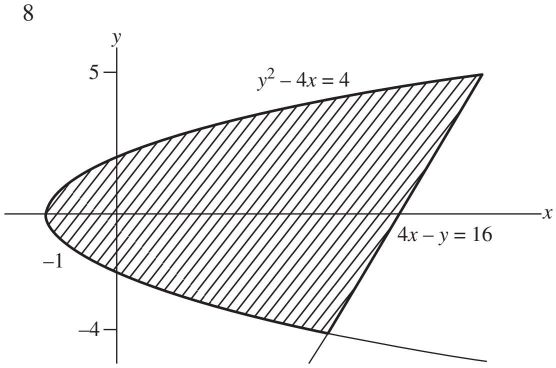
47. 18



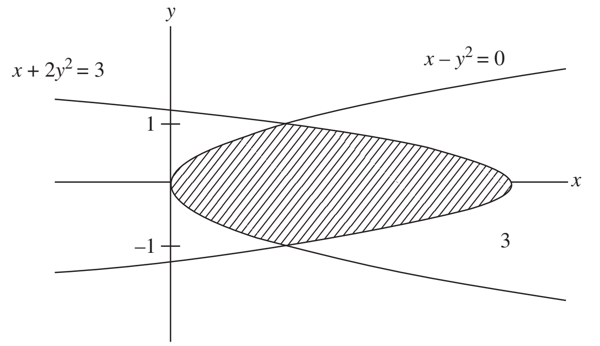
48. $\frac{9}{2}$



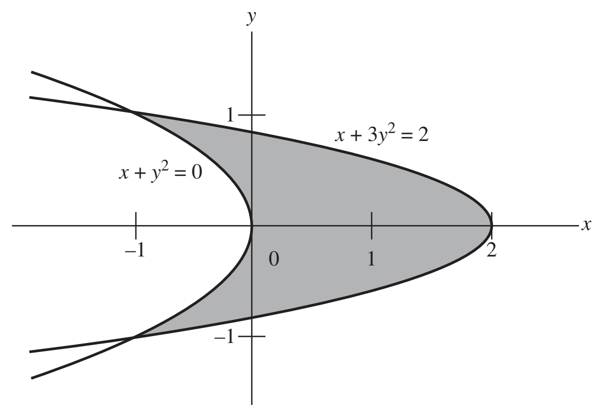
49. $\frac{243}{8}$



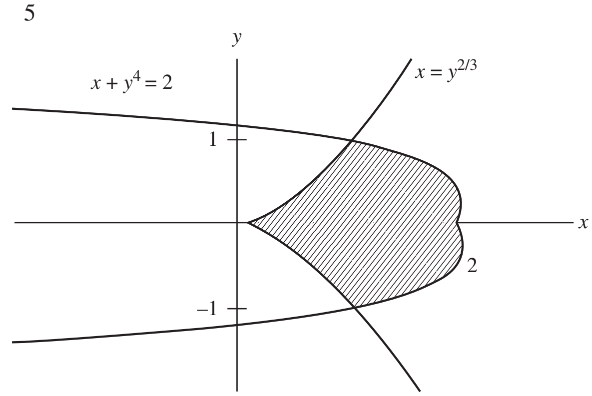
50. 4



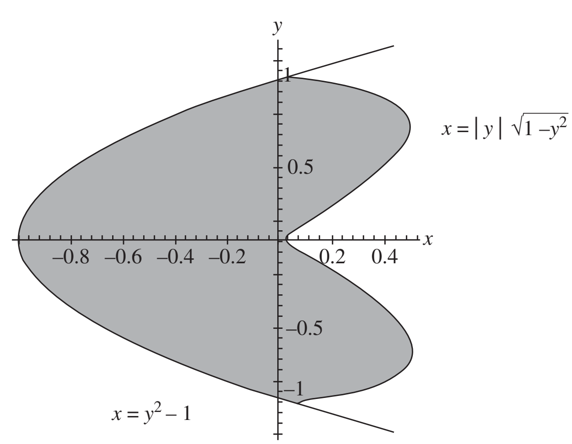
51. $\frac{8}{3}$



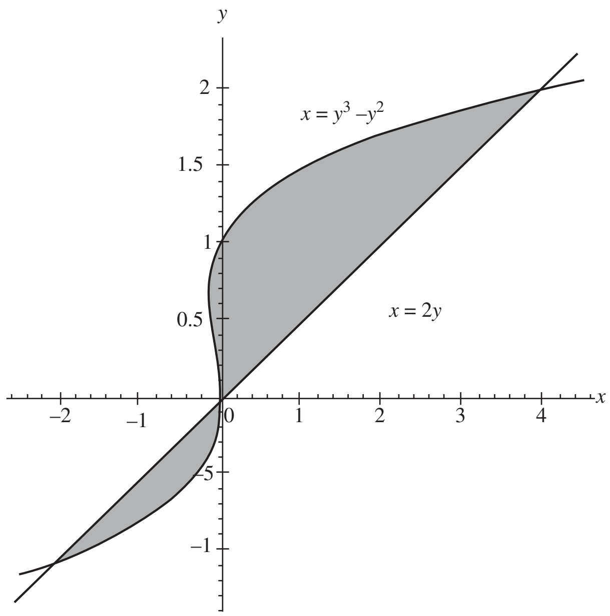
52. $\frac{12}{5}$



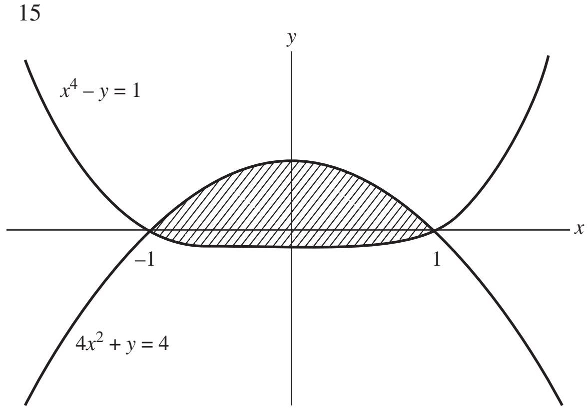
53. 2



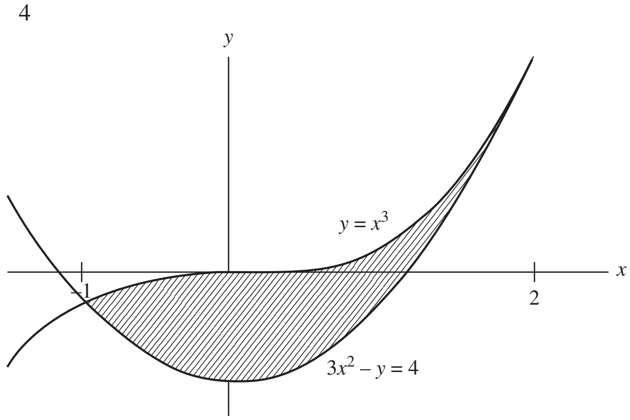
54. $\frac{37}{12}$



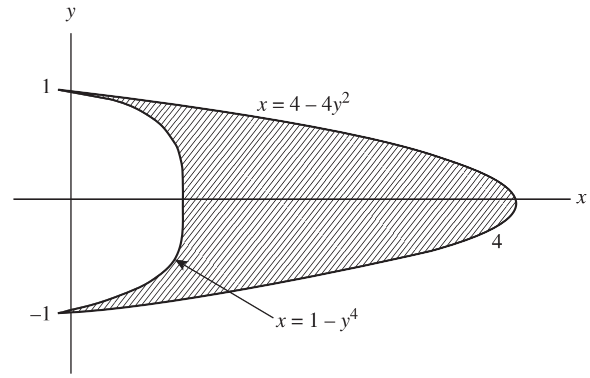
55. $\frac{104}{15}$



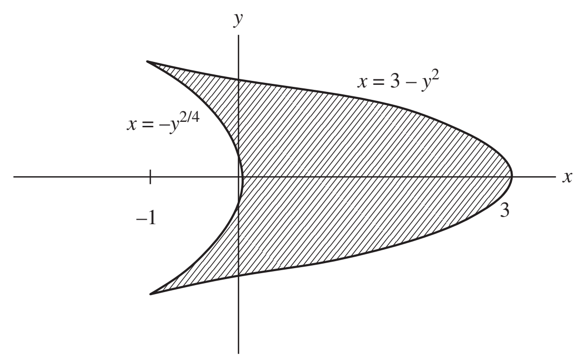
56. $\frac{27}{4}$



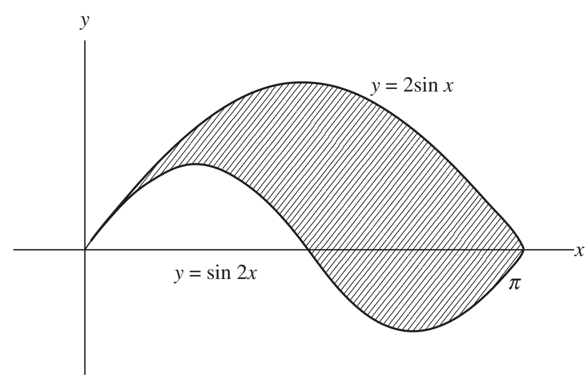
57. $\frac{56}{15}$



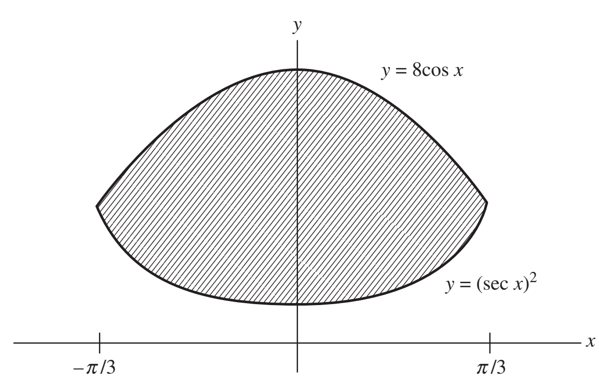
58. 8



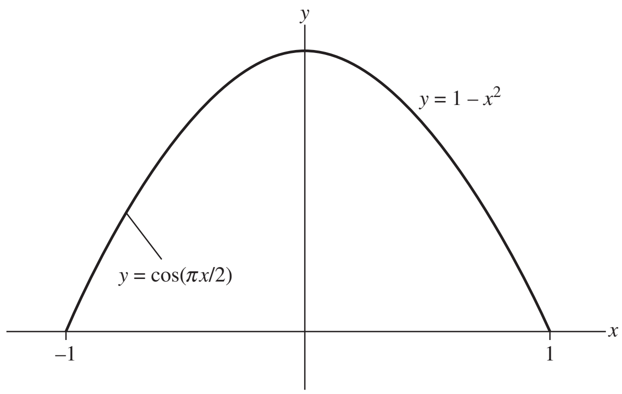
59. 4



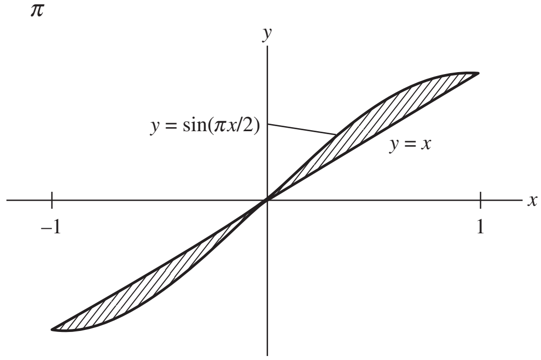
60. $6\sqrt{3}$



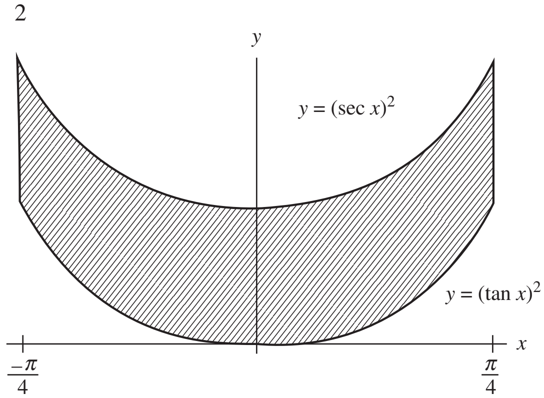
61. $\dfrac{4}{3}-\dfrac{4}{\pi}$



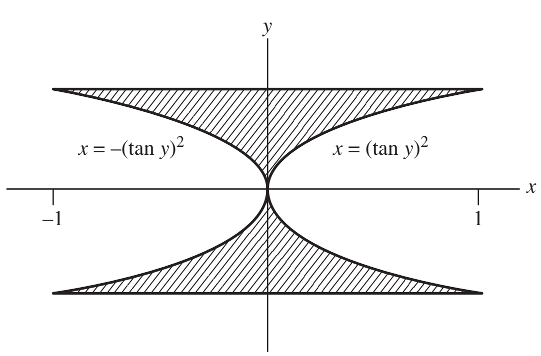
62. $\dfrac{4-\pi}{\pi}$



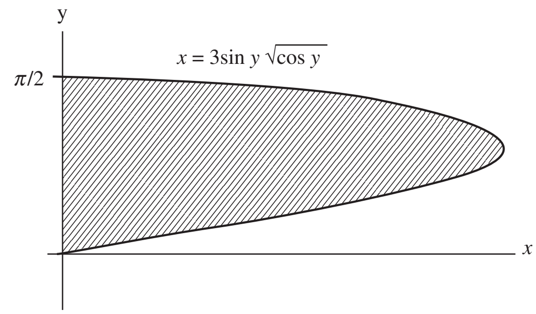
63. $\dfrac{\pi}{2}$



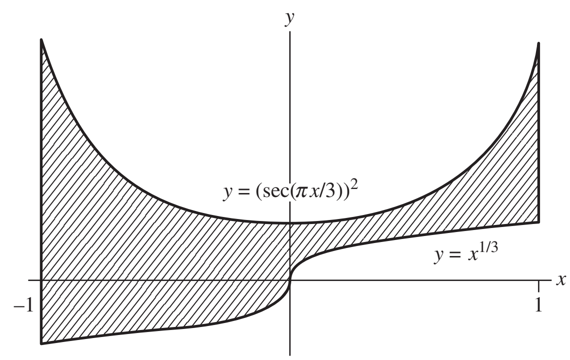
64. $4-\pi$



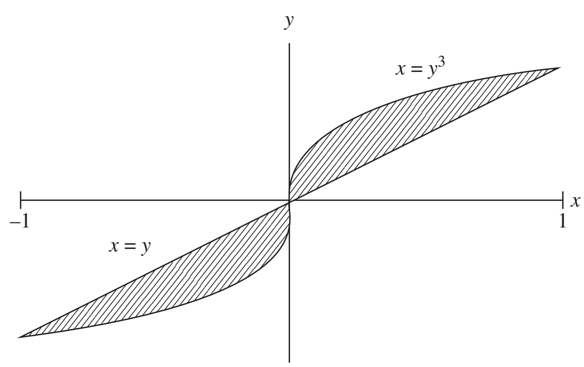
65. **2**



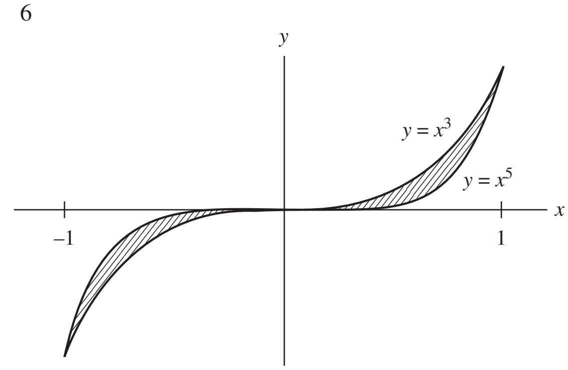
66. $\dfrac{6\sqrt{3}}{\pi}$



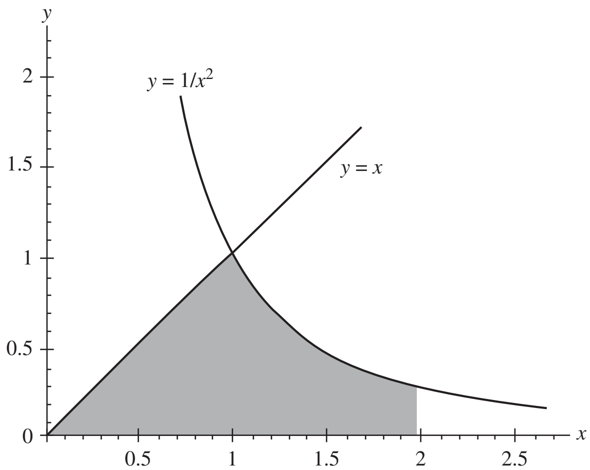
67. $\dfrac{1}{2}$



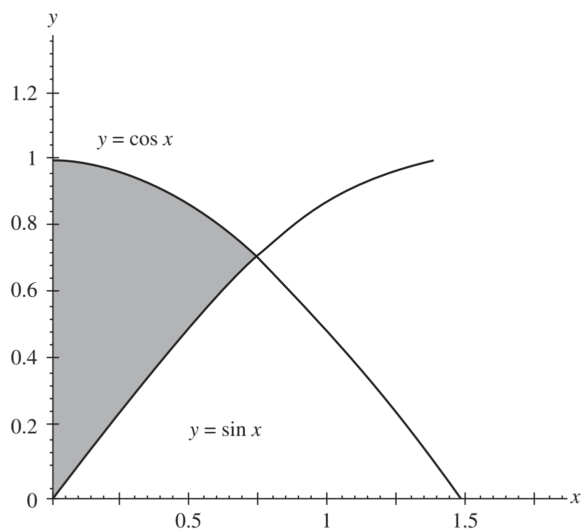
68. $\dfrac{1}{6}$



69. 1



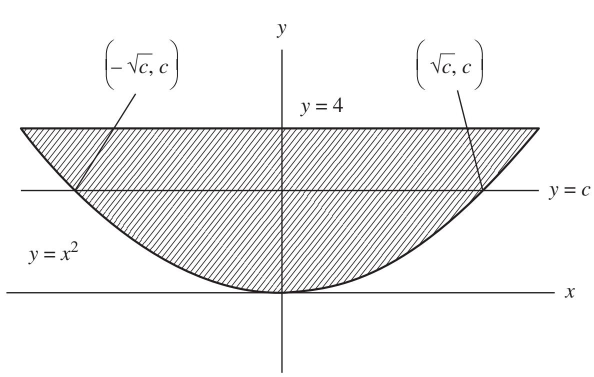
70. $\sqrt{2-1}$



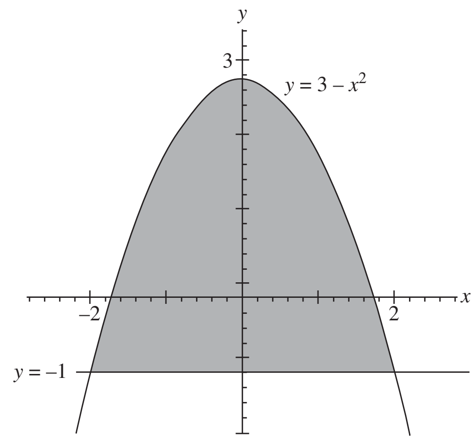
71. **(a)** $x = \pm\sqrt{c}$ and $y = c$

(b) C = $4^{2/3}$

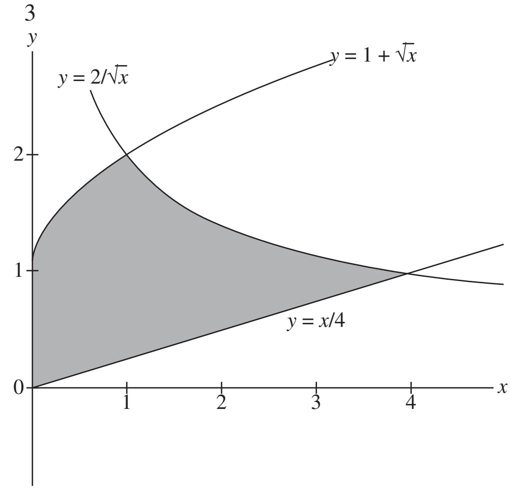
(c) $C = 4^{2/3}$



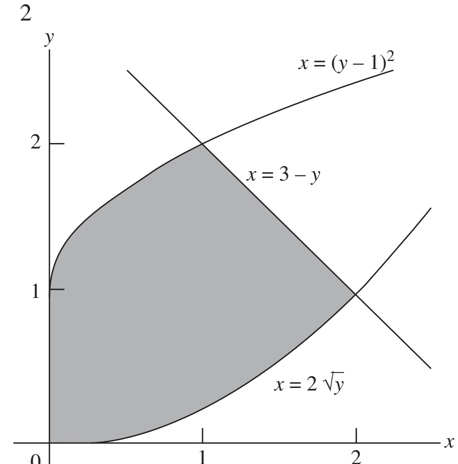
72. **(a)** $\frac{32}{3}$ **(b)** $\frac{32}{3}$



73. $\frac{11}{3}$



74. $\frac{5}{2}$



75. Area (Parabola) = $\frac{4a^3}{3}$

Area (Triangle) = 3/4 which is independent of *a*.

76. 4 **77.** 2 **78.** True

Answers to Practice Exercises

1. $y = -\ln\left[\dfrac{-2(x-2)^{3/2}(3x+4)}{15} - C\right]$

2. $\ln y = \dfrac{1}{2}e^{x^2} + C$

3. $\tan y = -\cos x - x\sin x + C$

4. $y^{3/2} = (2 - x^2)\cos x + 2x\sin x + C_1$

5. $(y+1)e^{-y} = -\ln|x| + C$

6. $(x-1)e^x + c$

7. $y = \dfrac{C_1(x-1)}{x}$

8. $\dfrac{y-1}{y+1} = C_1x^2$

9. $e^{x/2}\left(\dfrac{x^2}{4} + C\right)$

10. $e^{-x}(\sin x - \cos x) + Ce^{-2x}$

11. $y = \dfrac{1}{2} - \dfrac{1}{x} - \dfrac{C}{x^2}$

12. $y = x[\ln x]^2 + Cx$

13. $y = \dfrac{e^{-x} + C}{e^{-x} + 1} = \dfrac{e^{-x} + C}{1 + e^x}$

14. $y = 2x\,e^x - C^x + Ce^{-x}$

15. $xy = -y^3 + C$

16. $y = x^{-3}(\sin x + C)$

17. $y = (x+1)^{-2}\left(\dfrac{x^3}{3} + \dfrac{x^2}{2} + 1\right)$

18. $\dfrac{x^4 + 2x^2 + 1}{4x^2}$

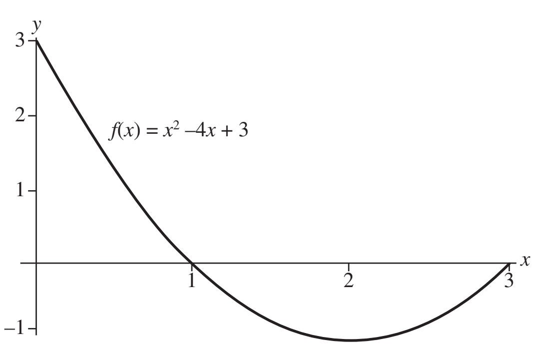
19. $y = \dfrac{1}{3} - \dfrac{4}{3}e^{-x^3}$

20. $y = \dfrac{-1 + \sin x}{x}$

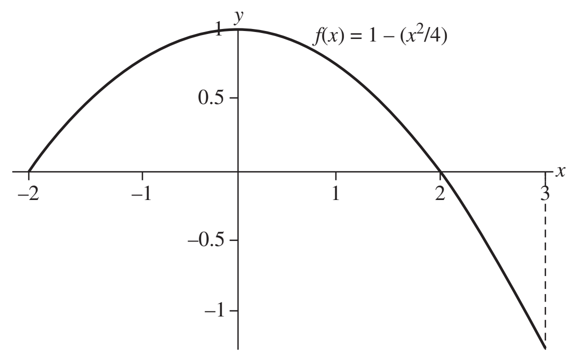
21. $y = x^2e^{-x}(3x - 3)$

22. $y^3 = \dfrac{2(y^2 + 2y + 2) - 4e^{y+1}}{x}$

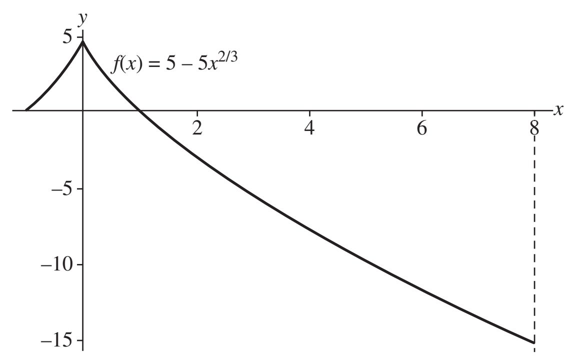
23. $\dfrac{8}{3}$



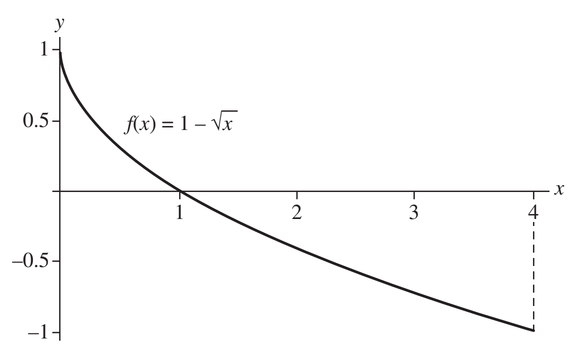
24. $\dfrac{13}{4}$



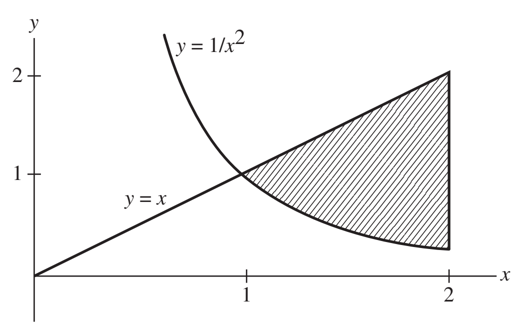
25. 62



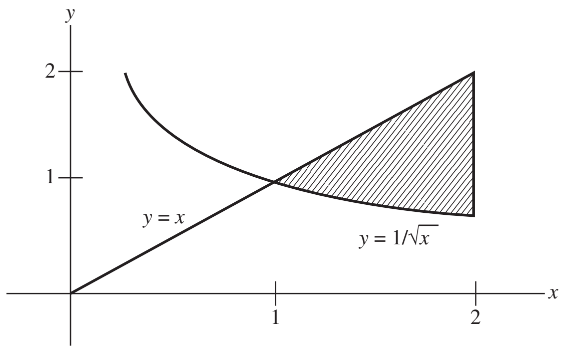
26. 2



27. 1

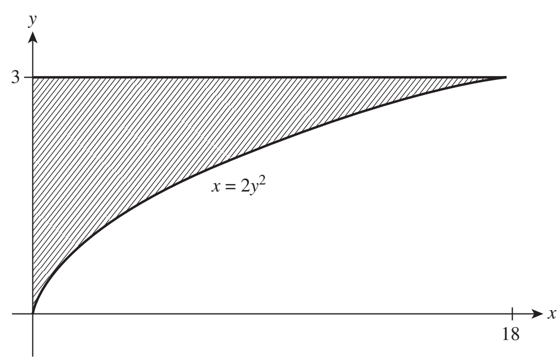


28. $\dfrac{7-4\sqrt{2}}{2}$

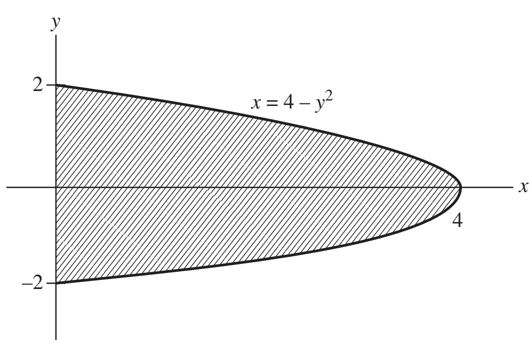


29. $\dfrac{1}{6}$ **30.** $\dfrac{9}{14}$

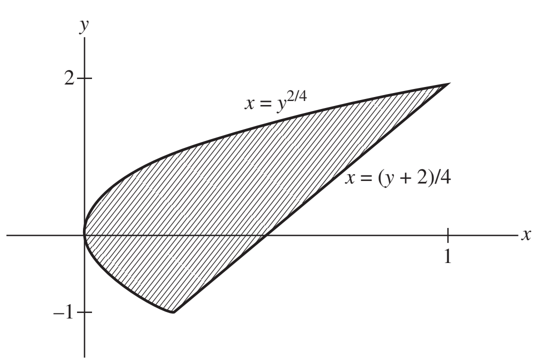
31. 18



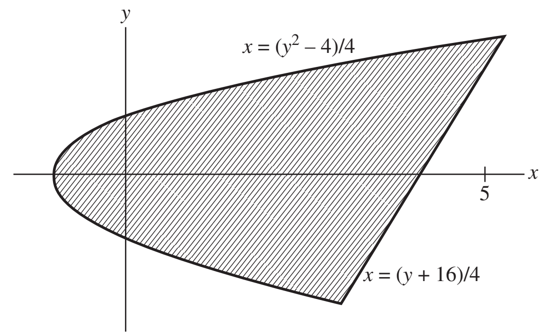
32. $\dfrac{32}{3}$



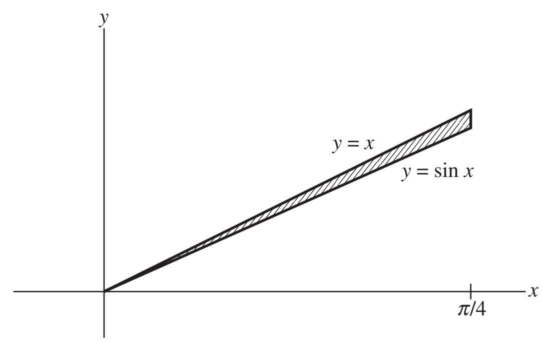
33. $\dfrac{9}{8}$



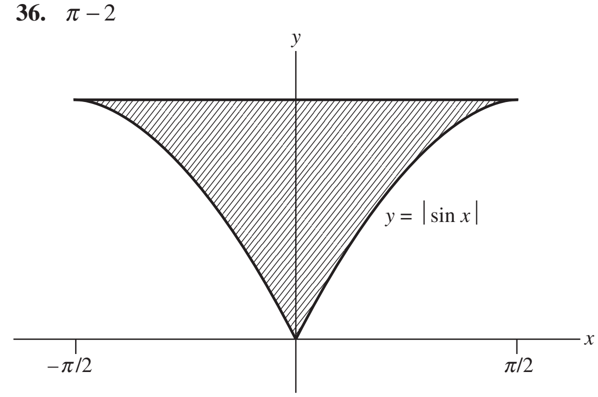
34. $\dfrac{243}{8}$



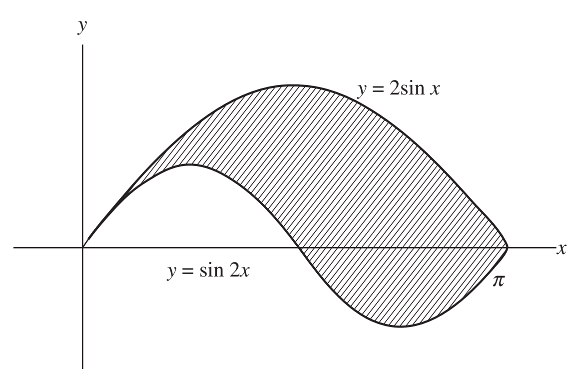
35. $\left(\dfrac{\pi^2}{32}+\dfrac{\sqrt{2}}{2}\right)-1$



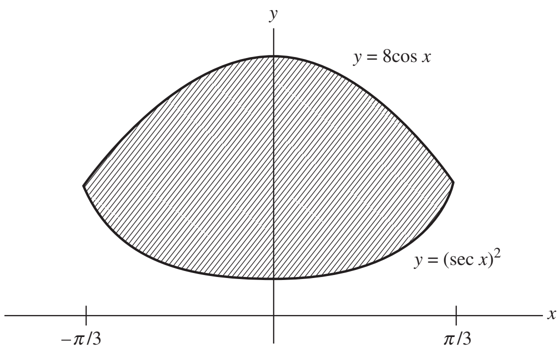
36. $\pi - 2$



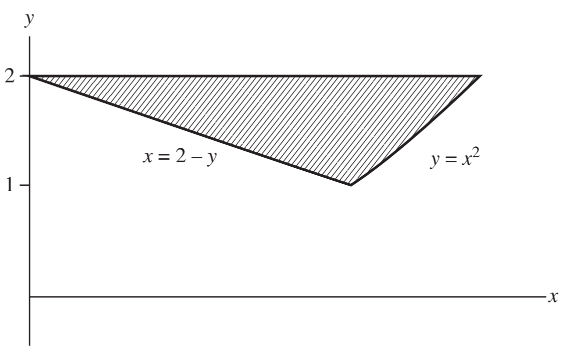
37. 4



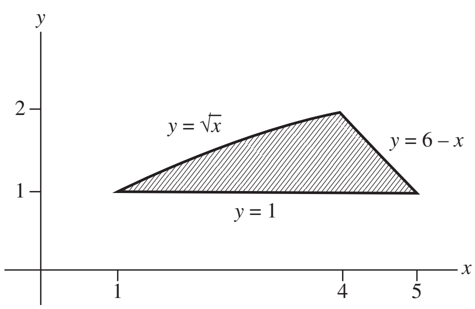
38. $6\sqrt{3}$



39. $\dfrac{8\sqrt{2}-7}{6}$



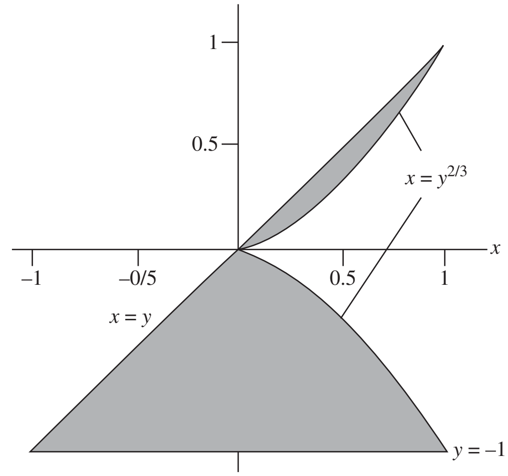
40. $\dfrac{13}{6}$



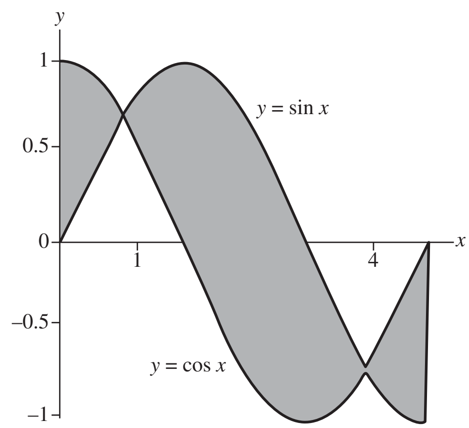
41. $\dfrac{27}{4}$

42. $\dfrac{a^2}{6}$

43. $\dfrac{6}{5}$



44. $4\sqrt{2}-2$



45. **(a)** b **(b)** b

46. **(a)** 2 **(b)** $\dfrac{2}{3}a$

47. $$f'_{av} = \frac{1}{b-a}\int_a^b f'(x)dx = \frac{1}{b-a}\left[f(x)\right]_a^b$$
$$= \frac{1}{b-a}[f(b)-f(a)] = \frac{f(b)-f(a)}{b-a}$$
so that average value of f' over $[a, b]$ is the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$, which is the average rate of change of f over $[a, b]$

48. Yes

Answers to Single Choice Questions

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (a)
11. (c)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (d)
21. (a)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (a)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (d)
31. (b)	**32.** (b)	**33.** (c)	**34.** (d)	**35.** (a)	**36.** (a)	**37.** (a)	**38.** (b)	**39.** (d)	**40.** (c)
41. (d)	**42.** (d)	**43.** (a)	**44.** (b)	**45.** (d)	**46.** (b)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (b)
51. (b)	**52.** (c)	**53.** (c)	**54.** (a)	**55.** (d)	**56.** (a)	**57.** (b)	**58.** (a)		

Answers to Multiple Choice Questions

1. (b), (c) and (d) **2.** (b) and (c) **3.** (a) (b) and (d) **4.** (a) and (c) **5.** (a) and (c)
6. (a) and (c) **7.** (a) and (d) **8.** (b) and (d) **9.** (a) and (c) **10.** (b) and (c)
11. (a), (b) and (d) **12.** (a), (b) and (d) **13.** (b) and (d) **14.** (b) and (d) **15.** (a) and (d)
16. (a) and (c) **17.** (a) and (d) **18.** (b), (c) and (d) **19.** (a) and (b) **20.** (a) and (d)
21. (b) and (c) **22.** (a) (b) (c) and (d) **23.** (a) and (b) **24.** (b) and (d) **25.** (b) and (d)
26. (b) and (d) **27.** (a), (b) and (c) **28.** (c) and (d)

Answers to Passage Type Questions

Passage 1
1. (b) 2. (b)
Passage 2
1. (c) 2. (b) 3. (a)
Passage 3
1. (b) 2. (d) 3. (a)
Passage 4
1. (b) 2. (a) 3. (d)
Passage 5
1. (c) 2. (a) 3. (b)
Passage 6
1. (b) 2. (b) 3. (b)
Passage 7
1. (b) 2. (a) 3. (c)

Answers to Matrix Match Type Questions

1. (a) $\rightarrow$ (q) (b) $\rightarrow$ (r) (c) $\rightarrow$ (q) (d) $\rightarrow$ (s)
2. (a) $\rightarrow$ (q) (b) $\rightarrow$ (p) (c) $\rightarrow$ (s) (d) $\rightarrow$ (r)
3. (a) $\rightarrow$ (p, q, s) (b) $\rightarrow$ (p, t) (c) $\rightarrow$ (p, q, r, t) (d) $\rightarrow$ (s)
4. (a) $\rightarrow$ (q) (b) $\rightarrow$ (r) (c) $\rightarrow$ (p) (d) $\rightarrow$ (s)
5. (a) $\rightarrow$ (r) (b) $\rightarrow$ (s) (c) $\rightarrow$ (p) (d) $\rightarrow$ (s)

Answers to Integer Type Questions

1. **(8)** **2.** (19) **3.** (3) **4.** $\left(\frac{1}{3}\right)$ **5.** $\left(\frac{125}{3}\right)$ **6.** (1) **7.** (108) **8.** (1) **9.** (6) **10.** (7)
11. (2) **12.** (5) **13.** (4) **14.** (91) **15.** (500) **16.** (1) **17.** (2) **18.** (12) **19.** (1) **20.** (5)
21. **(9)** **22.** (0) **23.** (8) **24.** (8) **25.** (2)

Answers to Additional and Advanced Exercises

1. 1

3. $1/\sqrt{2}$ units long by $1/\sqrt{e}$ units high, A=$1/\sqrt{2e} \approx$ 0.43 units^2

4. Absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units high.

5. $y = \left(\tan^{-1}\left(\frac{x+c}{2}\right)\right)^2$

6. $y - \ln y = (x+1)^3 + C$

7. $\sin(y^2) = 2\tan x + C_1$

8. $y = \pm\sqrt{\frac{-2}{\cos(x)} + c_1}$

9. $y = \ln(-e^{-(x+2)} + 2e^{-2})$

10. $e^{e\tan^{-1}(x)} + \ln 2$

11. $y = \left(2\sqrt{x} - 1\right)^2$

12. $y = \left[3\left(e^x - e^{-x}\right) + 1\right]^{1/3}$

13. **(a)** $y_1 - y_2$ **(b)** $\int 0\,dx = c$ **(c)** $y_1(x) = y_2(x)$ for $a < x < b$

14. $x^2(2y^2 + x^2) = C$

15. $\ln |x| - \frac{x}{y} = c$

16. $\ln|x| + e^{-y/x} = c$

17. $2\tan^{-1}\left(\frac{y}{x}\right) + \ln\left|y^2 + x^2\right| = c$

18. $\ln|x| - \ln\left|\sec\left(\frac{y}{x} - 1\right) + \tan\left(\frac{y}{x} - 1\right)\right| = c$

19. $\ln|x| + \ln\left|\sin\frac{y}{x}\right| = c$

20. $\frac{1}{2}$

21. $f(x) = \frac{x}{\sqrt{x^2+1}}$